TWO-PERSON FAIR DIVISION OF INDIVISIBLE ITEMS: BENTHAM VS. RAWLS ON ENVY

In the past forty years, there have been many books and articles on the fair allocation of divisible items, such as cake, land, or money. The fair allocation of indivisible items, such as physical property in a divorce, has also received attention.[1] The key property of fairness in this literature is envy-freeness, which the cake-cutting procedure, "I cut, you choose," satisfies. It guarantees each player at least half the value of a heterogenous cake, as each player values its parts, so neither player envies the other player for the portion it received.

Items that cannot readily be divided, such as marital property in a divorce, pose a more formidable problem of fair division. In this article, we focus on the properties of two-person fair allocation of indivisible items rather than algorithms to allocate the items or incentives of players to manipulate them.[2] These are important questions, but we think showing what combinations of properties are compatible or incompatible should precede questions of implementation and manipulation.

Assume two players wish to divide a finite set of indivisible items, over which they distribute the same number of points. If the utility of each player's bundle is the sum of the points it assigns to the items in it—so utilities are additive, and there are no synergies among subsets of items—we analyze what is a fair division of these items.

We begin by defining seven properties for the two-person fair division of indivisible items (an eighth will be defined in section II), based

[1] Different algorithms for allocating divisible and indivisible items have been analyzed and compared by Brams and Taylor and by Kilgour and Vetschera, among others. See Steven J. Brams and Alan D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution* (New York: Cambridge University Press, 1994); Steven J. Brams and Alan D. Taylor, *The Win-Win Solution: Guaranteeing Fair Shares to Everybody* (New York: W. W. Norton, 1999); and D. Marc Kilgour and Rudolf Vetschera, "Two-Player Fair Division of Indivisible Items: Comparison of Algorithms," *European Journal of Operational Research*, CCLXXI, 2 (December 2018): 620–31. For a recent review, see Christian Klamler, "The Notion of Fair Division in Negotiations," in D. Marc Kilgour and Colin Eden, eds., *Handbook of Group Decision and Negotiation*, 2nd ed. (Cham, Switzerland: Springer Nature, 2021), pp. 81–109.

[2] For an analysis of an indivisible-item algorithm and its manipulability, see Steven J. Brams, D. Marc Kilgour, and Christian Klamler, "How to Divide Things Fairly," *Mathematics Magazine*, LXXXVIII, 5 (December 2015): 338–48.

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on the utility of each player's bundle:

(1) ENVY-FREENESS (EF). An allocation is EF for a player if its utility for its bundle (assignment) is at least as great as its utility for the other player's bundle. The allocation is EF if it is EF for both players.
(2) EFFICIENCY OR PARETO-OPTIMALITY (PO). If each player's assignment in allocation S is at least as preferable as its assignment in allocation T, and in at least one case strictly preferable, then S is *Pareto-Superior* (PS) to T or Pareto-dominates T. An allocation T is *Pareto-Optimal* (PO) if there exists no allocation S such that S is PS to T.
(3) MAXIMINALITY (MM). An allocation is MM if there is no other allocation for which the minimum of the players' utilities is greater.
(4) MAXIMUM NASH WELFARE (MNW). An allocation is MNW if there is no other allocation for which the product of the players' utilities is greater.
(5) MAXIMUM TOTAL WELFARE (MTW). An allocation is MTW if there is no other allocation for which the sum of the players' utilities is greater.
(6) BALANCED (BL). An allocation is BL if each player receives the same number of items.
(7) LEXICOGRAPHIC OPTIMALITY (LO). For any allocation, write the players' utilities for their own assignments in ascending order. If each ascending utility for allocation S is at least equal to the corresponding ascending utility for allocation T, and in at least one case is strictly greater, then S is *Lexicographically Superior* (LS) to T. An allocation T is *Lexicographically Optimal* (LO) if there exists no allocation S such that S is LS to T.

Whereas an allocation is PO if there is no other allocation that each player values at least as much and one player values more, an allocation is LO if there is no other allocation for which the minimum and maximum assigned utilities are at least as great, and at least one assignment is greater. PS implies LS, but not vice versa. LO implies PO, but not vice versa.

In section I, we show that, if there is an EF allocation, there is an EF-PO-MM allocation, rendering these three properties compatible. But an EF-PO-MM allocation may not satisfy BL, MNW, MTW, or LO.

In section II, we show that if there is no EF allocation, it is always possible to satisfy EFX—a weaker form of EF to be defined in section II—but an EFX allocation also may fail any or all of BL, MNW, MTW, and LO. Moreover, if one player considers an item worthless (that is, assigns zero points to it), an EFX allocation may be Pareto-dominated by a non-EFX allocation that is MNW. We prove that, if there is no EF allocation, there must be an allocation that is EFX and MM.

In section III, we offer some concluding remarks on these compatibilities and incompatibilities and what seem to be the fairest allocations. We compare the views of the eighteenth-century utilitarian theorist Jeremy Bentham and the twentieth-century maximin theorist John Rawls,[3] suggesting that Rawls can be interpreted as favoring an EF or EFX allocation that is MM, and Bentham as favoring an MTW allocation but, because of its deficiencies, possibly finding the other maximum welfare allocation, MNW, acceptable.

I. PROPERTIES OF ENVY-FREE (EF) ALLOCATIONS

In the examples that follow, we call the two players A and B, and the items to be allocated $a, b, c, \ldots$ In our examples, the points the players assign to the items are non-negative integers $0, 1, 2, \ldots$ that sum to the same total for each player.

Thereby the players are on a par in terms of their entitlements—one player cannot "outbid" the other on every item. If one player's value for some item is greater, there must be another item that it values less than the other player.

We begin with an example that illustrates our notation and then show that an EF allocation may not be PO.

	a	b	c	d
A:	9	6	2	1
B:	1	2	6	9

Example 1. Sum of each player's points is 18.

In an allocation S of items to A and B, let S_A be A's bundle (assignment) of items, and let S_B be B's bundle. The utility of player A for S_A will be the sum of A's points for the items in S_A, or $u_A(S_A)$, and similarly for $u_B(S_B)$. We write the utilities of an allocation $S = (S_A, S_B)$ as $u[(S_A, S_B)] = [u_A(S_A), u_B(S_B)]$. Note that quantities in square brackets are utilities.

In Example 1, if $S_A = \{a\}$, or more simply a, and $S_B = \{b, c, d\}$, or more simply bcd, then $u(S_A, S_B) = [9, 17]$ because $u_A(S_A) = u_A(a) = 9$, and $u_B(S_B) = u_B(bcd) = 2+6+9 = 17$. This allocation is EF, because A is indifferent between bundle (item) a and bundle bcd (each bundle

[3]Jeremy Bentham, *An Introduction to the Principles of Morals and Legislation* (London: UCL Bentham Project, 1789/2017); John Rawls, *A Theory of Justice* (Cambridge, MA: Harvard University Press, 1971/1999).

gives A 9 points), whereas B prefers its 17 points from bundle *bcd* to 1 point from bundle *a*. In symbols, $u_A(S_A) = u_A(S_B) = 9$ and $u_B(S_B) = 17 > u_B(S_A) = 1$.

Observe that (ac, bd) is also EF, because $u(ac, bd) = [11, 11]$ so each player receives more than half the 18 points. In particular, A receives more points $(9 + 2 = 11)$ from its bundle than it would receive from B's bundle $(6 + 1 = 7)$, and B receives more points from its bundle $(2 + 9 = 11)$ than it would receive from A's bundle $(1 + 6 = 7)$.

But (ac, bd) is Pareto-dominated by (ab, cd), because $u(ab, cd) = [15, 15]$. Like (ac, bd), (ab, cd) is EF, but it is also PO, because there is no other allocation that gives each player utility at least 15 and one player more. In addition, (ab, cd) is MM, because there is no allocation that gives each player a utility greater than 15.[4] Hence, we call (ab, cd) an EF-PO-MM allocation.

EF allocations may not equally favor the players (in points), as illustrated by our next example.

	a	*b*	*c*	*d*
A:	9	4	3	2
B:	7	1	1	9

Example 2. Sum of each player's points is 18.

Observe that $u(a, bcd) = [9, 11]$, and $u(abc, d) = [16, 9]$. Neither (a, bcd) nor (abc, d) is BL, but both are EF, because each player's utility is at least half the total utility of 18. In addition, it is easy to verify that BL allocations (ab, cd) and (ac, bd) are also EF: $u(ab, cd) = [13, 10]$ and $u(ac, bd) = [12, 10]$.

Among the forgoing four EF allocations, (a, bcd) is PO, because there is no other allocation that gives B utility of 11 or more, so it is EF-PO. Allocations (ab, cd) and (ac, bd) are MM, because no other allocation gives both players utility greater than 10. However, because (ab, cd) Pareto-dominates (ac, bd)—and no other allocation Pareto-dominates (ab, cd)—only (ab, cd) is EF-PO-MM.

It is clear that (ab, cd), with utilities $u(ab, cd) = [13, 10]$, is LS to (a, bcd) with utilities $u(a, bcd) = [9, 11]$, because the players' utilities in ascending order are $[10, 13]$ and $[9, 11]$, and $10 > 9$ and $13 > 11$. Similarly, (abc, d) is LS to (a, bcd). In fact, even (ac, bd) is LS to (a, bcd). No other allocation gives ascending utilities that are at least equal to

[4] The allocations (a, bcd) and (abc, d), which give utilities $[9, 17]$ and $[17, 9]$ respectively, are EF-PO but not MM.

$[10, 13]$, so (ab, cd) is an EF-LO-MM allocation. But, as we will show later, it is not always the case that an EF-PO-MM allocation is also LO.

It is useful to distinguish PO and MM.

Proposition 1. PO and MM are independent properties of EF allocations.

Proof. In Example 2, we showed that (a, bcd) is EF-PO and (ac, bd) is EF-MM, but neither is EF-PO-MM. Hence, an EF allocation may satisfy PO without satisfying MM, and satisfy MM without satisfying PO, so the satisfaction of one property does not imply the satisfaction of the other, making PO and MM independent. □

Of course, Proposition 1 does not preclude an allocation from satisfying all three properties, as does allocation (ab, cd) in Example 2. It also does not imply that any EF allocation satisfies at least one of PO and MM. While all EF allocations in Example 2 are either PO or MM, this is not true for (ac, bd) in Example 1, which is EF but neither PO nor MM.

PO and MM measure, in different ways, how close two players come to benefiting, and benefiting equally, from an EF allocation. Fortunately, there is always an EF-PO-MM allocation in which these properties co-exist.

Proposition 2. If there is an EF allocation, then at least one EF allocation is EF-PO-MM.

Proof. Suppose there is an EF allocation, so each player values its bundle at least as much as the other player's bundle. Suppose that there is a possible switch of some items in the bundles so that one player values its new bundle at least as much and the other player values its new bundle more. Make this switch, and observe that the minimum of the two players' utilities has not decreased. Repeat. Because the number of items, and therefore the number of possible switches, is finite, at some point no such switch is possible, and the current allocation must be an EF-PO-MM allocation. □

Example 1 illustrates that switching b and c in (ac, bd), where $u(ac, bd) = [11, 11]$, yields $u(ab, cd) = [15, 15]$, and (ab, cd) is EF-PO-MM. In Example 2, beginning at the allocation (a, bcd), switch c to A, which increases the minimum—$u(ac, bd) = [12, 10]$ versus $u(a, bcd) = [9, 11]$—and then again increase the minimum by switching b and c—now achieving $u(ab, cd) = [13, 10]$—resulting in the EF-PO-MM allocation (ab, cd).

It is worth noting that the EF-PO-MM allocations in Examples 1 and 2 satisfy BL—each player receives two items. When two players take turns making choices, using what Brams and Taylor call "strict alternation," and there is an even number of items, BL will be satisfied.[5] In fact, strict alternation is a common procedure for dividing indivisible items, such as the marital property in a divorce or selecting players for two pickup teams in sports.

A prominent alternative to an EF-PO-MM allocation is an MNW allocation. The two allocations may coincide, but when they differ it is useful to compare their properties. An MNW allocation is always PO,[6] but it may fail to satisfy both EF and MM, as shown by our next example.

	a	*b*	*c*	*d*
A:	9	4	4	1
B:	7	1	2	8

Example 3. Sum of each player's points is 18.

The unique EF-PO-MM allocation is (ab, cd), with utilities $u(ab, cd) = [13, 10]$ and utility product $13 \times 10 = 130$. By contrast, the MNW allocation, which maximizes the product of the players' utilities, is (abc, d), because $u(abc, d) = [17, 8]$, and $17 \times 8 = 136$.

But the MNW allocation, while PO, is neither EF (B envies A for getting what it thinks is 10 when it gets only 8) nor MM (the EF-PO-MM allocation (ab, cd) gives a minimum utility of 10, whereas the MNW allocation gives B only 8). However, the MNW allocation does satisfy MTW, giving a maximum sum of $17 + 8 = 25$ vs. $13 + 10 = 23$ for the EF-PO-MM allocation, showing in this case that the two welfare properties, MNW and MTW, agree on an allocation (but this is not always the case, as we will see in section II).

What makes it difficult to say whether the EF-PO-MM or the MNW-PO allocation is "better" is that neither is LS with respect to the other. The ascending utilities of (ab, cd) are $[10, 13]$, and the ascending utilities of (abc, d) are $[8, 17]$, so there is no LS relation between them. Example 3 proves the following proposition.

Proposition 3. If an EF-PO-MM allocation differs from an MNW allocation, the MNW allocation may satisfy neither EF nor MM. There may be no LS relation between the PO-MNW allocation and the EF-PO-MM allocation.

Proposition 3 casts some doubt on the "unreasonable fairness" of MNW,[7] because it may fail EF, MM, and LO, even though it is PO (so, of course, is an EF-PO-MM allocation). As we will show in the next

[5] Brams and Taylor, *The Win-Win Solution*, *op. cit.*, ch. 2.

[6] Ioannis Caragiannis et al., "The Unreasonable Fairness of Maximum Nash Welfare," *ACM Transactions on Economics and Computation*, VII, 3 (September 2019): 305–22.

[7] *Ibid.*

section, however, an MNW allocation may Pareto-dominate an EFX allocation, the weaker form of EF that we discuss next.

II. ENVY-FREENESS AFTER THE REMOVAL OF ANY ITEM (EFX)

For indivisible items, there will often be no EF allocation when the players have similar preferences, which precludes an EF-PO-MM allocation. As an alternative, EFX has been conjectured but not proven to exist in the n-person fair division of indivisible items, although the conjecture has been proven for two-person fair division.[8]

(8) Envy-Freeness after the Removal of Any Item (EFX). Assume an allocation is not EF because, say, A envies B. Then it is EFX if the removal of any item from B's assignment eliminates A's envy. (The roles of A and B may be interchanged.)[9]

It is worth noting that EFX was not the first relaxation of EF to be proposed. A weaker notion of envy-freeness, EF1, says that the removal of one specific item—which might be the one that A ranks highest in B's bundle—eliminates A's envy. EF1 is easier to satisfy than EFX and, indeed, such an allocation always exists, even in the n-person case.

In the following example, it is apparent that there is no EF allocation, because if either player receives any two of the three items, the other player will be envious.

	a	b	c
A:	2	3	4
B:	3	2	4

Example 4. Sum of each player's points is 9.

Two allocations, (ab, c) with $u(ab, c) = [5, 4]$ and (c, ab) with $u(c, ab) = [4, 5]$, are EFX, because the removal of either item a or

[8] In fact, the conjecture has been proven for up to three players, but not in the general n-person case whenever players give more than two values to items; see Georgios Amanatidis et al., "Maximum Nash Welfare and Other Stories about EFX," *Theoretical Computer Science*, DCCCLXIII (April 2021): 69–85; and Ben Berger et al., *Proceedings of the AAAI Conference on Artificial Intelligence*, XXXVI, 5 (June 2022): 4826–33. An allocation is EFX in the n-person case if every pair of players satisfies the two-person EFX property (defined next). It is worth pointing out that several paradoxes crop up in the n-person division of indivisible items; see Steven J. Brams, Paul H. Edelman, and Peter C. Fishburn, "Paradoxes of Fair Division," this JOURNAL, XCVIII, 6 (June 2001): 300–14.

[9] The existence of an EF allocation does not preclude there being an EFX allocation, which would necessarily be different. As Example 5 shows, EF and EFX allocations may co-exist.

b eliminates the c-player's envy of the ab-player. On the other hand, other allocations are not EFX, such as (ac, b), for which $u(ac, b) = [6, 2]$, because the removal of item a or c from A leaves B envious, who gets 2 but envies A for getting what B thinks is 3 or 4.

The EFX allocations in Example 4, (ab, c) and (c, ab), do not satisfy MNW. The product of the players' utilities for these allocations is $5 \times 4 = 20$, whereas two non-EFX allocations—(b, ac) with $u(b, ac) = [3, 7]$, and (bc, a) with $u(bc, a) = [7, 3]$—both have a greater utility product ($3 \times 7 = 21$). Because that is the highest achievable utility product, these two allocations are MNW.

Like the unique EF-PO-MM allocation in Example 3, the two EFX allocations in Example 4 are EFX-PO-MM but not MNW. Neither the EFX nor the MNW allocations in Example 4 are related by LS—all of them are LO.

EF and EFX allocations may co-exist, as shown by the next example.

	a	b	c	d
A:	10	5	4	4
B:	10	7	5	1

Example 5. Sum of each player's points is 23.

The allocation (ad, bc), with utilities [14, 12], is EF. The allocation (a, bcd), with utilities [10, 13], is EFX, as A envies B but would not envy B if any of B's items were removed. Note that the EF allocation is MM, MNW, and MTW.

EFX allocations exhibit a significant drawback when a player gives zero points to an item, as illustrated by Example 6 (odd number of items) and Example 7 (even number of items).[10]

	a	b	c
A:	5	1	1
B:	4	3	0

Example 6. Sum of each player's points is 7.

The allocation (a, bc), where $u(a, bc) = [5, 3]$, is EFX, because the removal of item a eliminates B's envy of A. But (a, bc) is Pareto-

[10] If there is an item that *both* players consider worthless (that is, give zero points to), we exclude it from an allocation. But if only one player gives zero points to an item, we do not automatically award it to the other player, because the player who indicated it is worthless may value it positively but want to maximize the number of points it gives to items that it values more. Thereby we do not assume that players' point allocations are necessarily sincere; strategic factors may play a role in their allocations, which here we take at face value.

dominated by (ac, b), where $u(ac, b) = [6, 3]$. However, (ac, b) is not EFX—the removal of item c leaves B envious—even though it is MNW.

	a	b	c	d
A:	6	1	1	1
B:	5	3	1	0

Example 7. Sum of each player's points is 9.

The allocation (a, bcd), where $u(a, bcd) = [6, 4]$, is EFX, but it is Pareto-dominated by (ad, bc), where $u(ad, bc) = [7, 4]$. However, (ad, bc) is not EFX, whereas it is MNW.

In both Examples 6 and 7, one item (item a) is a "diamond" that is worth more to A, and to B, than all the other items combined (the "pebbles"). We call an example in which there is a single diamond that is more valued by both players than all the pebbles combined a *diamond-pebble situation.*[11]

In a diamond-pebble situation, there are no EF allocations, and the only EFX allocations are those that assign the diamond to one player and all the pebbles to the other. In particular, if A gets the diamond in an EFX allocation, B must get all the pebbles, including the one that it considers worthless (item c in Example 6 and item d in Example 7).

As we showed in Examples 6 and 7, EFX allocations (a, bc) and (a, bcd) are Pareto-dominated by non-EFX allocations that are MNW. To be sure, there are alternative EFX allocations in which B obtains the diamond—(bc, a), with $u(bc, a) = [2, 4]$, in Example 6, and (bcd, a), with $u(bcd, a) = [3, 5]$ in Example 7—that are PO. But they are neither MNW nor MM and, therefore, not as appealing as the non-EFX allocations, which give $[6, 3]$ and $[7, 4]$ in Examples 6 and 7 and are also LO.

The underlying reason why the EFX allocations in Examples 6 and 7 are not PO is that one pebble in these allocations is 0-valued by B but positive-valued by A. If A is given this pebble, the resulting allocation, which is not EFX, is PO as well as MNW.

Proposition 4. In a diamond-pebble situation, there are no EF allocations and two EFX allocations. In an EFX allocation, one player receives the diamond and the other all of the pebbles. If there is a pebble that is 0-valued by, say, B but positive-valued by A, then the EFX allocation in which A is assigned the diamond is Pareto-dominated by a non-EFX allocation.

[11] In a divorce, the diamond might be the house, and all the other indivisible property the pebbles.

Proof. There are no EF allocations because the player who does not receive the diamond will envy the one who does. An allocation in which one player receives the diamond and at least one pebble cannot be EFX, as the other player's envy will not be eliminated by removal of the pebble. Thus, the EFX allocations are exactly those in which one player receives the diamond and the other all of the pebbles. If there is a pebble that is 0-valued by B but not A, the EFX allocation in which A is assigned the diamond is Pareto-dominated by the non-EFX allocation in which the 0-valued pebble is transferred from B to A. □

One consequence of Proposition 4 is that, in an allocation problem with an even number of items, there may be no EF or EFX allocation that satisfies BL. Indeed, in a diamond-pebble situation with $n \geq 3$ pebbles, where n is odd, there are no EF allocations, and in any EFX allocation one player receives the diamond and the other all $n \geq 3$ pebbles.

Because of Proposition 4, the existence of 0-valued items in a diamond-pebble situation may make EFX and PO incompatible properties, as shown by Example 8, which also illustrates a connection to MM.

	a	b	c	d
A:	5	2	0	2
B:	6	2	1	0

Example 8. Sum of each player's points is 9.

The EFX allocations in Example 8 are (a, bcd) and (bcd, a), with utilities $[5, 3]$ and $[4, 6]$, respectively. Neither EFX allocation is PO; (a, bcd) is Pareto-dominated by (ad, bc), with utilities $[7, 3]$, and (bcd, a) is Pareto-dominated by (bd, ac), with utilities $[4, 7]$. Note that the MM allocations are (bcd, a) and (bd, ac); the latter, with utilities $[4, 7]$, Pareto-dominates the former, with utilities $[4, 6]$.

When an EFX allocation is Pareto-dominated by a non-EFX allocation, Proposition 4 does not preclude the existence of a second EFX allocation that is LS to the non-EFX allocation.

	a	b	c	d
A:	5	2	1	1
B:	6	2	1	0

Example 9. Sum of each player's points is 9.

In Example 9, the allocation (a, bcd), where $u(a, bcd) = [5, 3]$, is EFX. It is Pareto-dominated by the non-EFX allocation (ad, bc) where $u(ad, bc) = [6, 3]$. But a second EFX allocation, (bcd, a), gives utilities $[4, 6]$. It is LO-MM, and therefore PO, proving the next proposition.

Proposition 5. In a diamond-pebble situation with a 0-valued item, an EFX allocation may be LS to a non-EFX allocation that Pareto dominates another EFX allocation.

In Example 9, the EFX-PO-MM allocation is also MNW (utility product $4 \times 6 = 24$ vs. $6 \times 3 = 18$ for the non-EFX allocation). However, in a diamond-pebble situation, it is not always the case that an EFX allocation is MNW.

	a	b	c
A:	6	3	2
B:	6	4	1

Example 10. Sum of each player's points is 11.

Notice that both players value the diamond the same, but they value the two pebbles differently.

Allocation (ac, b), which gives utilities $[8, 4]$, has the largest utility product of any allocation (32) and so is MNW. It also has the largest utility sum and so is MTW. By contrast, the EFX allocations (a, bc) and (bc, a), with utilities $[6, 5]$ and $[5, 6]$, respectively, both have utility product 30. These two allocations are EFX-PO-MM, but they are not LS to (ac, b), which proves the next proposition.

Proposition 6. In a diamond-pebble situation, an EFX-PO-MM allocation may be different from, and not LS to, an allocation that is MNW and MTW.

This proposition echoes Proposition 3 that, together with Proposition 6, shows that neither EF nor EFX (if there is no EF allocation) may be MNW.

We complete this section with a theorem that unexpectedly links MM, EF, and EFX allocations. Because it is considerably more difficult to prove than our propositions, we have put its proof in the appendix so as not to break the flow of the text.

Theorem 1. If the minimum utility of an MM allocation is at least half of the total utility, then every MM allocation is EF, and at least one is EF-PO-MM. If that minimum is less than half of the total utility and there are no 0-valued items, then all MM allocations are EFX, and at least one is EFX-PO-MM.

We note that an MM allocation must always exist, and every MM allocation must yield the same minimum utility. Theorem 1 states that, if that minimum is at least half of the total utility, then every MM allocation is EF and at least one is PO; if it is less than half, and there are no 0-valued items, then every MM allocation is EFX, and at least one is PO. (Example 4 shows that it is possible for the minimum utility at an MM allocation to be less than half of the total available.)

We think this is a significant finding, because it means that an allocation that is neither EF nor EFX cannot be MM, even if it is, say, an MNW allocation. As a case in point, the minimum utility of the EFX allocation in Example 10 is 5, whereas that of the MNW allocation, which is not EFX, is 4.

We note that Theorem 1 depends on our assumption that each player's item utilities have the same sum. To illustrate what happens when this assumption fails, consider the following example:

	a	*b*	*c*	*d*
A:	10	10	10	10
B:	1	1	1	1

Example 11. Sum of A's points is 40. Sum of B's points is 4.

Any MM allocation must assign 1 item to A and 3 to B. No such allocation can be EF or even EFX, as A will envy B, and deleting any item from B's bundle will not remove this envy. There are no 0-valued items. Thus the conclusion of Theorem 1 fails for Example 11.

To recapitulate, Theorem 1 shows that

(1) If there are any EF allocations, then at least one MM allocation is EF-PO-MM.
(2) If there are no EF allocations and no 0-valued items, then at least one MM allocation is EFX-PO-MM.

We also noted that these EF or EFX allocations may fail MNW, MTW, and BL.

So far, we have not discussed in detail compatible and incompatible properties of MTW, which we associate with Bentham's principle that "it is the greatest happiness of the greatest number that is the measure of right and wrong." Putting aside the moral implications of "right and wrong," we associate Bentham's principle with MTW and illustrate it with the following example:

	a	*b*	*c*	*d*
A:	9	4	4	1
B:	4	3	3	8

Example 12. Sum of each player's points is 18.

The allocation (ab, cd) is EF-PO-MM-BL, with utilities $[13, 11]$. The utility sum is 24. But an unbalanced allocation, (abc, d), with utilities

[17, 8], is MTW, with a utility sum of 25. Not only is (*abc*, *d*) unbalanced, but it is not EF (B receives only 8 out of 18 points), MM (its minimum utility is 8 vs. 11 for the EF-PO-MM-BL allocation), or MNW (its utility product is 136 vs. 143 for the EF-PO-MM-BL allocation). Yet another unbalanced allocation, (*a*, *bcd*), with utilities [9, 14], is EF, but it has a smaller utility sum (23 vs. 24 and 25), utility product (126 vs. 136 and 143), and is not MM.

In short, an MTW allocation may fail several of our properties. If there is an EF-PO-MM allocation that is also MNW, as there is in Example 12, it is clearly preferable. If there is no EF allocation, there is a trade-off between an EFX-PO-MM allocation, on the one hand, and an MNW-PO allocation on the other, if it is different. We regard the former allocations to be Rawlsian because, as Theorem 1 shows, they are always MM, whereas the latter are more Benthamite without the manifest failings of MTW.

III. CONCLUSIONS

The two-person fair division of indivisible items is a ubiquitous problem, from allocating marital property in a divorce to putting together competitive teams in a sports league. Insofar as two players can allocate points to the items being divided, and these points are additive and determine the values of bundles, it is not obvious which allocation is the fairest.

First, there may be different EF allocations that favor different players. This leaves open which allocation is the fairest, especially if one allocation is PO and the other is MM, which we showed was possible. Happily, if there is an EF allocation, there is always one that is PO and MM as well, rendering all three properties compatible.

Second, if there is no EF-PO-MM allocation, which we consider the ideal, and if there are no items that one player thinks worthless (0-valued), there is always an EFX-PO-MM allocation. But these EF or EFX allocations may not satisfy MNW, MTW, or BL. In addition, if there is a 0-valued item, an EFX allocation may be Pareto-dominated by a non-EFX allocation. In this case, there may exist another EFX allocation that is MNW and is LS to the non-EFX allocation. But it is also possible that there is no EFX-PO-MM allocation—every EFX allocation is Pareto-dominated by a non-EFX allocation. On the other hand, at least one EFX-PO-MM allocation must exist if there are no 0-valued items.

Although incompatibilities among some of the desirable properties we postulated suggest—in the absence of an EF-PO-MM allocation—that there is no "perfect" two-person fair division of indivisible items, EFX and MNW divisions are the most compelling alternatives for the two-person fair division of indivisible items. By contrast, we argued that an MTW allocation is not attractive, especially when it differs from an EF, EFX, or MNW allocation.

An EF or EFX allocation, when it satisfies MM, reflects the Rawlsian view of helping the worse-off player, whereas an MNW allocation—when it differs from an EF or EFX allocation—reflects, at least to some degree, the Benthamite view of MTW without some of its flagrant deficiencies. To be sure, there is no conflict when EF or EFX allocations coincide with MTW allocations, but when they do not, the choice is not so clear as to which allocation is the fairest.

Beyond the technical results of our analysis, we think there are four broader philosophical implications:

(1) It is too simple to say that one must make a choice between Bentham, the utilitarian, and Rawls, the maximinalist. There is middle ground between these two theorists, with one compromise position being to choose an allocation that maximizes the product of the players' total utilities (MNW) instead of the sum (MTW).
(2) But MNW does not necessarily eliminate one player's envy of the other, because there may be no EF allocation. However, an EFX allocation moves the envious player toward an EF allocation via the removal of any item from the envied player's bundle, which may be the item that the envious player least values. Setting aside this item, therefore, allows for at least a partial solution to the problem of envy.
(3) If one player places zero value on an item and the other player does not, an EFX allocation may be Pareto-dominated by one that is not EFX. By giving the item in question to the player who values it positively, one loses EFX to satisfy PO. In this situation, the trade-off is between greater overall satisfaction (PO) and reduced envy (EFX).
(4) On the positive side, we proved that if there is no EF allocation and no 0-valued items, there is always an EFX allocation that is MM, which is exactly the property that Rawls championed.

APPENDIX

This appendix is a proof of Theorem 1:

Theorem 1. If the minimum utility of an MM allocation is at least half of the total utility, then every MM allocation is EF, and at least one is EF-PO-MM. If that minimum is less than half of the total utility and there are no 0-valued items, then all MM allocations are EFX, and at least one is EFX-PO-MM.

We normalize the players' utilities so that a player's total available utility is 1. Thus, for any allocation (S_A, S_B), $u_A(S_A \cup S_B) =$

$u_A(S_A) + u_A(S_B) = 1$ and $u_B(S_A \cup S_B) = u_B(S_A) + u_B(S_B) = 1$. Note that, if $u_A(S_A) \geq \frac{1}{2}$, then $u_A(S_A) \geq u_A(S_B)$, so A does not envy B. Similarly, if $u_B(S_B) \geq \frac{1}{2}$, then B does not envy A. In particular, if $\min\{u_A(S_A), u_B(S_B)\} \geq \frac{1}{2}$, then (S_A, S_B) is an EF allocation.

An MM allocation is an allocation for which the minimum utility, $\min\{u_A(S_A), u_B(S_B)\}$, is a maximum. At least one MM allocation must exist, all of them must have the same minimum utility, and at least one must be PO. If an MM allocation satisfies

$$\min\{u_A(S_A), u_B(S_B)\} \geq \frac{1}{2},$$

then that allocation must be EF. Moreover, if an EF allocation exists, then the minimum utility at an MM allocation must equal at least $\frac{1}{2}$, so there must be an EF-PO-MM allocation.

To prove the Theorem, we suppose there is no EF allocation and no 0-valued items, and consider—to achieve a contradiction—an allocation (S_A, S_B) that is MM but not EFX. Without loss of generality, we assume

(1) $u_A(S_A) \leq u_B(S_B)$.

Claim 1. $u_A(S_A) < u_A(S_B)$.

Proof. To obtain a contradiction, assume that $u_A(S_A) \geq u_A(S_B)$. Then, since $u_A(S_A \cup S_B) = 1$, we have $u_A(S_A) \geq \frac{1}{2} \geq u_A(S_B)$. Thus A does not envy B. On the other hand, since $u_B(S_B) \geq u_A(S_A)$ by (1), it follows that $u_B(S_B) \geq u_A(S_A) \geq \frac{1}{2} \geq u_B(S_A)$. Thus B does not envy A, which contradicts our assumption that no EF allocation exists. □

Note that $u_A(S_A) < \frac{1}{2}$, as a direct consequence of Claim 1 and $u_A(S_A \cup S_B) = u_A(S_A) + u_A(S_B) = 1$.

Claim 2. $u_B(S_B) \geq 1/2$.

Proof. To obtain a contradiction, suppose that $u_B(S_B) < \frac{1}{2}$. Then $u_B(S_A) > \frac{1}{2} > u_B(S_B)$. Now consider the allocation (S_B, S_A)—that is, reverse the assignments of A and B. We have shown that $u_B(S_A) > u_B(S_B)$ which, together with Claim 1, shows that the allocation (S_B, S_A) gives both players greater utility than (S_A, S_B), contradicting our assumption that (S_A, S_B) is MM. □

Claim 3. For any item $x \in S_B$, $u_B(S_B \setminus x) \leq u_A(S_A)$.

Proof. To obtain a contradiction, assume that for some $x \in S_B$, $u_B(S_B \setminus x) > u_A(S_A)$. Then consider the allocation (S'_A, S'_B) defined by $S'_A = S_A \cup x$ and $S'_B = S_B \setminus x$. Because there are no 0-valued items, $u_A(S'_A) = u_A(S_A) + u_A(x) > u_A(S_A)$. Also, $u_B(S'_B) = u_B(S_B \setminus x) > u_A(S_A)$ by assumption. Thus, $\min\{u_A(S'_A), u_B(S'_B)\} > u_A(S_A) = \min\{u_A(S_A), u_B(S_B)\}$, which contradicts our original assumption that (S_A, S_B) is MM. □

Claim 4. There is some $x \in S_B$ such that $u_A(S_B \setminus x) > u_A(S_A)$.

Proof. By Claim 2, B does not envy A. Because (S_A, S_B) is not EF, A must envy B. The claim is a direct consequence of the assumption that this allocation is not EFX. □

We are now ready to prove Theorem 1. Recall that, to obtain a contradiction, we assumed that the allocation (S_A, S_B) is MM but not EFX.

Let $x \in S_B$ be as guaranteed in Claim 4. Consider a new assignment: give the bundle $S_B \setminus x$ to A and the bundle $S_A \cup x$ to B. Then the minimum utility of the new assignment is

$$(2) \quad \min(u_A(S_B \setminus x), u_B(S_A \cup x)) = \min(u_A(S_B \setminus x), 1 - u_B(S_B \setminus x)).$$

By Claim 4, $u_A(S_B \setminus x) > u_A(S_A)$. By Claim 3, $1 - u_B(S_B \setminus x) \geq 1 - u_A(S_A)$, which is strictly larger than $u_A(S_A)$ by Claim 1. Therefore, the minimum in (2) is strictly larger than $u_A(S_A)$, the minimum utility for the allocation (S_A, S_B). This finding contradicts the assumption that the original assignment is MM, proving that if there are no EF allocations and no 0-valued items, an MM allocation must be EFX. The proof is completed by noting that at least one MM allocation must be PO, so an EFX-PO-MM allocation must exist.

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