

On Quantifying into Predicate Position: Steps towards a New(tralist) Perspective¹

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In the *Begriffsschrift* Frege drew no distinction—or anyway signalled no importance to the distinction—between quantifying into positions occupied by what he called *eigennamen*—singular terms—in a sentence and quantification into predicate position or, more generally, quantification into open sentences—into what remains of a sentence when one or more occurrences of singular terms are removed. He seems to have conceived of both alike as perfectly legitimate forms of generalisation, each properly belonging to logic. More accurately: he seems to have conceived of quantification *as such* as an operation of pure logic, and in effect to have drawn no distinction between first-order, second-order and higher-order quantification in general.

In the 20th century the prevailing view, largely under the influence of writings of and on Quine, became quite different. The dominant assumption was of the conception of quantification encapsulated in Quine's famous dictum that "to be is to be the value of a variable". The dictum, as we know, wasn't really about existence but rather about ontological *commitment*. Subsequently, it was better expressed as "to be *said to be* is to be the value of a variable". Writing in a context when ontic fastidiousness was more fashionable than now, Quine was preoccupied with possibilities of masked ontological commitments and, conversely, of merely apparent ontological commitments. His proposal was: regiment the theory using the syntax of individuation, predication and quantification and then see what entities you need to regard as lying in the range of the bound variables of the theory if it is to rank as true. You are committed, as a theorist, to what you need to quantify over in so formulating your theory. (As to the prior question of how we are supposed to recognise the *adequacy*—or inadequacy—of such a regimentation, Quine was largely modestly silent.)

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The Quinean proposal can seem almost platitudinous. But once it is accepted, first- and second-order quantification suddenly emerge as standing on a very different footing. First-order quantification quantifies over *objects*. No one seriously doubts the existence of objects. By contrast, second-order quantification seems to demand a realm of *universals*, or *properties*, or *concepts*. And of such entities Quine canvassed an influential mistrust: a mistrust based, initially, on their mere abstractness—though Quine himself later, under pressure of the apparent needs of science, overcame his phobia of the abstract—but also on the ground that they seem to lack clear criteria of identity—a clear basis on which they may be identified and distinguished among themselves. It was the latter consideration which first led Quine to propose that the range of the variables in higher-order logic might as well be taken to be *sets*—abstract identities no doubt, but ones with a clear criterion of identity given by the axiom of extensionality—and then eventually to slide into a view in which "second-order logic" became, in effect, a misnomer—unless, at any rate, one regards set theory as logic.² By 1970 he had come to his well-known view:

Followers of Hilbert have continued to quantify predicate letters, obtaining what they call higher-order predicate calculus. The values of these variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic set theory's staggering existential assumptions are cunningly hidden now in the tacit shift from schematic predicate letters to quantifiable set variables.³

Those remarks occur in Quine's paper, "The Scope of Logic" in the sub-section famously entitled: Set Theory in Sheep's Clothing! By the end of that paper, Quine has persuaded himself, and probably most of his readers too, that Frege and others such as Russell and Hilbert who followed him in allowing higher-order quantification, have simply muddied the distinction between logic properly so regarded—the theory of the valid patterns of inference sustained by the formal characteristics of thoughts expressible using singular reference, predication, quantification and identity—and set theory which, to the contrary, is properly regarded as a branch of mathematics.

My principal concern in what follows will be to outline what I think is the proper, and overdue, reaction to—and against—the conception of quantification which drives so much of the Quinean philosophy of logic: the conception which sees quantification into a particular kind of syntactic position as essentially involving a *range of entities* associated with expressions fitted to

² The Quinean tradition has, of course, absolutely no regard for Frege's notion of a *logical object*.

³ W. V. Quine, *Philosophy of Logic* (Cambridge, Mass.: Harvard University Press 1970) at p.68

occupy that kind of position and as providing the resources to generalise about such entities. This view of the matter is so widespread that it has become explanatory orthodoxy. For instance in the entry under "Quantifier" in his splendid *Oxford Dictionary of Philosophy*, Simon Blackburn writes:

Informally, a quantifier is an expression that reports a quantity of times that a predicate is satisfied in some class of *things*, i.e. in a 'domain' [my emphasis, p. 313.],

while the corresponding entry in Anthony Flew's and Stephen Priest's *Dictionary of Philosophy* observes that

The truth or falsity of a quantified statement.... cannot be assessed unless one knows what totality of objects is under discussion , or where the values of the variables may come from. (Flew and Priest *A Dictionary of Philosophy*, p. 338.)

I believe that this conception—of quantification as essentially *range-of-associated-entities-invoking*—is at best optional and restrictive and at worst a serious misunderstanding of what quantification fundamentally is; more, that it squanders an insight into the nature of the conceptual resources properly regarded as logical which Frege already had in place at the time of *Begriffsschrift*.

Why is this an issue apt for discussion in an anthology whose focus is the epistemology of mathematics? What has the question of the proper conception of higher-order quantification got to do with mathematical knowledge? Part of the answer lies in the integral role played by higher-order logic in the contemporary programme in the philosophy of mathematics—sometimes called neo-Fregeanism, or neo-logicism but I prefer *abstractionism*—which seeks to save as much as possible of the doomed project of Frege's *Grundgesetze* by replacing his inconsistent Basic Law V with a number of more local *abstraction principles* designed to provide the deductive resources for derivations of the fundamental laws of, for example, classical number theory and analysis in systems of second-order logic.⁴ There has been much discussion of the status and credentials of abstraction principles for their part in this project. But one thing is clear: if the *logic* used in the abstractionist programme is indeed, as Quine thought, nothing but set theory in disguise, then execution of the various local abstractionist projects, however technically

⁴ A brief sketch of the current situation with respect to the prospects of technical recovery of arithmetic, analysis, and set theory on an abstractionist basis is provided in the Appendix.

successful, will be of greatly diminished philosophical interest. A reduction of arithmetic, or analysis, to a special kind of axiom cum *set theory* will hardly be news! It is therefore, generally speaking, a presupposition of the significance of the abstractionist programme that there be a quite different way of thinking about higher-order logic than the Quinean way—and that this different way, whatever exactly it is, should be consonant with the general spirit of logicism: the thesis that logical knowledge and at least basic mathematical knowledge are, in some important sense, of a single epistemological kind.⁵

The connection with the abstractionist programme is, however, only part of the reason why the epistemologist of mathematics should interest herself in the nature of higher-order logic. It is probably fair to say that contemporary philosophers of mathematics number many more structuralists than neo-logicists, and among structuralists there is an increasing trend of placing weight on higher-order logics both as sources of categorical axioms — presumed to provide determinate characterisations of the structures of which pure mathematics is conceived as the special science —and as a safe medium for their investigation. For this programme, too, pure mathematical knowledge is heavily invested in and conditional on the good standing of the concepts and deductive machinery of higher-order logic.

To develop a new overarching conception of quantification which is powerful enough to make sense of the classical logic of the *Begriffsschrift* yet stands contrasted with the Quinean orthodoxy in the way prefigured is a tall order. The task is part philosophical and part technical. My goal in what follows has to be limited. I shall attempt no more than to provide some primary motivation for the thought that such an alternative—non set-theoretic, properly logical—conception does indeed exist, and to outline some of the cruces which will have to be negotiated in developing it properly. In short, my aim here is no more than to outline a project, as a spur to further work.

I. Basic Idea and Project Outline

⁵ The kind in question is sometimes gestured at by the inflammatory term, “analytic”. A more useful characterisation will appeal to a class of concepts whose content allows of explanation by means that (ancestrally) presuppose a familiarity only with logical concepts, and a class of truths formulable using just such concepts which can be recognised to be true purely on the basis of those explanations and logical deduction.

The proposal is to explain and vindicate what we may term a *neutralist*⁶ conception of higher-order quantification—and indeed of quantification more generally. Here is a quotation from a recent paper by Agustin Rayo and Stephen Yablo that I think gives us a pointer in the right direction:

If predicates and the like needn't name to be meaningful—to make their characteristic contribution to truth-value—then we have no reason to regard them as presupposing entities at all. And this indeed appears to be Quine's view. But now he goes on to say something puzzling:

One may admit that there are red houses, roses, and sunsets, but deny, except as a popular and misleading manner of speaking, that they have anything in common (W. V. Quine, *From a Logical point of View*, Harper & Row: New York 1953, at p. 10).

Quine is right, let's agree, that "there are red houses, roses and sunsets" is not committed to anything beyond the houses, roses and sunsets, and that one cannot infer that "there is a property of redness that they all share." But why should "they have something in common"—or better, "there is something that they all are"—be seen as therefore misleading? If predicates are noncommittal, one might think, the quantifiers *binding* predicative positions are not committal either. After all, the commitments of a quantified claim are supposed to line up with those of its substitution instances. Existential generalisations are *less* (or no more) committal than their instances, and universal generalisations are *more* (or no less) committal. "There is something that roses and sunsets are" is an existential generalisation with "roses and sunsets are red" as a substitution instance. So the first sentence is no more committal than the second. But the second is not committed to anything but roses and sunsets. So the first isn't committed to anything but roses and sunsets either.⁷

There is a principle implicit in this passage that I want to endorse. Here is a formulation:

(Neutrality) Quantification into the position occupied by a particular type of syntactic constituent in a statement of a particular form cannot generate ontological commitment to a kind of item not *already* semantically associated with the occurrence of that type of constituent in a true statement of that form.

This is less than the whole gist of the quoted passage, since Rayo and Yablo are also claiming—in a nominalist spirit—that the ontological commitments of "Roses and sunsets are red" go no further than roses and sunsets. What I am endorsing, by contrast, is a more modest claim: precisely, that quantification is *neutral* as far as ontological commitment is concerned—that the commitments of quantified statements go no further than the requirements of the truth of their instances *whatever the latter requirements are*. So the view I am suggesting is also open to

⁶ This apt term was Fraser MacBride's suggestion

⁷ Agustin Rayo and Stephen Yablo, "Nominalism Through De-nominalisation", *Nous* 35, (2001) pp. 74-92. A similar thought is expressed in Hale and Wright, *The Reason's Proper Study* (Oxford: the Clarendon Press 2001), p. 431-2 (Postscript, problem #11.) The basic idea is anticipated at Wright, *Frege's Conception of Numbers as Objects*, (Aberdeen: Aberdeen University Press 1983; 2nd edition forthcoming with Oxford University press) at p. 133.

someone who thinks that simple predications of "red" *do* commit us to, say, a universal of redness, or to some other kind of entity distinctively associated with that predicate. But the crucial point is that Quine is not on board in any case. For in Quine's view, while mere predication *is* free of distinctive ontological commitment, quantification into predicate position must commit a thinker to an *extra* species of entity—in the best case, sets; in the worst, attributes or universals. This, I think, is a major mistake.

The basic idea I wish to set against Quine's is that quantification should be viewed as a device for *generalisation of semantic role*. Given any syntactic category of which an instance, *s*, can significantly occur in a context of the form, [...*s*...], quantification through the place occupied by '*s*' is to be thought of as a function which takes us from [...*s*...], conceived purely as a content, to another content whose truth conditions are given as satisfied just by a certain kind (and quantity) of distribution of truth values among contents of the original kind. A quantifier is a function which takes us from a statement of a particular form to another statement which is true just in case some range of statements—a range whose extent is fixed by the quantifier in question—which share that same form are true. The central task for the project I am proposing is to sharpen and develop this rather inchoate idea

Four issues loom large on the agenda. The primary task will be

—To develop a definite proposal, or proposals, about how quantification, neutrally conceived, can work.

There are then three further interrelated tasks. The second is

—To explore whether neutralism can provide a satisfactory account of—that is, sustain all the demands on quantification made by—classical impredicative higher-order logics.

Should matters stand otherwise, the third task will be

—To explore what kinds of higher-order logic *can* be underwritten by neutralism.

The fourth task will be

—To determine whether neutralism can sustain the epistemological and technical demands imposed on higher-order logic by the abstractionist programme, and by other programmes in the foundations of mathematics which make special demands on, or have special needs for, higher-order theories.

There are, of course, other potentially 'ontologically liberating' conceptions of higher-order quantification. Notable among them are substitutional construals and Boolos' conception of such

quantifiers as in effect devices of plural generalisation, ranging over the same items as first-order quantifiers but ranging over them in bunches, as it were.⁸ The shortcomings of these approaches are well known: Boolos has no ready means of interpreting quantification over relations, and substitutionalism limits legitimate generalisation to the range of cases captured by the expressive resources of a particular language. Neutralism must do better. It must provide a uniform construal of higher-order generalisation, irrespective of the grade of predication involved; and it must allow language-unbounded generalisation. Well executed, it will not of itself put to bed Quine's "set-theory in sheep's clothing" canard. But it will do so in conjunction with the thesis that a satisfactory account of predication need involve no commitment to sets.

II Fixing the meanings of the quantifiers

The obvious—I think inevitable—direction to explore is to suppose that the conception of quantification we seek must ultimately be given by direct characterisation of the *inferential role* of the quantifiers; more specifically, that what is needed are explicit statements of natural deductive introduction- and elimination- rules for the various possible {syntactic-type: quantifier-type} pairings. In order to be effective—to be safe in inference and to establish genuine concepts—such stipulations must of course meet certain general constraints: at a minimum consistency, but also arguably conservativeness and harmony.⁹ Some of the issues here are on the abstractionist agenda already, in connection with the task of characterising which are the a priori acceptable abstraction

⁸ See especially his "To be is to be the value of a variable (or to be some values of some variables)" *Journal of Philosophy* 81 (1984) pp. 430-449.

⁹ Familiarly, these notions are open to various construals. Here I intend only the general idea that the patterns of inference, in and out, licensed by putatively meaning-constitutive rules should have a certain theoretical coherence: that what one is licensed to infer from a statement by such (elimination) rules should be exactly — no more and no less — what is prepared for by the (introduction) rule that justifies one in inferring to it. Non-conservativeness — for example, to the egregious degree illustrated by Arthur Prior's connective "tonk" — is the penalty of violating such constraints one way round. But disharmony in the converse direction — for example, that manifested by a connective, *tunk*, which combines the disjunction-elimination and conjunction-introduction rules — results in safe yet *unintelligible* inferential practice.

Inferentialism is a broad church, and there are many ways of developing the basic thought that the most fundamental logical laws are constitutive of the logical operations that they configure, and that this consideration is somehow key when it comes to explaining our knowledge of them. I cannot pursue these issues here, nor attempt to respond to the recent opposition to the basic thought developed by writers such as Timothy Williamson in his "Understanding and inference", *Proceedings of the Aristotelian Society*, supp. vol. 77 (2003) pp. 249-293, and "Conceptual truth", *Proceedings of the Aristotelian Society*, supp. vol. 80 (2006) pp. 1-41. The aim of the present paper must be restricted to a conditional: if inferentialism can contrive a satisfactory account of the epistemology of basic logic at all, then it can play this part in relation to higher-order quantification theory in particular, and in a way that can liberate our understanding of it from the Quinean shackles.

principles.¹⁰ In any case, the proposal will be to view suitable rules as *constituting meanings* for the quantifiers and—simultaneously—as showing how they can assume (something close to) their standard inferential roles without being saddled with the "range-conception" of their function which drives the Quinean critique.

So: what might be the shape of suitable such rules? If quantification is to be conceived as an operation which generalises on semantic role in a uniform way, irrespective of what the semantic role of the particular kind of expression concerned actually is, then it might seem to be a key thesis for neutralism that the various kinds of quantifiers—and par excellence the universal and existential quantifiers—each admit of a *single* account which goes through independently of the syntactic category of the expressions which the bound variables replace. And indeed I do not know that such an account is not attainable. However while natural deductive formulations of the quantifier rules for higher-order logic, for instance, should, of course, follow their first-order cousins as closely as possible, it is not clear that neutralism *demand*s strict uniformity of explanation. It should be good enough merely if there is some suitable common pattern exhibited in e.g. the different kinds of universal quantification, even though the statements of the proof-rules for, say, first- and second-order universal quantification display important local differences and cannot be arrived at simply by applying a more general uniform account to the two syntactic classes of expression concerned. Consider for instance the standard first-order Universal

Introduction rule:

$$\begin{array}{l} \Gamma \Rightarrow A(t) \\ \Gamma \Rightarrow (\forall x)Ax \end{array}$$

This provides that, if we can establish $A(t)$ on a certain pool of assumptions, Γ , then—provided the proof meets certain additional requirements—those same assumptions suffice for the universal quantification through 't'. The additional requirements in question are, of course, standardly characterised syntactically—for the kinds of formal language usually envisaged, it suffices to specify that the sentences in Γ be free of (unbound) occurrences of 't'—but their intent is to ensure that the proof in question has a certain *generality*: that the derivation of $A(t)$ in no way turns on

¹⁰ See the discussion in Bob Hale and Crispin Wright, "Implicit Definition and the A Priori" in Paul Boghossian and Christopher Peacocke, eds., *New Essays on the A Priori* (Oxford: the Clarendon Press, 2000), pp. 286-319, (reprinted in *The Reason's Proper Study*.)

the choice of 't' but would succeed no matter what (non-empty) term we were concerned with. That's the very feature, of course, that makes it legitimate to generalise. So in order for a proposed introduction rule for a second-order operator to count as governing a genuinely universal quantifier, the same feature must be ensured. Thus we would expect to retain the basic pattern

$$\begin{aligned}\Gamma &\Rightarrow \Phi(F) \\ \Gamma &\Rightarrow (\forall X)\Phi X\end{aligned}$$

to which would then be added constraints to ensure that the derivability of $\Phi(F)$ from the set of assumptions in Γ in no way turns on the choice of the predicate F but would go through no matter what predicate was involved.

However it is clear that, save where F is atomic, the intended generality cannot be ensured merely by stipulating that Γ be free of occurrences of F . For F may be an open sentence of arbitrary complexity. If its satisfaction-conditions are logically equivalent to those of some syntactically distinct open sentence, F^* , then the derivation of $\Phi(F)$ from Γ may in effect lack the requisite generality—since Γ may incorporate special assumptions about F^* —even though the strict analogue of the first-order restriction is satisfied. Γ may just be the assumption, $FaVGa$, for example. In that case, propositional logic will take us to $GaVFfa$. And now, since ' $FaVGa$ ' is free of any occurrence of the 'parametric' predicate, $G\dots VF\dots$, the envisaged second-order UI rule would license the conclusion, $FaVGa \Rightarrow (\forall X)Xa$.

Obviously we do not want to outlaw all generalisation on semantically complex predicates. The way to ensure the generality we need is to require that the proof of the premise-sequent, $\Gamma \Rightarrow \Phi(F)$, be such that it will still succeed if the occurrence of ' F ' in its conclusion is replaced by an atomic predicate. That way we ensure that any semantic or logical complexity in F is unexploited in the proof, which consequently is forced to have the requisite generality. A similar requirement on UI will indeed be apposite at first-order too if the language in question permits the formulation of complex—for instance, definite descriptive—singular terms. But my point is merely that the spirit of neutralism will not be violated if the eventual rigorous statement of the second-order rule proves to differ in a number of respects from its first-order counterpart: in order to ensure that both are concerned with universal quantification, properly so regarded, it will be enough that they exhibit the common structure illustrated and that their differences flow from the

respectively different demands imposed by the need for generality, in the light of the differences between the syntactic types of expression concerned.

Semantic complexity in expressions quantified into raises another issue that it is worth pausing on briefly. In a language featuring definite descriptive singular terms, some form of free-logical restriction on the standard first-order rules may be required to ensure that the validity of the transitions in question is not hostage to reference failure. We might impose on the UE rule for instance

$$\begin{array}{l} \Gamma \Rightarrow (\forall x)Ax \\ \Gamma \Rightarrow A(t) \end{array}$$

the restriction that 't' may not be just any term but must be such that, for some atomic predicate, F, we have in addition that F(t); so that a fuller statement of the rule would look like this:

$$\begin{array}{l} \Gamma \Rightarrow (\forall x)Ax; \Delta \Rightarrow Ft \\ \Gamma, \Delta \Rightarrow A(t) \end{array}$$

The point, intuitively, would be that while an empty term may feature in true extensional contexts—for instance, "there is no such thing as Pegasus"—its lack of reference will divest any *atomic* predication on it of truth. Now, the relationship between sense and reference for the case of predicates (if we permit ourselves for a moment to think of predicates as having reference at all) is manifestly different to that which obtains in the case of terms. There can be no presumption that a complex singular term refers just in virtue of its having a sense, whereas in the general run of cases the association of a predicate, complex or simple, with a determinate satisfaction-condition ensures it against (any analogue of) reference failure. But the point is not exceptionless: the predicate "... is John's favourite colour" as it occurs in

That car is John's favourite colour

may, despite being a significant open-sentence of English, associated with determinate satisfaction-conditions, actually fail of reference—or anyway suffer an appropriately analogous failing to that of an empty but meaningful singular term—in just the same circumstances, and for the same reasons, as the singular term, "The favourite colour of John". A fully general natural deductive formulation of higher-order logic would have to allow for this kind of case. However it could not happily do so merely by transposing the reformulated first-order UE rule above, as

$$\begin{aligned} \Gamma \Rightarrow (\forall X)\Phi X, \Delta \Rightarrow Ft \\ \Gamma, \Delta \Rightarrow \Phi(F) \end{aligned}$$

where 't' is required to be atomic, since that would be tantamount to the requirement that, before F could be legitimately regarded as an instance of a second order universal quantification, we need it to be *instantiated*—for there to be something which is F. But that is not the right requirement: what we need is not that something be F but rather, putting it intuitively, that there be something which being F is. (By contrast, John might have a favourite colour that was unexemplified by anything.)

It needs further discussion how this wrinkle might best be addressed. It would be natural to propose that we might mimic the first-order restriction, but require that the collateral premise ascend a level, so that what is required is that F be correctly characterized by some atomic *second-order* predicate (in effect, a quantifier.) Thus the statement of the rule would be

$$\begin{aligned} \Gamma \Rightarrow (\forall X)\Phi X, \Delta \Rightarrow \Psi F, \text{'}\Psi\text{' atomic} \\ \Gamma, \Delta \Rightarrow \Phi(F) \end{aligned}$$

But what might be a suitable choice for 'Ψ'? The existential quantifier would be safe enough in the role, but would once again leave us without the resources to instantiate on legitimate but unsatisfied predicates. What *is* the quantifier, actually, that says of a predicate F that its sense and the world have so co-operated as to make it true that there is such a thing as being F?

The issue—to give a proper account of the distinction for predicates, unexampled in the case of singular terms, between (something akin to) lacking reference and being unsatisfied,—is of some independent interest.¹¹ Once properly clarified, there would be no objection to introducing a primitive second-level predicate to mark it, and framing the UE rule in terms of that. But since it is, of course, an informal philosophical distinction, we'd be bound to acknowledge a consequential compromise in the ideal of a fully rigorous syntactic capture of the rules (though admittedly a kind of compromise we already make in the formulation of free first-order logic.) My present point again, however, is merely that it should not necessarily be regarded as

¹¹ One suggestion (put forward by Bob Hale in correspondence) is that an assurance against the counterpart of reference-failure for predicates is provided by satisfaction of the generalized Law of Excluded Middle, $(\forall x)(Fx \vee \neg Fx)$. But one would need some principled way to head off the response of an awkward customer that if John has no favourite colour, then it is true of my shirt that it is not John's favourite colour.

compromising the generality of a neutralist account of Universal Elimination if the details for the second-order case turn out to be somewhat different from those appropriate in the first-order case. What is crucial is the common pattern of inference and that the additional constraints on that pattern, respectively appropriate in the first- and second-order cases, should be intelligible as addressed to common requirements: viz. genuine generality, and the need—where the languages in question may throw up examples of such—to address the risk of (something akin to) reference failure.

III Extreme Neutralism

Let me stress again the sense in which what is aimed at is a neutralist account of quantification. The central thesis is not that higher-order quantification ought to be construed in such a way that it implicates no special ontological commitment. Rather it is that quantification is not *per se* ontologically committing: Quine's idea, that the ontological commitments of a theory are made fully manifest by the range of kinds of quantification that a minimally adequate formulation of the theory requires—the idea rather unhappily encapsulated in the slogan 'to be is to be the value of a variable'—is simply a mistake, encouraged by a preoccupation with first-order quantification in a language all of whose names refer. The Neutrality Principle has it that statements resulting from quantification into places occupied by expressions of a certain determinate syntactic type need not and should not be conceived as introducing a type of ontological commitment not already involved in the truth of statements configuring expressions of that type. In particular, if there is a good account of the nature of predication which frees it of semantic association with entities such as universals, properties or concepts, then quantification into predicate position does not introduce any such commitment.

Again: quantifiers are merely devices of generalisation—devices for generating statements whose truth conditions will correspond in ways specific to the kind of quantifier involved to those of the statements quantified into—the instances. Applied to second-order quantifiers in particular, the neutralist thesis is thus that they do not differ from those of first-order logic by the implication of a type of ontological commitment not already present in the former. If second-order quantifiers demand a special kind of—perhaps problematic—entity, that will be because that same demand is

already present merely in the idea of atomic predication in which, of course, in its characteristic recourse to predicate-schemata, first-order logic is already up to its eyeballs. There is thus, in the neutralist view, no interesting difference in the ontological commitments of first- and second-order quantification, and no interesting difference between the ontological commitments of either and those of the quantifier-free atomic statements to which they are applied. If there is a residual sense in which only first-order logic is properly styled as logic, it is not at any rate an additional implicit ontology that disqualifies second-order logic from that honorific title.

We should note that the spirit of this proposal is consistent with a particularly extreme form of incarnation. From a strict neutralist point of view, there can be no *automatic* objection to existential quantification into the place occupied by "the largest number" in

(a) There is no such thing as the largest number

Plainly "the largest number" is not being used referentially in that sentence. So its existential generalisation, which merely involves generalisation of whatever its semantic role there is, will not—absurdly—imply the existence of something which there is no such thing as. What it will imply is merely that some statement is true for whose truth it suffices that some—it doesn't matter which—statement of the form

There is no such thing as #

be true, where '#' holds a place for an expression whose semantic role is that of "the largest number" in (a). Regular—existence implying—first-order existential quantification will thus be a special case of neutral first-order existential quantification, falling out of instances where the term quantified into *is* functioning referentially (i.e. functioning in such a way that its failure to refer will divest the containing context of truth.)

I think we should grant that this extreme neutralism is a *prima facie* defensible option. Of course it may run into trouble, but there is nothing in neutralism *per se* that closes it off. There are good reasons, nevertheless, to fashion an account of the quantifiers which allows the connections, in the first-order case, with the idea of a range of entities, and with ontological commitment, which Quine took to be central. In that case, we will prefer—for languages involving possibly empty terms—to employ something like the approach illustrated by the free-logical rule proposed for UE above.

IV A Neutralist heuristic

So to recap: our proposal is that the meanings of the quantifiers of all orders, neutrally conceived, are to be viewed as grounded in the inferential practices licensed by their introduction and elimination rules, that they are as logical as any other concepts of comparable generality which are associated in this way with a distinctive role in the inferential architecture of subject-predicate thought as such, and that there is no better reason to view them as associated with distinctive ontological commitments than there is so to regard the conditional or negation. Still, old habits die hard and the temptation may remain to press for an informal elucidation of what, in using operators so characterised, we are really *saying*— what kind of claim is being made by someone who offers a higher-order generalisation if he is not—or need not— be “quantifying over” properties or the like? The felt need remains for a *heuristic*—a way of thinking about the gist of higher-order generalisation that frees it of the Quinean associations and makes some kind of sense of the practices codified in the rules. We run a risk, of course, in offering such a heuristic, of confusing it with a theory of the actual meaning of the statements at issue—a risk, by the way, which is also run by formal model-theory. Such a heuristic, or model theory, may very well incorporate ideological and ontological assumptions which are, properly speaking, no part of the content of the targeted class of statements. Still the result may be helpful. No one can deny the utility and fruitfulness of possible-worlds semantics, for example, as a model for modal discourse, but relatively few believe that possible worlds semantics uncovers its actual content. I acknowledge, naturally, that there is an issue of wherein the utility and fruitfulness consists if actual meanings are not recovered. I’ll make a suggestion about that shortly.

What can be offered by way of an informal heuristic for quantification when neutrally viewed? Here is one suggestion. Begin with *substitutional* quantification as normally conceived. A substitutional quantifier is assigned a range of significance with respect to a given determinate language: we first individuate a certain grammatical type of expression, and then specify that, for instance, the result of existential generalisation into a place in a particular statement occupied by an expression of that type is to be true just in case some sentence of the language differing from that statement only at most by the occurrence in that place of a different expression of the same grammatical type is a true sentence. The corresponding explanations for other kinds of quantifiers

proceed in the obvious way. Two salient points are these: first, that in characterising the truth conditions of statements formed by substitutional quantification, we resort—in the meta-language—to *non-substitutional* quantification over *sentences* in the object language; and second, and very familiarly, that the scope of the generality that substitutional quantifiers enable us to achieve is bounded by the actual vocabulary—in particular, by the range of expressions of the particular grammatical kind concerned—of the object language in question. Hence the familiar complaint that the generality provided by substitutional quantification is not the true generality that quantification should involve.

The neutralist heuristic I have in mind takes off from these two points. First, in explicating the truth conditions of quantified sentences, we now resort—in the meta-language—to quantification not over sentences but over the things that sentences express: over *thoughts* (propositions). Second, we waive the restriction to particular (limited) languages. We conceive of thoughts as structured entities with a structure corresponding to that of the deep-semantics of a properly canonical expression for them, and quantification is then seen as a device of generalisation into the positions configured in structured entities of this kind.¹² Thus, for instance, the existential generalisation of any significant syntactic constituent in the expression of a thought takes us to a thought which is true just in case some thought is true differing at most only in the syntactic constituent that would figure in that place in a canonical expression of it.

Correspondingly for universal quantification.

This is a notion of quantification whose generality is unbounded by the expressive resources of any particular actual language—it provides for generalised commitments in a way

¹² In so conceiving of thoughts as structured I of course follow the lead of Evans in the following memorable passages:

It seems to me that there must be a sense in which thoughts are structured. The thought that John is happy has something in common with the thought that Harry is happy, and the thought that John is happy has something in common with the thought that John is sad. This might seem to lead immediately to the idea of a language of thought... However, I certainly do not wish to be committed to the idea that having thoughts involves the subject's using, manipulating or apprehending *symbols* — ...I should prefer to explain the sense in which thoughts are structured, not in terms of their being composed of several distinct elements. but in terms of their being a complex of the exercise of several distinct conceptual *abilities*.while sentences need not be structured, thoughts are essentially structured. Any meaning expressed by a structured sentence could be expressed by an unstructured sentence. But it is simply not a possibility for the thought that *a* is *F* to be unstructured — that is, [for the thinking of it] not to be the exercise of two distinct abilities.”

Gareth Evans, *The Varieties of Reference* (Oxford: Oxford University Press 1982) pp. 100-102.

which embraces arbitrary *extensions* of such a language—and which manifestly carries only the ontological commitments already carried by the use of the syntactic type of expression quantified into. However it is true that the heuristic also involves quantification over *thoughts*. That may provoke the misgiving that it parts company with the neutralist intent at the crucial point and so misses its mark. For was not the idea to explicate an ontologically non-committal conception of higher-order quantification, free of any association with a range of entities?

Not exactly. Rather, the idea was, in the first instance, to explicate a conception of quantification into predicate — or indeed into any syntactic kind of — position which would avoid associating occurrences of expressions of that syntactic kind with a *distinctive kind* of entity, presumed to provide their semantic values. According to the heuristic, all quantification, of whatever order, into whatever syntactic kind of place in a statement, is quantification ‘over’ — generalisation with respect to — thoughts. So the heuristic saves a key element of the *Begriffsschrift* conception — its indifference to the distinctions of order that, after Quine, came to seem so crucial.

But there is a more fundamental point. The examples of regular set-theoretic semantics for first-order languages, and of worlds semantics for modal languages, remind us of the point I mentioned a moment ago, that model-theory, whether in formal, rigorous form or that of informal heuristics, need not — and often had better not — be thought of as conveying the actual content—the *actual conceptual resources deployed in*—the statements of the explicated language. In the present instance, that content remains fixed operationally by the introduction- and elimination-rules for the quantifiers. What the heuristic does is provide an informal theory which, as it were, maps, or reflects, the truth-conditions of the statements whose contents are so fixed without pretending to recover the understanding of them possessed by competent speakers. Although it would take us too far afield to explore the idea thoroughly here, I suggest that the right way to think of this mapping relationship is that it consists in an identity of truth-conditions *modulo* the assumption of the ontology of the heuristic or model theory. The very simplest example of such a

relationship is provided by the relation between the statement that Socrates is a man and the statement that Socrates is a member of the set of men — a relation of necessary equivalence in truth-conditions *modulo* a modest ontology of sets (sameness of truth value in all worlds, if you will, in which the ontology of the model theory is realised.) The same view may be held concerning the statements that the number of planets is the same as the number of clementines in the fruit-bowl and that there are exactly as many planets as clementines in the fruit-bowl— a necessary equivalence in truth conditions *modulo* the ontology of the finite cardinals.¹³ And it is, so I would argue, the same with unreconstructed modal claims and the construals of them offered in worlds semantics. In each case, a theoretical explication of content is achieved via an equivalence in truth-conditions under the hypothesis of the ontology of the explicating theory. Where T is a statement of the theory concerned, S a statement whose content is to be explicated, and E its explicans, the explicating relationship is of the schematic form

$$T \Rightarrow (S \Leftrightarrow E),$$

and explication is achieved, when it is, via the (conceptual) necessity of appropriate instances of this schema.

The distinction between explication of this kind and some yet closer relation of content that we might want to bring under the old fashioned term, “analysis”, is something we need generally if we are to understand the proper role — both the prospects and the limitations — of interpretative semantic theory in philosophy. It is a distinction that is, ironically, enforced by the precedent of set-theoretic semantics for first-order theories yet completely missed by the normal, Quinean way of receiving higher-order logics’ standard set-theoretic semantics. My suggestion would be that we avail ourselves of it, in the form I have adumbrated, to understand the relationship between the sketched heuristic and (higher-order) quantification: a quantified statement and the kind of paraphrase we may give of it in terms of the heuristic are equivalences

¹³ Of course the abstractionist view of this particular equivalence is that it holds unconditionally.

modulo the ontology of a suitable theory of thoughts as structured entities — a theory which, speaking for myself, I think there is ample reason to endorse and to develop but of which it is open to us to take an agnostic, or even fictionalist view, while still deriving illumination from the content mappings it lets us construct.

V Comprehension

In standard formulations of higher-order logic, comprehension axioms are used to tell us which predicates—open sentences—are associated with an entity belonging to the domain of quantification in question and are thus safe for quantification. They are therefore to be compared with postulates in a free first-order logic telling us which of the terms in a language in question have reference to an object. It goes with neutralism, as I've been outlining it, that there need be—more accurately: that for extreme neutralism, at least, there is— *no role* for comprehension axioms. If quantification is viewed as generalisation of semantic function, whatever it is, where discharging the semantic function in question just means contributing appropriately to a range of intelligible thoughts, there can be no hostage, comparable to reference failure, which must be redeemed before quantification is safe. If an expression belongs to a determinate syntactic kind, and plays the semantic role appropriate to an expression of that kind in the canonical expression of a given intelligible thought, then quantification will be safe without additional assumption.

Still, one will need circumspection about its *significance*, which will be constrained by the actual semantic role of the expression quantified into in the statement concerned. Existential quantification, for instance, as we have noted, may not *per se* carry the kind of existential commitment with which it is normally associated. But one may still wish to make sure that it does, that only *referential* singular terms are eligible for existential generalisation. And, as we have noted, there may be similarly desirable restrictions to be placed on those expressions which are to be eligible for higher-order generalisation. The informal example of "... is Johns favourite colour" is a reminder that, once we go past the atomic case and allow ourselves the luxury of forming new predicates by quantification into or other forms of operation (in this case, definite

description) on predicates, we may find ourselves forming significant expressions which nevertheless stand in some kind of parallel to reference-less singular terms.

There is much more to say, but we can, I think, now table a group of natural proposals about the specific form that introduction- and elimination rules for second-order universal and existential quantification may assume consonantly with an inferentialist conception of the meaning of those operations and the neutralist heuristic. The universal rules are, unremarkably, these:

UE

$$\Gamma \Rightarrow (\forall X)\Phi X, \text{ where 'F' is good}$$

$$\Gamma \Rightarrow \Phi(F)$$
UI

$$\Gamma \Rightarrow \Phi(F), \text{ for arbitrary atomic 'F'}$$

$$\Gamma \Rightarrow (\forall X)\Phi X$$

while those for the existential quantifier are

EI

$$\Gamma \Rightarrow \Phi(F), \text{ where 'F' is good}$$

$$\Gamma \Rightarrow (\exists X)\Phi X$$
EE

$$\Gamma \Rightarrow (\exists X)\Phi X; \Delta, \Phi(F) \Rightarrow P, \text{ for arbitrary atomic 'F'}$$

$$\Gamma, \Delta \Rightarrow P$$

Departures from extreme neutralism will be signalled by activating the "where 'F' is good" clause, while the effect of one's favourite comprehension restrictions may be achieved by specific interpretations of what goodness consists in. The effect of full, impredicative second order comprehension may be accomplished by counting all meaningful open-sentences as good, save possibly those afflicted by (the analogue of) reference-failure, and by finding no obstacle to meaningfulness in the free formation of open sentences involving the second-order quantifiers themselves. More on this below.

We should note at this point, however, that the *generosity* of standard Comprehension axioms in classical higher-order logic is hard to underwrite by means of the proposed heuristic. A standard formulation of classical Comprehension will provide that, for each formula, Φ , of the language in n argument places, there is a relation in n places, X , such that for any n -tuple, $\langle x_1, \dots, x_n \rangle$, X holds of $\langle x_1, \dots, x_n \rangle$ just in case $\Phi \langle x_1, \dots, x_n \rangle$. At first blush, this seems perfectly consonant with neutralism: existential generalisation into predicate position is legitimated just when we have an intelligible formula of the language (or some extension of it) to provide for a

witness. But the appearance is deceptive. Comprehension as standardly formulated has, for example, this instance:

$$(\forall x)(\exists X)(\forall y)(Xy \leftrightarrow y=x)$$

- intuitively, that for any thing, there is a property—a way things can be—which a thing exemplifies just in case it is identical with that thing. So we get a range of properties specified simultaneously as a bunch, in a way that is parametric in the variable, ‘x’. This is consonant with a connection between second-order quantification and the intelligibility of the thoughts that constitute its instances only if each object in the first-order domain in question—each object in the range of ‘x’—is itself a possible object of intelligible singular thought.

To see the point sharply, consider how we might set about establishing the above instance of comprehension using the higher-order rules outlined above and the standard first-order rules. Presumably we have as a theorem at first-order:

$$(\forall x)(\forall y)(y=x \leftrightarrow y=x)$$

So instantiating on ‘x’, we obtain

$$(\forall y)(y=a \leftrightarrow y=a)$$

Then, assuming ‘...=a’ is good, the proposed **EI** rule allows the move to

$$(\exists X)(\forall y)(Xy \leftrightarrow y=a)$$

And finally, since each of these theses is justified purely by the logic, and ‘T’ is empty throughout, we can generalise on ‘a’ to arrive at the target claim:

$$(\forall x)(\exists X)(\forall y)(Xy \leftrightarrow y=x)$$

However manifestly the way we have arrived there depends entirely on the good standing of the open sentence, ‘...=a’, which in turn is naturally taken to presuppose the existence of a suitable singular mode of presentation of the object concerned. The ‘generous’ instance of Comprehension is established only relative to that presupposition.

With uncountably infinite domains, however, the presupposition is doubtful. Only an infinite notation could provide the means canonically to distinguish each classical real number from every other in the way that the standard decimal notation provides the means canonically to distinguish among the naturals. So, on plausible assumptions, no finite mind can think

individuating thoughts of every real. Yet the instance of Comprehension cited implies that to each of them corresponds a distinctive property. So standard Comprehension strains the tie with intelligible predication, crucial to the neutralist heuristic as so far understood—at least, it does so if classically uncountable populations of objects are admitted. Quantification over uncountable totalities admits of no evidently competitive interpretation in terms of distribution of truth-values through a range of intelligible singular thoughts.

That there would be likely to be difficulties when it comes to reconciling a broad conception of quantification as distributive of truth-values among intelligible thoughts with the classical model-theoretic (set-theoretic) conception of higher-order quantification was of course obvious from the start, since the classical conception associates each n th order of quantification with a domain of ‘properties’ (sets) of the n th cardinality in the series of Beth numbers and nothing resembling human thought, nor any intelligible idealisation of it, is going to be able to encompass so many ‘ways things might be’. But the point developed above concerns uncountable domains of *objects*, not higher-order domains, and calls into question whether the kind of conception of quantification which I am canvassing as possibly suitable for the abstractionist programme in the philosophy of mathematics is actually fit for purpose — assuming that the project is to embrace the pure mathematics of certain uncountable domains, including *par excellence* the classical real numbers. To stress: the difficulty is with the heuristic, not (yet) with the inferentialist conception of quantification *per se*. But there is clearly matter here for further study and invention.

VI Incompleteness

There is a salient objection to the neutralist approach — indeed to any inferentialist account of higher-order quantification — which we can defer no longer. It springs from Gödel’s results about the incompleteness of arithmetic. Here is an intuitive statement of it.¹⁴

As is familiar, second-order arithmetic is, like first-order arithmetic, shown by Gödel’s findings to be incomplete. But there are crucial differences. The completeness of first-order logic ensures that every sentence that holds in every model of first-order arithmetic can be derived from the first-order Peano Axioms by its deductive resources. So the incompleteness of first-order arithmetic means that some truths of arithmetic do not hold in every model of the first-order Peano Axioms. However the second-order Peano axioms are categorical: they have exactly one (standard) model and any truth of arithmetic holds in it. Hence the incompleteness of second-order arithmetic entails that second-order logic is incomplete: not every sentence that holds in every (i.e. the unique) model of the second-order Peano Axioms can be derived from them in second-order logic. So doesn’t this point somehow enforce a model-theoretic perspective on the idea of second-order logical validity? For surely we are now forced to allow that there are valid second-order logical sentences that which cannot be established by second-order deduction. Yet what sense can be made of this, if we are inferentialist about the meanings of the quantifiers? How can we regard the meanings of the quantifiers as fully fixed by the inference rules and at the same time allow that there are valid sentences expressible using just second-order logical vocabulary which those rules fail to provide the means to establish?

Well, are we forced to say this? Let’s go carefully. Let 2PA be a suitable axiom set for second-order arithmetic, and let G be the Gödel sentence for the resulting system. Let $2G^*$ be the conditional, $2PA \rightarrow G$. Clearly this cannot be proved in 2PA. And the usual informal reasons for regarding G as a truth of arithmetic weigh in favour of regarding this conditional as a truth of arithmetic. But they are not reasons for regarding it as a truth of logic — it is shot through with non-logical vocabulary. However consider its universal closure $U2G^*$. This is expressed purely second-order logically but it too cannot be provable in 2OL, for if it were, we would be able to prove $2G^*$ in 2PA. Is there any reason to regard $U2G^*$ as a logical truth/validity? If there is, that

¹⁴ The discussion to follow is greatly indebted to Marcus Rossberg. The issues are pursued in his “....” [Forthcoming.]

will be a *prima facie* reason to regard second-order validity as underdetermined by the second-order inference rules, including those for the quantifiers—a body-blow, seemingly, against inferentialism.

There is indeed such a reason. *Third-order* logic allows us to define a truth-predicate for second-order arithmetic and thereby to mimic rigorously in a formal third-order deduction the informal reasoning that justifies the conclusion that G holds good of any population of objects that satisfy 2PA. So 2G* can be proved using just 3OL resources, without special assumption about the arithmetical primitives. There is therefore also a legitimate generalisation to U2G*. So now we have a sentence expressible by means purely of second-order logical vocabulary, and establishable as a theorem of third-order logic but underivable in second-order logic.¹⁵ If third-order logic is logic, this sentence is a logical truth that configures no logical concepts higher-than second-order yet cannot be established by the inference rules of 2OL.

The situation generalises. From a non- model-theoretic perspective, the incompleteness entailed by Gödel's results for each *n*th order of higher-order logics consists in the deductive non-conservativeness of its hierarchical superiors over it—each *n*th order logic provides means for the deduction of theorems expressible using just the conceptual resources of its predecessor but which cannot be derived using just the deductive resources of its predecessor. But then at no *n*th order is it open to us to regard the meanings of the quantifiers at that order as fully characterised by the *n*th order rules — otherwise, what could possibly explain the existence of truths configuring just those meanings but underivable by means of the rules in question?

There are various possible responses.¹⁶ One might attempt a case that 3OL is *not* logic, so that some of its consequences, even those expressed in purely second-order logical vocabulary, need not on that account be rated as logical truths.¹⁷ But the prospects for this kind of line look pretty forlorn in the present context— after all, the neutralist conception of quantification embraces *any* kind of open sentence, including in particular any resulting from the omission from

¹⁵ For details, see section 3.7 and 4.1 of Daniel Leivant, “Higher-Order Logic” in D.M Gabbay, C. J. Hogger, J.A. Robinson, J. A., and J. Siekmann, (eds.) *Handbook of Logic in Artificial Intelligence and Logic Programming Volume 2: Deduction Methodologies* (Oxford: OUP, 1994) pp. 230-321.

¹⁶ See Rossberg *op. cit.*

¹⁷ By way of a parallel, the at least countable infinity of the universe may be expressed in purely logical vocabulary. Still, it may coherently be regarded as a necessary *mathematical* truth, rather than a logical one.

complete sentences of first-order predicates. The resulting class of expressions ought to be open to generalisation in just the same sense as first-order predicates. How could the one kind of operation be ‘logical’ and the other not?

However, I think it is the very generality of the neutralist conception of quantification that points to the correct inferentialist response to the problem. That response is, in essentials, as follows. Epistemologically, it is a mistake to think of higher-order quantifiers as coming in conceptually independent layers, with the second-order quantifiers fixed by the second-order rules, the third-order quantifiers fixed by the third-order rules, and so on. Rather it is the *entire series* of pairs of higher-and higher-order quantifier rules which collectively fix the meaning of quantification at each order: there are *single* concepts of higher-order universal and existential generalisation, embracing all the orders, of which it is possible only to give a schematic statement — as indeed we effectively did in the rules proposed earlier, in which, although I then announced them as “second-order”, and accordingly represented the occurrences of ‘F’ within them as schematic for predicates of objects, no feature was incorporated to bar the construal of ‘F’ as schematic for predicates of any particular higher-order, including — if one is pleased to go so far — transfinite ones. Higher-order quantification is a uniform operation, open to a single schematic inferentialist characterisation, and there is no barrier to regarding each and every truth of higher-order logic as grounded in the operation of rules that are sanctioned by that characterisation.

VII Impredicativity

Classical second-order logic allows full *impredicative* comprehension. Impredicative comprehension is this process. In an open sentence, "...F...", configuring one or more occurrences of a (simple or complex) predicate, we first form a new predicate by binding one or more occurrences of F with a quantifier—say " $(\forall X)\dots X\dots$ "—and then treat this new predicate as falling within the scope of its own quantifier. The manoeuvre is essential to a number of fundamental theorems in classical foundations, including the second-order logical version of Cantor's theorem and various key lemmas in the abstractionist recovery of arithmetic and analysis.

Can neutralism underwrite impredicative quantification? The standard objection to impredicative quantification of all orders is epistemological—that it introduces a risk of vicious

circularity, or ungroundedness, into the meaning of expressions which involve it. One reason for thinking this which we may discount for present purposes is endorsed in various places by Michael Dummett.¹⁸ Dummett's objection is that determinacy in the truth-conditions of statements formed by means of quantification depends upon a prior definite specification of the range of the quantifiers involved, and that such a definite specification may be impossible to achieve if the quantifiers are allowed to be impredicative, since their range may then include items which can only be specified by recourse to quantification over the very domain in question. I think *this* form of worry involves a mistake about the preconditions of intelligible quantification. It simply isn't true that to understand a quantified statement presupposes an understanding of a domain specified independently—as is easily seen by the reflection that such an independent specification could hardly avoid quantification in its turn.¹⁹ But even waiving that, Dummett's objection can hardly be germane in a context in which we are precisely trying to view higher-order quantification as *range-free*. Range-free quantification involves no domain.

So is there a problem at all? Well, we can take it that if the quantifiers allow of satisfactory inferential role -style explanations at all, then higher-order quantified sentences will be intelligible and have determinate truth conditions whenever their instances do. So any genuine problem to do with impredicativity here must have to do with the intelligibility of the enlarged class of instances that it generates—the new *open sentences* that impredicative quantification allows us to formulate. The problem, if there is one, will have to concern whether or not such sentences have determinate satisfaction-conditions.

There is a lot to think about here, but it is clear—at least insofar as the project is informed by the proposed heuristic—that there is no good objection to one basic kind of case. Consider this example

Federer is tactically adroit
 Federer has pace and stamina

¹⁸ See especially various passages in chapter 18, “Abstract Objects”, of his *Frege: Philosophy of Mathematics*, (London: Duckworth 1991)

¹⁹ A point emphasised in Bob Hale's “Dummett's Critique of Wright's Attempt to Resuscitate Frege”, *Philosophia Mathematica* 3 (1994), pp. 122-47. The issues are further discussed in my “On the (Harmless) Impredicativity of Hume's Principle”, first published in *Philosophy of Mathematics Today*, ed. M. Schirn (Oxford: the Clarendon Press 1998), pp. 339-68. Both papers are reprinted in Hale and Wright, *The Reason's Proper Study*.

Federer has a complete repertoire of shots
 Federer is gracious in victory

.....
 Federer is everything a tennis champion should be

The last is naturally understood as an impredicative second -order quantification—after all, being everything a champion should be is certainly one thing, among the rest, that a champion should be! But notice that this introduces no indeterminacy into the satisfaction conditions of "...is everything a tennis champion should be" since it remains necessary and sufficient for the application of that predicate that a player have a range of *predicatively specifiable* characteristics, like those illustrated by the first four items in the list. That is, it is necessary and sufficient for Federer to be everything a tennis champion should be—including that very characteristic—that he have a range of *other* characteristics, not including that characteristic, whose satisfaction conditions can be grasped independently.

The example is one of a predicate formed by means of impredicative quantification which is fully intelligible because its satisfaction conditions can be explicated without recourse to impredicative quantification. Russell in effect tried to assimilate all cases of impredicative quantification to this model by his famous Axiom of Reducibility, which postulated that every (propositional) function of whatever kind is equivalent to a predicative function—one in whose specification no higher-order quantification need be involved. But of course he was unable to provide a general compelling motive for believing the Axiom. The point remains that impredicative quantification need not *per se* pose any special obstacle for neutralism. The crucial question for second-order impredicative quantification in particular is just whether allowing predicates formed by impredicative quantification results in predicates of whose satisfaction conditions there is no clear (non-circular) account. It is clear that there is no reason in general to suppose that has to be so. When it is not so, such new predicates will be open to intelligible neutral quantification in turn.

But how general is that run of cases? Do all the important examples of impredicative quantification allowed in classical higher-order logic fall on the right side of that distinction? What about the proof (strictly in third-order logic, I suppose) of the second-order logical version of Cantor's Theorem (and how, on a range-free conception of quantification, is that proof to be

interpreted in any case?) Do all the types of impredicative quantification demanded by the abstractionist programme fall on the right side of the distinction? Frege's definition of the Ancestral of a relation is impredicative. What are the implications of that for its role in the abstractionist programme?

We need answers to these questions, but clear answers will take some digging out. Of course, issues about impredicativity have long been regarded as pivotal in the debates about the foundations of mathematics for more than a century. But the discussion of them has traditionally been taken in the context of—and clouded by—the opposition between platonist and constructivist philosophies of mathematics. Not the least of the merits of the neutralist view, it seems to me, is that it lets us relocate these issues where they belong, in the proper account of quantification and the epistemology of logic.

I have offered the beginnings of a case that there is a legitimate conception of higher-order quantification which views it, like first-order quantification, as an operation grounded purely in the structure of subject-predicate thought and consequently as logical in whatever sense first-order quantification is logical. I confidently assert that *some* version of second-order logic, so conceived, is pure logic if anything is pure logic. But the question remains how much of *classical* second-order logic can be recovered under the aegis of this conception, when properly worked out, and whether the resulting system(s) can be meet for the needs of technical philosophical programmes, such as Abstractionism or Conceptual Realism based on categoricity results.

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Appendix: Abstractionist Mathematical Theories

On a thumbnail, the technical part of the Abstractionist project is to develop branches of established mathematics in higher-order—in practice, second-order—logic, with sole additional axioms consisting of *abstraction principles* (often simply termed *abstractions*.) These are axioms of the form:

$$\forall a \forall b (\Sigma(a) = \Sigma(b) \leftrightarrow E(a,b)),$$

where a and b are variables of first- or higher-order (typically, second-order), ‘ Σ ’ is a singular-term forming operator denoting a function from items in the range of ‘ a ’ and ‘ b ’ to objects, and E is an equivalence relation over items of the given type.²⁰

Arithmetic

Hume's Principle is the abstraction:

$$\forall F \forall G (N_x:Fx = N_x:Gx) \leftrightarrow (F \text{ 1-1 } G),$$

where $F \text{ 1-1 } G$ is an abbreviation of the second-order statement that there is a one-to-one relation mapping the F 's onto the G 's. Hume's principle thus states that the number of F s is identical to the number of G s if and only if F stands in such a relation to G . Adjoined to a suitable impredicative higher-order logic, it is technically sufficient for classical number-theory.

Real Analysis

— can be obtained by starting with Hume's Principle plus full, impredicative second order logic. We use the *Pairs* abstraction:

$$(\forall x)(\forall y)(\forall z)(\forall w) (\langle x,y \rangle = \langle z,w \rangle \leftrightarrow x=z \ \& \ y=w)$$

to get the ordered pairs of the finite cardinals so provided. An abstraction over the *Differences* between such pairs:

$$\text{Diff}(\langle x,y \rangle) = \text{Diff}(\langle z,w \rangle) \leftrightarrow x+w = y+z,$$

provides objects with which we can identify the *integers*. Define addition and multiplication on these integers. Next, where m , n , p and q are any integers, we form *Quotients* of pairs of integers in accordance with this abstraction:

$$Q\langle m,n \rangle = Q\langle p,q \rangle \leftrightarrow (n=0 \ \& \ q=0) \vee (n \neq 0 \ \& \ q \neq 0 \ \& \ m \times q = n \times p)$$

identifying a *rational* with any quotient $Q\langle m,n \rangle$ whose second term n is non-zero. Define addition and multiplication and thence the natural linear order on the rationals so generated. Move on to the Dedekind-inspired *Cut Abstraction*:

$$(\forall F)(\forall G)(\text{Cut}(F) = \text{Cut}(G) \leftrightarrow (\forall r)(F \leq r \leftrightarrow G \leq r))$$

where ‘ r ’ ranges over rationals, and the relation, “ \leq ”, holds between a property, F , of rationals and a specific rational number, r , just in case any instance of F is less than or equal to r under the constructed linear order on the rationals. Identify the *real numbers* with the cuts of those properties F , G which are both bounded above and instantiated in the rationals.²¹ It can be

²⁰ Of course, the conception of S as denoting an operation on *things* of a certain order—in general, the orthodox conception of a higher-order function — and the idea of E as an equivalence relation on such things need to be rethought if one is going in for a range-free conception of higher-order quantification along the lines of the present discussion.

²¹ This bracing progression is developed in Stewart Shapiro's "Frege Meets Dedekind: A Neologicist Treatment of Real Analysis", *Notre Dame Journal of Formal Logic* 41 (2000) pp. 335-364. An alternative, closer in spirit to Frege's insistence that a satisfactory foundation for a mathematical theory must somehow

shown that the reals so arrived at constitute a complete, ordered field, exactly as is classically conceived as distinctive of them.

Set Theory

No comparable abstractionist successes can be documented in this area so far. The most well researched proposal for a set theory founded on abstraction takes the usual second-order logical setting with a single abstraction principle, George Boolos's so-called 'New V':

$$\forall F \forall G (\{x: Fx\} = \{x: Gx\}) \leftrightarrow (\text{Bad}(F) \ \& \ \text{Bad}(G)) \vee (\forall x)(Fx \leftrightarrow Gx)$$

— informally, the abstracts of F and G are the same just in case either F and G are both bad, or are co-extensive. Here, 'badness' is identified with having the size of the universe — in effect, with sustaining a bijection with the universal concept, $x=x$ — and the effect of the proposal, in keeping with the tradition of limitation of size, is to divide the abstracts it yields into well-behaved ones, corresponding to 'good' concepts, for which identity is given by co-extensionality, as for 'naive' sets, and a single bad object corresponding to all the universe-sized concepts whether co-extensive or not. We can think of the *sets* as the former group.

As Boolos shows,²² a surprising number of the laws of classical set-theory can be verified of the sets (though not necessarily holding in the full domain) delivered by New V. Extensionality, Null set, Pairs, Regularity, Choice (global), Replacement, Separation, and Union are all forthcoming. But neither Infinity — that is, the existence of an infinite *set* (as oppose to the infinity of the domain) — nor Power Set can be obtained.

How these crucial shortcomings might be remedied — if indeed any satisfying remedy is possible at all — is a matter of current research. It will be evident that in order to obtain an infinite set via New V, we need first to obtain a concept larger than that by reference to which the sought-for set is to be defined; at a minimum, then, we first need an uncountable domain in order to generate a countably infinite set. As for Power Set, the requirement becomes that, for any infinite set so far obtained, a concept can be found large enough to outstrip the cardinality of its power set: that is, for any F, there exists a G such that the sub-concepts of F, as it were, can be injected into G but not vice versa. A natural thought is to try to augment the comprehension afforded by New V with the objects generated by an abstraction affording an uncountable domain, for example the Cut abstraction principle given above. A principle of this form may be given for any densely ordered infinite domain, and will always generate more cuts than elements in such a domain — indeed exactly 2^n many cuts if n is the cardinality of the original domain. So if we could show that any infinite concept allowed of a dense order, and could justify limitless invocation of appropriate Cut principles, then Cut abstraction would provide resources to generate limitlessly large concepts, and New V would generate limitlessly large well-behaved sets in tandem.

But the provisos are problematic. There are issues about abstractionism's right to the limitless invocation of Cut principles.²³ And the crucial lemma, that any infinite domain of objects allows of a dense order, is standardly proved²⁴ in ways that presuppose the existence of

integrate its potential applications into the basic explanations, is elegantly developed by Bob Hale in "Reals by Abstraction", *Philosophia Mathematica* 8 (2000), pp. 100—123 (reprinted in *The Reason's Proper Study*.)

²² See George Boolos "Saving Frege from Contradiction", *Proceedings of the Aristotelian Society* 87 (1986/7), pp. 137-51, and "Iteration Again", *Philosophical Topics* 42 (1989), pp. 5-21.

²³ See the exchange between Roy Cook "The state of the economy: Neo-logicism and inflation", *Philosophia Mathematica* 10 (2002), pp. 43—66, and Bob Hale "Abstraction and Set Theory", *Notre Dame Journal of Formal Logic* 41 (2000) pp. 379-98

²⁴ For example in Cook *op. cit.*

cardinals — hence sets — larger than the domain in question: the very point at issue. Finally, New V itself is anyway in violation of certain of the conservativeness constraints to which abstractions arguably ought to be subject, by virtue of its very entailment of the particularly strong form of Choice that it delivers — the global well-ordering of the universe (including, of course, any non-sets that it contains.)²⁵

Although it would certainly be a mistake to dismiss the prospect of any significant contribution by abstractionism to the epistemology of set theory,²⁶ my own suspicion is that the inability of the approach to justify more than a tiny fraction of the conventionally accepted ontology of sets may prove to be an enduring feature. Abstraction principles articulate identity-conditions for the abstracts they govern, and so explain what kind of things they are. It is down to the characteristics of the abstractive domain— the field of the equivalence relation on the right hand side — how much comprehension is packed into this explanation. If it turns out that any epistemologically and technically well-founded abstractionist set theory falls way short of the ontological plenitude we have become accustomed to require, we should conclude that nothing in the nature of sets, as determined by their fundamental grounds of identity and distinctness, nor any uncontroversial features of other domains on which sets may be formed, underwrites a belief in the reality of that rich plenitude. The question should then be: what, do we suppose, does?

²⁵ The downside of this result is emphasised in Shapiro, S., and A. Weir, "New V, ZF and abstraction", *Philosophia Mathematica*, vol. 7 (1999), pp. 293-321.

²⁶ A reader who wants to think further about the prospects for an abstractionist recovery of a strong set-theory (perhaps, at a minimum, something equivalent to ZFC) should turn, —in addition to the sources referenced in other notes in the present essay, — to Stewart Shapiro, 'Prolegomena to any future neo-logicist set theory: Abstraction and indefinite extensibility', *British Journal for the Philosophy of Science* 54 (2003), pp. 59-91, Kit Fine *The Limits of Abstraction* (Oxford: the Clarendon Press 2002) and John P. Burgess, *Fixing Frege* (Princeton, N. J.: Princeton University Press 2005.)