For 4th International Lauener Symposium on Analytical Philosophy, Bern, May 27 2010
In Honour of Professor Sir Michael Dummett

Whence the Paradox? Axiom V and Indefinite Extensibility

Crispin Wright
Northern Institute of Philosophy, University of Aberdeen
and New York University

In a well-known passage in the last chapter of *Frege: Philosophy of Mathematics* Michael Dummett suggests that Frege's major "mistake"—the key to the collapse of the project of *Grundgesetze*—consisted in "his supposing there to be a totality containing the extension of every concept defined over it; more generally [the mistake] lay in his not having the glimmering of a suspicion of the existence of indefinitely extensible concepts" (Dummett [1991, 317]). Now, claims of the form,

Frege fell into paradox because…….

are notoriously difficult to assess even when what replaces the dots is relatively straightforward. Offerings have included, for instance, that —

(A) Unrestricted quantification: Frege fell into paradox because he allowed himself to quantify over a single, all-inclusive domain of objects (Russell, Dummett);

(B) Impredicative objectual quantification: Frege fell into paradox because he allowed himself to define courses-of-values using (first order) quantifiers ranging over those very courses-of-values (Russell, Dummett);

(C) Impredicative higher-order quantification: Frege fell into paradox because he allowed himself to formulate conditions on courses-of-values using (higher order) quantifiers ranging over those very conditions (Russell, Dummett);

(D) Inflation: Frege fell into paradox because Basic law V is <u>inflationary</u>, i.e. defines its proper objects by reference to an equivalence on concepts that partitions the higher-order domain into too many cells (Boolos, Fine).

And while it is indeed clear that Frege did do all these things, — and prior to that, what it <u>is</u> to do them, — the diagnoses presented are nevertheless problematic. However with

(E) Frege fell into paradox because he didn't have even a glimmer of a suspicion of the existence of indefinitely extensible concepts,

matters may seem yet worse. This diagnosis may seem not to get so far as proposing *any* definite account of Frege's "blunder" (as Dummett was pleased to regard it) at all, even a controversial one.

The notion of indefinite extensibility has been connected in recent philosophy of mathematics with many issues, including not merely the proper diagnosis of the paradoxes, but the legitimacy of unrestricted quantification, the content of quantification (if legitimate at all) over certain kinds of populations, the legitimacy of classical logic for such quantifiers, the proper conception of the infinite, and the possibilities for (neo-) logicist foundations for set theory, But my project in this short note must be limited. It will be merely to begin to say what I think is right about this particular diagnostic suggestion of Dummett's: to offer a characterization of indefinite extensibility and to explain a connection of the notion so characterized with paradox. A full enough plate.

1 **Indefinite extensibility intuitively understood.**

I don't think Dummett would — or could — object to the suggestion that Russell anticipated him. Russell [1906] begins with an examination of the standard paradoxes, and concludes:

> . . . the contradictions result from the fact that . . . there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

Indeed, citing this very passage, Dummett [1993, 441], writes that an

> "*indefinitely extensible* concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it"

According to Dummett, an indefinitely extensible concept *P* has a "principle of extension" that takes any definite totality *t* of objects each of which has *P*, and produces an object that also has *P*, but is not in *t* (see also Dummett [1991, 316-319]).

But what does "definite" mean in that? Presumably a concept *P* is *Definite* for Dummett's purpose in those passages just if it is not indefinitely extensible! If so, then Dummett's remarks won't do as a definition, even a loose one, since they appeal to its complementary "definite" to characterize what it is for a concept to be indefinitely extensible. And Russell, of course, does no better by speaking unqualifiedly of "any class of terms all having such a property", since he is taking it for granted that classes, properly so regarded, are "wholes"—are Definite.

This problem of implicit circularity in the intuitive characterization of indefinite extensibility is a serious one. Indeed, it is the major difficulty in forming a clear idea of the notion, and one I want to try to solve. But it would be premature to lose confidence in the notion over it. The three classic paradoxes of classes and the transfinite — Burali,[1] Cantor, and Russell — surely make very salient the pattern that Russell and Dummett both discern:

(1) *Burali-Forti*. Think of the ordinals in an intuitive way, simply as order-types of well-orderings. Let *O* be any Definite collection of ordinals. Let O' be the collection of all ordinals α such that there is a $\beta \in O$ for which $\alpha \leq \beta$. It is easy to see that O' is well-ordered under the natural ordering of ordinals. Let γ be the order-type of O'. So γ is itself an ordinal. Let γ' be the order-type of $O' \cup \{\gamma\}$. That is γ' is the order-type of the well-ordering obtained from O' by tacking an element on at the end. Then γ' is an ordinal number, and γ' is not a member of *O*. So *ordinal number* is indefinitely extensible.

(2) *The Russell paradox*. Let *R* be any set of sets that do not contain themselves; so if $r \in R$ then $r \notin r$. Then *R* does not contain itself. So the concept, *set that does not contain itself*, is indefinitely extensible—any set of such sets omits a set, namely itself. A fortiori, *set* itself is indefinitely extensible, since any Definite collection—set—of sets must omit the set of all of its members that do not contain themselves.

---

[1] As Dummett [1991, 316] puts it,

> if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. Any [D]efinite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognise as being an ordinal.

(3) *The Cantor paradox*. Let C be a collection of cardinal numbers. Let C' be the union of the result of replacing each $\kappa \in C$ with a set of size κ. The collection of subsets of C' is larger than any cardinal in C. So *cardinal number* is indefinitely extensible.

While these examples are not completely uncontestable — someone could challenge the set-theoretic principles (Union, Replacement, Power-set, etc.) that are implicitly invoked in the constructions, for instance — I think it reasonable to agree with Russell and Dummett that the concepts in question do have some kind of "self-reproductive" feature which the notion of indefinitely extensibility gestures at. The question is whether we can give a more exact, philosophically robust characterization.

2 Indefinite extensibility and the ordinals: Russell's Conjecture and 'Small' cases.

We can make a start by following up on a suggestion of Russell. Russell [1906, 144] wrote that it "is probable" that if *P* is any concept which demonstrably "does not have an extension", then "we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the concept *P*". The conjecture is in effect that if *P* is indefinitely extensible, then there is a one-to-one function from the ordinals into *P*.

If Russell is right, then any indefinitely extensible concept determines a collection at least as populous as the ordinals — so, one might think, surpassing populous! And in that case one might worry whether the connection made by Russell's Conjecture is acceptable. For Dummett at least has characteristically taken it that both the natural numbers and real numbers are indefinitely extensible totalities in just the same sense that the ordinals and cardinals are, with similar consequences, in his opinion, for the understanding of quantification over them and the standing of classical logic in the investigation of these domains. Moreover in the article (Dummett [1963]) which contains his earliest published discussion of the notion, Dummett argues that the proper interpretation of Gödel's incompleteness theorems for arithmetic is precisely to teach that *arithmetical truth* and *arithmetical proof* are each indefinitely extensible concepts—yet neither presumably has an even more than countably infinite extension, still less an ordinals-sized one. (The ordinary, finitely based language of second-order arithmetic presumably suffices for the expression of any arithmetical truth.) It would be disconcerting to lose contact with one of the leading friends of indefinite extensibility so early in the discussion. But then who is mistaken, Russell or Dummett?[2]

The issue will turn out to be crucial to the proper understanding of indefinite extensibility. To fix ideas, I am going to consider in some detail the so-called Berry paradox, the paradox of "the smallest natural number not denoted by any expression of English of fewer than 17 words". This is an English expression that, on plausible assumptions, should denote a natural number—but it contains 16 words. So its referent—the smallest natural number not denoted by any expression of English of fewer than 17 words—is denoted by an English expression of 16 words. Is this a paradox of indefinite extensibility?

Let's try to state the paradox more carefully. Define an expression *t* to be *numerically determinate* if *t* denotes a natural number and let *C* be the set—if there is one—of all numerically determinate expressions of English. Consider the expression *b*: "the smallest natural number not denoted by any expression in *C* of fewer than 17 words." Assume: (1) that *b* is a numerically determinate expression of English and (2) that *C* — the set of all such expressions — indeed exists. Then contradiction follows from (1) and (2) and the empirical datum that *b* has 16 words (counting the contained occurrence of '*C*' as one word).

The analogy with the classic paradoxes may look good, a principle of extension seemingly inbuilt into a concept leading to aporia when applied to a totality supposedly embracing all instances of the concept. But, as emerges if we think the process of "indefinite

[2] It is relevant to recall that Russell [1908] himself, in motivating a uniform diagnosis of the paradoxes, included in his list of chosen examples some at least where the "self-reproductive" process seems bounded by a relatively small cardinal. For instance the Richard paradox concerning the class of decimals that can be defined by means of a finite number of words makes play with a totality which, if indeed indefinitely extensible, is at least no greater than the class of decimals itself, i.e. than $2^{\aleph_0}$. Was Russell simply unaware of this type of example in 1906, when he proposed the Conjecture discussed above? Or did he not in 1906 regard the Richard paradox and others involving "small" totalities as genuine examples of the same genre, then revising that opinion two years later?

extension" through, it is not quite right.

To see why, let an initial collection *D* just consist of the ten English numerals, "zero" to "nine". Introduce a (one word) name "*d*" for *D* (counting it as part of "English"), and consider "the smallest natural number not denoted by any member of *d* of fewer than 17 words". Call this 16-worded expression "w_1"—it will denote 10. w_1 is a numerivally determinate exprssion of English, but not in *D*. Let *D*1 be $D\cup\{w_1\}$. Give *D*1 a one-word name. Now do a Berry on *D*1, producing w_2. Let *D*2 be $D1\cup\{w_2\}$. Give D2 a one-word name (counting it as part of English). Do the Berry construction again. Keep going . . . Now let *D*ω be the union of *D*, *D*1, *D*2, . . . What happens next?—what happens when we apply the Berry construction to *D*ω?

The answer, clearly, is that it aborts. For reflect that 0 to 9 are all denoted by single-word members of *D*; 10 is denoted by the 16-worded "the smallest natural number not denoted by any member of [write in the one-word name of *D*] of fewer than 17 words"; 11 is denoted by the "the smallest natural number not denoted by any member of [write in the one-word name of *D*1] of fewer than 17 words"; 12 is denoted by the "the smallest natural number not denoted by any member of [write in the one-word name of *D*2] of fewer than 17 words"; and so on. So the "the smallest natural number not denoted by any member of [write in the one-word name of *D*ω] of fewer than 17 words" *has no reference*—for *every* natural number is denoted by a member of *D*ω of fewer than 17 words.

It seems fair to say is that — with the relevant idealizations of what counts as English — there is a *kind* of indefinite extensibility about the concept*, numerically determinate expression of English*. But it is a *bounded* indefinite extensibility, as it were—indefinite extensibility up to a limit (ordinal). If union is a Definiteness preserving operation, there will eventually be, in such bounded cases, a *definite* collection of entities of the kind in question that does not in turn admit of extension by the original operation. So they will not be *indefinitely* extensible, at least not in the spirit of our initial characterization.

[3 'Small' indefinitely extensible concepts? Case study (2): arithmetical truth

Another example. As noted earlier, Dummett [1963] contends that Gödel's incompleteness theorem shows that *arithmetical truth* is indefinitely extensible. Given any Definite collection C of arithmetical truths, one can construct a truth—the Gödel sentence for C—that is not a member of C.

This is apt to impress as a puzzling claim. If "definite collection" means *set*, and if the latter concept is understood as in classical mathematics, then Dummett is wrong—*arithmetical truth* is not indefinitely extensible. Following Tarski, one can give a straightforward explicit definition of "arithmetical truth". It then follows from the *Aussonderungsaxiom* that there is a set of all arithmetical truths. There is no "Gödel sentence" for this set.

Still, we know what Dummett has in mind. It is straightforward to initiate something that looks like a process of "indefinite extension". Just let A_0 be the theorems of some standard axiomatization of arithmetic. For each natural number n, let A_{n+1} be the collection A_n together with a Gödel sentence for A_n. Presumably, if A_n is Definite, then so is A_{n+1}, and, of course, A_n and A_{n+1} are distinct. Unlike the case of the Berry paradox, this construction can be continued into the transfinite. Let A_ω be the union of $A_0, A_1, \ldots$ Arguably, A_ω is Definite. Indeed, if A_0 is recursively enumerable, then so is A_ω; if A_0 is arithmetic, then so is A_ω. Thus, we can define $A_{\omega+1}, A_{\omega+2}, \ldots$ Then we take the union of those to get $A_{2\omega}$, and onward, Gödelising all the way (so to speak).

On the usual, classical construal of the extent of the ordinals, however, this process too does not continue without limit. In fact it must "run out" well before the first uncountable ordinal. Let λ be an ordinal and let us assume that we have defined A_λ. The foregoing construction will take us on to the next set $A_{\lambda+1}$ only if the collection A_λ has a Gödel sentence. And this is possible only if A_λ is *arithmetic*. Clearly, it cannot be the case that for every (countable) ordinal λ, A_λ is arithmetic. For there are only countably many arithmetic sets (at most one for each formula in the language of arithmetic), but there are uncountably many (countable) ordinals.

Faced with such examples, there is in principle the option of just insisting on the Russell-conjecture but maintaining that there is no such ordinal as κ. But, for the classical mathematician at least, the conclusion should be that the notion of arithmetical truth is not fully indefinitely extensible; we cannot run on indefinitely through the ordinals in iterating Gödel sentences.]

4 Indefinite extensibility explicated.

Let's take stock. Russell's Conjecture, that indefinitely extensible concepts are marked by the possession of extensions into which the ordinals are injectible, still stands. Apparent exceptions to it, like *numerically determinate expression of English* [and *arithmetical truth*], are not really exceptions. For the principles of extension they involve are not truly *indefinitely* extensible but stabilize after some series of iterations isomorphic to a proper initial segment of the ordinals. Or at least they do so if the ordinals are allowed their full classical extent.

That said, though, the point remains that Russell's Conjecture, even if extensionally correct, is not the kind of characterization of indefinite extensibility we should like to have. To get a clear sense of the shortfall, reflect that if Russell's Conjecture provided a full account, it would be a *triviality* that the ordinals are indefinitely extensible. Whereas what is wanted is a perspective from which we can explain *why* Russell's Conjecture is good, if indeed it is—equivalently, a perspective from which we can characterize exactly what it is about *ordinal* that *makes* it the paradigm of an indefinitely extensible concept.

So step back. Any indefinitely extensible totality *P* is intuitively unstable, "restless", or in "growth". Whenever you think you have it safely corralled in some well-fenced enclosure, suddenly—hey presto!—another fully *P*-qualified candidate pops up outside the fence. The primary problem in clarifying this figure is to dispense with the metaphors of "well-fenced enclosure" and "growth". Obviously a claim is intended about sub-totalities of *P* and functions on them to (new) members of *P*. Equally obviously, we need to qualify for which type of sub-totalities of *P* the claim of iterative extensibility within *P* is being made. Clearly it cannot be sustained for absolutely *any* sub-totality of *P*: if for example, we continue to take it that *ordinal* is a paradigm of indefinite extensibility, we do not want to claim that *ordinal* itself picks out a sub-totality of the relevant kind (though of course there are issues, which will occupy us later, about whether one can avoid that claim). Nor would it help to restrict attention to proper sub-totalities: *ordinal other than three* does not pick put the right kind of sub-totality either. If we could take it for granted that the notion of indefinite extensibility is in clear standing and picks out a distinctive type of totality, or concept, then we could characterize the relevant kind of sub-totality exactly as Dummett did—they are the *Definite* sub-totalities. For the indefinite extensibility of a totality, if it consists in anything, precisely consists in the fact that any Definite sub-totality of it is merely "proper". But now we again confront the circularity worry: unless there is some direct route into the intended notion of Definiteness other than via "not indefinitely extensible" we make no explanatory progress.

Is there a way forward? Well, in order, at least temporarily, to finesse the "which sub-totalities?" issue, let's start with an explicitly relativised notion. Let *P* be a concept of items of a certain type τ. Typically, τ will be the (or a) type of individual objects. Let Π be a concept of concepts of type τ items. Let us say that *P* is *indefinitely extensible with respect to* Π if and only if there is a function *F* from items of the same type as *P* to items of type τ such that if *Q* is any sub-concept of *P* such that ΠQ then

(1) *FQ* falls under the concept *P*,
(2) it is not that case that *FQ* falls under the concept *Q*, and
(3) $\Pi Q'$, where *Q'* is the concept instantiated just by *FQ* and by every item which instantiates *Q* (i.e., $\forall x[Q'x \equiv (Qx \vee x{=}FQ)]$; in set-theoretic terms, *Q'* is $(Q\cup\{FQ\})$.

Intuitively, the idea is that the sub-concepts of P of which Π holds have no maximal member. For any sub-concept Q of P such that ΠQ, there is a proper extension Q' of Q such that $\Pi Q'$.

This relativised notion of indefinite extensibility is quite robust, covering a lot of different examples. I'll quickly review six.

> (*Natural number*) Px iff x is a natural number ; ΠX iff there are only finitely many X's; FX is the successor of the largest X. So *natural number* is indefinitely extensible with respect to *finite*.
>
> (*Real number*) Px iff x is a real number; ΠX iff there are only countably many X's. Define FX using a Cantorian diagonal construction. So *real number* is indefinitely extensible with respect to *countable*.
>
> (*Arithmetical truth*) Px iff x is a truth of arithmetic; ΠX iff the X's are recursively enumerable. FX is a Gödel sentence generated by the X's, or the straightforward statement that the X's are consistent. Then, if every member of X is true of the natural numbers, then so is the sentence FX. And, of course, FX is not one of the X's. So *arithmetical truth* is indefinitely extensible with respect to *recursively enumerable*.
>
> (*Ordinal number*) Px iff x is an ordinal ; ΠX iff each of the X's is an ordinal and the X's are themselves isomorphic to an ordinal (under the natural ordering). In other words, ΠX iff each X is an ordinal and the X's have (or exemplify) a well-ordering type. (In still other words, ΠX iff the X's are a set of ordinals.) FX is the successor of the union of the X's. So *ordinal number* is indefinitely extensible with respect to the property of being *isomorphic to an ordinal* (or exemplifying a well-ordering type).
>
> (*Set*) Px iff x is a set; ΠX iff the X's form a set, i.e., $\exists y \forall x\ (x \in y \equiv Xx)$. FX is just the set of X's that are not self-members. So *set* is indefinitely extensible with respect to the property of *constituting a set*.
>
> (*Cardinal number*) Px iff x is a cardinal number; ΠX iff the X's form a set. Given such an X, take the union of a totality containing exactly one exemplar set of each X cardinal (using choice and replacement); FX is the powerset of that. So *cardinal number* is indefinitely extensible with respect to the property of *constituting a set*.

Most instances of relativised indefinite extensibility are unremarkable. They do not, as far as they go, shed any philosophical light on the paradoxes. But our goal remains to define an unrelativised notion of indefinite extensibility, a notion that covers *ordinal*, *cardinal*, and *set* and at least purports to shed some light on the paradoxes, in the sense that the latter should emerge as somehow turning on the indefinite extensibility of the concepts concerned. So what next?

Three further steps are needed. Notice to begin with that the listed examples sub-divide into two kinds. There are those where—helping ourselves to the classical ordinals—we can say that some ordinal λ places a lowest limit on the length of the series of Π-preserving applications of *F* to any *Q* such that Π*Q*. Intuitively, while each series of extensions whose length is less than λ results in a collection of *P*'s which is still Π, once the series of iterations extends as far as λ the resulting collection of *P*'s is no longer Π, and so the "process" stabilises. This was the situation noted with *numerically determinate expression of English* in our discussion of the Berry paradox, and is also the situation of the first three examples above. But it is not the situation with the last three cases: in those cases, by contrast, there is no ordinal limit to the Π-preserving iterations. With (*Ordinal number*), this is obvious, since the higher-order property Π in that case just is the property of having a well-ordering type. Indeed, let λ be an ordinal. Then the first λ ordinals have the order type λ and so they have the property. The "process" thus does not terminate or stabilize at λ. With (*Set*) and (*Cardinal number*), we get the same result if we assume that for each ordinal λ, a totality that has order type λ is a set and has a cardinality.

Let's accordingly refine the relativised notion to mark this distinction. So first, for any ordinal λ say that *P* is *up-to-λ-extensible with respect to* Π just in case *P* and Π meet the conditions for the relativised notion as originally defined but λ places a limit on the length of the series of Π-preserving applications of *F* to any sub-concept *Q* of *P* such that Π*Q*. Otherwise put, λ iterations of the extension process on any Π*Q* "generates" a collection of *P*'s which form the extension of a non-Π sub-concept of *P*. Next, say that *P* is *properly indefinitely extensible* with respect to Π just if *P* meets the conditions for the relativised notion as originally defined and there is no λ such that *P* is up-to-λ- extensible with respect to Π. Finally, say that *P* is *indefinitely extensible* (simpliciter) just in case there is a Π such that *P* is properly indefinitely extensible with respect to Π.

My suggestion, then, is that the circularity involved in the apparent need to characterize indefinite extensibility by reference to *Definite* sub-concepts/collections of a target concept *P* can be finessed by appealing instead at the same point to the existence of some species—Π—of sub-concept of *P*/collections of *P*'s for which Π-hood is *limitlessly* preserved under iteration of the relevant operation. This notion is, naturally, relative to one's conception of what constitutes a limitless series of iterations of a given operation. No doubt we start out innocent of any conception of serial limitlessness save the one implicit in one's first idea of the infinite, whereby any countable potential infinity is limitless. Under the aegis of this conception, *natural number* is properly indefinitely extensible with respect to *finite* and so, just as Dummett suggests, indefinitely extensible simpliciter. The crucial conceptual innovation which transcends this initial conception of limitlessness and takes us to the ordinals as classically conceived is to add to the idea that every ordinal has a successor the principle that every infinite series of ordinals has a limit, a first ordinal lying beyond all its elements—the resource encapsulated in Cantor's Second Number Principle. If it is granted that this idea is at least partially—as it were, initial-segmentally—acceptable, the indefinite extensibility of *natural number* will be an immediate casualty of it. (Critics of Dummett who have not been able to see what he is driving at are presumably merely taking for granted the orthodoxy that the second number principle is at least partially acceptable.)

5. Indefinite extensibility and the paradoxes.

Very well. Roughly summarized, then, the proposal is that *P* is indefinitely extensible just in case, for some Π, any Π sub-concept of *P* allows of a *limitless* series of Π-preserving enlargements. It is striking that there seems to be nothing inherently paradoxical about this idea. So what is the connection with paradox—how is the indefinite extensibility of *set*, *ordinal* and *cardinal* linked with the classic paradoxes that beset those notions?

The immediate answer is that in each of these cases there is powerful intuitive cause to regard *P* <u>itself</u> as having the property Π. For example, in case *P* is *ordinal*, and Π*Q* holds just if the *Q*'s *exemplify a well-order-type*, it seems irresistible to say that *ordinal* is itself Π. After all, the ordinals are well-ordered. But then the relevant principle of extension kicks in and dumps a new object on us that both must and cannot be an ordinal—must because it corresponds, it seems, to a determinate order-type, cannot because the principle of extension always generates a non-instance of the concept to which it is applied. So we have the Burali-Forti paradox.

The question, therefore, is what seduces us in the first place into so fashioning our intuitive concepts of *set*, *ordinal*, and *cardinal* that they seem to be indefinitely extensible with respect to Π's which are, seemingly, characteristic of those very concepts themselves? For *that*, in a nutshell, is the *pensée fausse*, the great "blunder". These remarks of Dummett [1991, 315-316] suggest a key insight:

> . . . to someone who has long been used to finite cardinals, and only to [finite cardinals], it seems obvious that there can only be finite cardinals. A cardinal number, for him, is arrived at by counting; and the very definition of an infinite totality is that it is impossible to count it. . . . [But this] prejudice is one that can be overcome: the beginner can be persuaded that it makes sense, after all, to speak of the number of natural numbers. Once his initial prejudice has been overcome, the next stage is to convince the beginner that there are distinct cardinal numbers: not all infinite totalities have as many members as each other. When he has become accustomed to this idea, he is extremely likely to ask, 'How many transfinite cardinals are there?'. How should he be answered? He is very likely to be answered by being told, 'You must not ask that question'. But why should he not? If it was, after all, all right to ask, 'How many numbers are there?', in the sense in which 'number' meant 'finite cardinal', how can it be wrong to ask the same question when 'number' means 'finite or transfinite cardinal'? A mere prohibition leaves the matter a mystery. It gives no help to say that there are some totalities so large that no number can be assigned to them. We can gain some grasp on the idea of a totality too big to be counted . . . but once we have accepted that totalities too big to be counted may yet have numbers, the idea of one too big even to have a number conveys nothing at all. And merely to say, 'If you persist in talking about the number of all cardinal numbers, you will run into contradiction', is to wield the big stick, but not to offer an explanation.

The insight is into the interconnection, in the case in point, between the indefinite extensibility built into *cardinal number* and the temptation to say that the concept falls under—*ought* to fall under—the relevant Π. We only get the indefinitely extensible series of transfinite cardinals up and running in the first place by first insisting on one-one correspondence between concepts as necessary and sufficient for sameness, and hence existence, of cardinal number in general—not just in the finite case— and then allowing each-cardinal-numbered concept to be subject to the reasoning of Cantor's Theorem. The conception of cardinal number embracing both the finite and the spectacular array of transfinite cases thus only arises in the first place when it is taken without question that concepts in general—or at least any that sustain determinate relations of one-one correspondence—have cardinal numbers, identified and distinguished in the light of those relations. That is how the intuitive barrier to the question, how many natural numbers are there, is overcome. But then the lid is off Pandora's box: for the intuitive barrier to the question, how many *cardinal* numbers are there is overcome too. *Cardinal*, it seems, has to be both indefinitely extensible with respect to *has a cardinal number*[3] and an instance of it.

It is straightforward to transpose this diagnosis to our paradigm, the ordinals, taken intuitively as the order-types of well-orderings. Consider an imaginary Heroine being introduced to the ordinal numbers. Suppose that she has been introduced to the finite ordinals, but not the infinite ones. She wonders about the order-type of the finite ordinals, and realizes that she has no ordinal for this—yet. So she thinks that there is no ordinal of finite ordinals. But we tell her that the finite ordinals do indeed have an ordinal, just not one that she has encountered already. She thus encounters ω, and she formulates the notion of "countable ordinal". Heroine then learns about $\omega+1$, 2ω, ω^2, ω^ω, ε_0, etc. So now she naturally asks about the order-type of the countable ordinals, and she encounters the same problem. We tell her that the countable ordinals do have an ordinal—just not one that she has encountered already. So she learns about ω_1. Heroine is a quick study, and she recognizes the pattern: every initial segment of the ordinals has an order-type—just not one featuring in the segment itself, but rather the next one after all those. But now she notices that the ordinals themselves are well-ordered, and so she inquires after the order-type of *all* ordinals. This time it seems we have the option neither of telling her that the ordinals do indeed have an order-type—just not one among those she has encountered already—nor of denying that they have any order-type. She asked about the order-type of ALL ordinals, and

[3] I am assuming the equivalence of the second-order properties, *has a cardinal number* and *determines a set*.

since the ordinals are well-ordered, there ought to be one. But if this order-type exists, it too is an ordinal and must therefore occur somewhere among the ordinals whose collective order-type she asked about. *Ordinal* has been so explained to her as to be both indefinitely extensible with respect to *has instances which exemplify an well-order-type*—and at the same time itself an instance of this Π. Nemesis in the form of Burali-Forti ensues.

We'll turn to the case of *set*— more specifically, to the notion of *extension* characterized by Basic Law V — in a moment.

In sum: the suggestion is that the classic paradoxes of the transfinite arise not with indefinite extensibility as such—at least, not if that is characterized as I have proposed—but with a particular twist taken by the examples concerned. They are cases where we myopically load a concept with a principle of indefinite extension whose trigger-concept—the relevant Π—can be denied of the concept in question only by making an arbitrary exception to a connection—e.g. that well-ordered collections have order-types, that concepts which sustain relations of one-one correspondence have cardinals, that well-defined collections comprise sets—which is integral to the way we want to think of the ontology of the instances of the concept concerned as determined. Of course it may seem perverse to caption the making of an exception necessary to avoid contradiction as arbitrary. But, as Dummett said, intimidation is one thing, and explanation is another.

6 Indefinite extensibility and Basic Law V

It remains to focus these thoughts specifically on Frege's ill-fated axiom. For our purposes we can restrict attention to the case of courses of values whose ranges are concepts and values truth-values—to the case of *extensions of concepts*. So the axiom becomes, in effect, this:

$$(\forall P)(\forall Q)(\{x{:}Px\}=\{x{:}Qx\} \leftrightarrow (\forall x)(Px \leftrightarrow Qx))$$

Extensionality and Naïve Comprehension can be read off straightaway: extensions arte identical just when their associated concepts are co-extensive; and every concept has one. (<u>Proof</u>: take '*P*' for 'Q', detach the left-hand-side of the biconditional, and existentially generalise on one occurrence of '$\{x{:}Px\}$'.) So *absolutely any* concept of extensions is associated with its own extension; compare—absolutely any well-ordered series has an ordinal number, absolutely any concept that sustains determinate relations of 1-1 correspondence has a cardinal number.

Now recall the mechanics of indefinite extensibility paradoxes. Bracketing for the moment the requirement of limitless iterability, it follows from our earlier characterization that P is indefinitely extensible *only if* there are Π and F such that for any sub-concept Q of P, if ΠQ, then $P(FQ)$ but not $Q(FQ)$. Notice that *this* property of P is enough to trigger paradox in any case where ΠP. For then, taking 'P' for 'Q', we immediately have that $P(FP)$ and not $P(FP)$—full (limitless) indefinite extensibility is not required.[4] But the point to be emphasized is that we can near enough read off from Law V that *extension* will satisfy this evil mix of conditions just from its immediate provisions noted. Let ΠX hold just if the instances of X comprise an extension; then, by Naïve Comprehension, every concept X is such that ΠX—including *extension* itself. Contradiction therefore just awaits the construction of a suitable F to fit the template, and for that we have only to find a schematic concept whose instances embrace only extensions that fall under a given concept Q but whose own extension cannot perforce be an instance of Q. *Is an instance of Q and not of any concept of which it is the extension* will do nicely.

It may be said on Frege's behalf that whereas the Burali-Forti paradox is a very immediate consequence of Cantor's second number principle as applied to the ordinals, whereby every initial sequence of them is accorded a least ordinal bound, the construction of a disaster–inducing F for Law V is relatively unobvious — it took a clever man to spot how to do this and make trouble. Nevertheless from the perspective of the notion of indefinite extensibility that I have been trying to clarify, it does not seem unduly harsh, to conclude that what really lets the Serpent into Paradise is in both cases a kind of intellectual *greed*: the determination that

[4] So there might in principle be paradoxes of 'small' indefinite extensibility.

absolutely any well-ordered series should have an ordinal number, with absolutely no limits, is exactly what gets the classical transfinite ordinals off and running. Once we allow exceptions, what can be said about how far they go? Where should the series stop? If we try saying, "Well, the claim holds in every case except when the sequence in question is that of all ordinals", that seems unprincipled at best— an exception driven purely by the belated realization of imminent paradox, and for which nothing in the way the concept is explained prepares us. But there is worse to follow: we just locked ourselves out of Paradise, for the restriction is consistent e.g. with the only ordinals there are being finite. And exactly so also with the claim, implicit in Law V, that every concept (of extensions) determines an extension. One might likewise try saying, "Well, except when the concept is *extension* itself." But then, again, one will need to know independently how far the universe of extensions extends before one can correctly respect the prescribed limit. And once again, the restriction as formulated will be consistent with that being much less far than one had in mind.

There are differences between the cases. In particular, for Law V there is a range of (by now, well-explored) manoeuverings which allow ΠP—allow that every concept, including *extension* itself, determines an extension—and, by predicativity restrictions of one level or another, or by outlawing the Russell concept, one way or another, from the range of the second order quantifiers, seek to block the inference to contradiction. No comparable range of manoeuvres seem open for Burali-Forti. But if indefinite extensibility as explained here is a well-conceived notion, then the perspective it encourages will be one from which these maneuverings will seem to disrespect the (moral of the) analogy between the two cases, rather than diminish it.

There is much more to say, of course. The hope dies hard that there might be some principled way of restricting Law V yet getting a 'decent' amount of set-theory—for, instance, a theory comprehending the iterative hierarchy up to the first inaccessible. The point remains principles of limitless, greedy comprehension—whether of ordinals or sets (courses-of-values)—of the kind illustrated by Law V and Cantor's Second Number Principle, are almost as if designed to activate the initial conditions for paradoxes of indefinite extensibility. That is what I take to be the central point of Dummett's diagnosis.

REFERENCES

Cantor, G. [1833], *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Leipzig, Teubner.

Dummett, M. [1963]: "The philosophical significance of Gödel's theorem", *Ratio , 5*, 140-155.
[1991]: *Frege: Philosophy of mathematics*, Cambridge, Massachusetts: Harvard University Press.
[1993]: *The seas of language*, Oxford: Oxford University Press.

Rogers, H. [1967], *Theory of recursive functions and effective computability*, New York, McGraw-Hill.

Russell, B. [1906]: "On some difficulties in the theory of transfinite numbers and order types", *Proceedings of the London Mathematical Society 4*, 29-53.
[1908]: "Mathematical logic as based on the theory of types*", American Journal of Mathematics 30*, 222-262.

APPENDIX

Additional examples of relative indefinite extensibility[5]

1. *Px* iff *x* is a finite ordinal number; Π*X* iff there are only finitely many *X*'s; *FX* is the successor of the largest *X*. So *finite ordinal* is indefinitely extensible with respect to *finite*.

2. *Px* iff *x* is a countable ordinal (i.e., countable well-ordering type); Π*X* iff there are only countably many *X*'s; *FX* is the successor of the union of the *X*'s. So *countable ordinal* is indefinitely extensible with respect to *countable*.

3. Let κ be any regular cardinal number,[9] and define *Px* iff *x* is an ordinal smaller than κ. Π*X* iff there are fewer than κ-many *X*'s; *FX* is the successor to the union of the *X*'s. So,

[5] Thanks here to Stewart Shapiro

for each regular cardinal κ, the concept *ordinal smaller than κ* is indefinitely extensible with respect to *has fewer than κ instances*.

(A converse holds. A cardinal κ is regular if and only if *ordinal smaller than κ* is indefinitely extensible with respect to *has fewer than κ instances* using the indicated "successor of union" function.)

4. Let κ be any infinite cardinal, and define *Px* iff *x* is an ordinal smaller than κ. Π*X* iff there are fewer than κ-many *X*'s; *FX* is the smallest ordinal λ such that *P*λ & ¬*X*λ. So, for each infinite cardinal κ, the concept *ordinal smaller than κ* is indefinitely extensible with respect to *has fewer than κ instances*.

5. Let κ be a strong inaccessible. *Px* iff *x* is an ordinal smaller than κ; Π*X* iff there are fewer than κ-many *X*'s. *FX* is the powerset of the union of the *X's*. So, for each strong inaccessible κ, the concept of *ordinal smaller than κ* is indefinitely extensible with respect to Π via this function. (Here, again, there is a converse.)

6. *Px* iff *x* is the Gödel number of a truth of arithmetic; Π*X* iff the *X*'s are arithmetic, i.e., iff there is a formula Φ(*x*) with only *x* free such that Φ(*x*) iff *Xx*. *FX* is a fixed point for ¬Φ(*x*): the Gödel number *n* of a sentence Ψ such that (Ψ ≡ ¬Φ(*n*)) is provable in ordinary Peano arithmetic (and so true). So *truth of arithmetic* is indefinitely extensible with respect to the property of being *arithmetic*.

7. Let *A* be a productive set of natural numbers (see Rogers [1967, 84]). *Px* iff *x*∈*A*; Π*X* iff *X* is recursively enumerable. So, for each productive set *A*, the concept of *being a member of A* is indefinitely extensible with respect to the property of being *recursively enumerable*.