THE PURE LOGIC OF GROUND

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Abstract. I lay down a system of structural rules for various notions of ground and establish soundness and completeness.

Ground is the relation of one truth holding *in virtue of* others. This relation is like that of consequence in that a necessary connection must hold between the relata if the relation is to obtain but it differs from consequence in so far as it required that there should also be an explanatory connection between the relata. The grounds must *account* for what is grounded. Thus even though P is a consequence of P & P, P & P is not a ground for P, since it does not account for the truth of P.

It is the aim of this paper to develop a semantics and proof theory for the pure logic of ground. The pure logic of ground stands to ground as Gentzen's structural rules stand to consequence. One prescinds from the internal structure of the propositions under consideration and simply asks what follows from what in virtue of the formal features of the underlying relation. Thus the claim that ground is transitive, that if P is a ground for Q and Q a ground for R then P should be a ground for R, is plausibly regarded as part of the pure logic of ground; but the claim that P is a ground for P & P will be part of the applied as opposed to the pure logic of ground, since it turns on the logical properties of &.

The question of the logic of ground has been considered by a number of other authors (even if not exactly under this head) and so it may be helpful to bring out some of the more distinctive aspects of my own approach.[1] I have, in the first place, attempted to be rigorous and systematic in the development of the logic. Rather than simply laying down various principles that I regard as true, I have specified various formal systems of which these principles are a part. These systems, in familiar fashion, thereby become the object of metalogical study; they can be assessed for consistency, assigned a semantics, compared for proof-theoretic strength, etc.

There are two other, more particular, aspects of my approach. The first lies in its conceptual basis. Most other philosophers have worked almost exclusively with a concept of *strict* ground, under which a truth is not capable (or, at least, not normally capable) of being a ground for itself.[2] But I believe that there is also an important concept of *weak* ground, under which a truth will automatically be a ground for itself. Very roughly, we may say that strict grounds must move us down in the explanatory order while weak grounds must not move us up. Thus P can weakly ground P, but P & P cannot weakly ground P given that P strictly grounds P & P.

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[1] The authors include Audi (2011), Batchelor (2010), Correia (2005, 2011), Rosen (2010), Schnieder (2006, 2011), Fine (2011), and, much earlier, Mulligan *et al.* (1984).

[2] An exception is Correia (2011). He explicitly works with a notion of ground-theoretic equivalence and implicit in his approach, I believe, is a notion of weak ground. He also provides a formal system and a semantics.

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The notion of weak ground is in a number of ways more natural than the notion of strict ground and there is a considerable interest in developing the logic of the notion in its own right. But its interaction with the notion of strict ground is also of considerable interest; and it turns out that the most natural way of developing a logic of strict ground is by combining it with the logic of weak ground. What would otherwise be anomalies in the formulation are thereby removed.

The other more particular aspect of my approach lies in its development of a fact-based semantics for the various logics. We suppose that truths are made true by the facts ('out there' in the world); and we then provide truth-conditions for statements of ground in terms of the facts that make their component statements true. Thus the truths A and B will weakly ground C if any fact that verifies A, when combined with a fact that verifies B, will result in a fact that verifies C.

The development of a semantics for the various systems has some obvious advantages. It enables us to prove soundness and completeness. It provides us with the means of establishing results (concerning nonderivability, for example) that might otherwise be hard to prove. It provides us with a basis for coming up with new principles (and, indeed, it was by considering which principles were valid in the semantics that I was able to detect some deficiencies in previous formulations of the logics). And to the extent that the semantics is natural, it provides us with some reason to think that the concept is itself natural and not just some 'made up' thing.

One curious feature of the semantics is that it brings together two traditions for thinking about ontological explanation that have often thought to be opposed. According to the truth-making tradition, we think in terms of the existence of a fact (or the like) grounding or *making true* the truth of a proposition (or the like). According to the more recent ground-theoretic tradition, we think in terms of one or more truths grounding another, and no attempt is made to link our representation of the world with the world itself. But on the present approach, the truth-making tradition can be seen to provide a semantical underpinning for the statements of ground as conceived under the ground-theoretic tradition. It is by reference to facts and what they make true that we provide a semantical account of what it is for one truth to be grounded by others; and so just as the ontology of worlds can be seen to provide us with an illuminating model of modality, the ontology of facts can be seen to provide us with an illuminating model of ground. Far from being opposed, the two approaches, when properly viewed, are seen to be complementary.

I begin the paper by drawing the relevant distinctions between *full* and *partial* ground on the one side and between *strict* and *weak* ground on the other (§1). I then present a system for the logic of ground in which all four ground-theoretic operators—strict-full, weak-full, strict-partial, and weak-partial—are present (§2). What is perhaps most distinctive of the logic is its absence of a rule of Weakening. The relationship of ground may not be preserved under the addition of grounds, since the new grounds may not be relevant to the truth of what is grounded. I provide a fact-based semantics for the logic of ground and establish that the system is sound for the semantics (§3). It is evident from the semantics that the notions of weak ground are conceptually simpler than those of strict ground (just as the notions of necessity and possibility are conceptually simpler than those of contingency or compatibility). The next two sections (§§4–5) prepare the way for the proof of completeness (§6). Partial or strict statements of ground need to be appropriately 'witnessed'; and showing that this does not lead to trouble requires us to establish a number of proof-theoretic facts about the system and about how, in particular, the derivations of partial statements of ground can be put into a certain canonical form. We conclude by determining various fragments of our system, in which only some of the ground-theoretic operators are present (§6).

I hope in subsequent work to extend the logic and its semantics to the truth-functional connectives and the quantifiers and to show how the resulting framework has application to a wide range of topics. I also hope to provide further clarification and defense of the different notions of ground. But the present investigations, limited and preliminary as they may be, do suggest that there is no essential difficulty in understanding the behavior of these notions from a purely logical or semantical point of view.

§1. Distinctions of ground. I treat ground as a relationship between truths—a number of truths will *ground* another. Intuitively, this relationship will hold when the given truths *account for* the other truth or when the other truth obtains *in virtue of* the given truths. Various explanatory ties of accounting for or of obtaining in virtue of might be distinguished but I suspect that, for present purposes, these distinctions will not matter and that the 'pure' structural logic of ground will be the same regardless of how the explanatory connection is conceived.

There are also different views one might have of the relata. One might take them to be true propositions or to be facts, where these are to be distinguished from true propositions. My own preference is to treat *ground* as a sentential operator like 'or' or 'if - then -'; it combines with different sentences (those indicating the grounds and what is grounded) to produce a sentence (specifying the relationship of ground). But again, for present purposes, nothing much will turn on the issue.

There is a familiar distinction between *full* and *partial* ground. A number of truths will fully ground another when they are sufficient on their own to ground its truth. Thus P and Q will be a full ground for P & Q. One truth partly grounds another when it is of help in fully grounding the other. Thus P will be a partial ground for P & Q, as will Q, since P with Q, or Q with P, is a full ground for P & Q.

We shall need to make another, less familiar, distinction between *strict* and *weak* ground. We might express weak ground by means of the locution 'for - for -'. Thus for John to marry Mary is for John to marry Mary, for John to marry Mary is for Mary to marry John, and for John to marry Mary is for John to marry Mary and (for) Mary to marry John'. In each of these cases, we may say that the truth or truths on the right *weakly* ground the truth on the left. But we cannot say that they ground in our original sense, since John's marrying Mary, or Mary's marrying John, do not *account* for John's marrying Mary.

In general, whenever a number of truths weakly ground a given truth, whatever explanatory role can be played by the given truth can also be played by their grounds. Thus if John's marrying Mary accounts for the existence of the married couple John and Mary, then Mary's marrying John also accounts for the existence of the married couple. Or if John's marrying Mary accounts for John's marrying Mary or Bill's marrying Sue then Mary's marrying John will also account for John's marrying Mary or Bill's marrying Sue.

We might think of the strict grounds as moving us down in the explanatory hierarchy. They always takes us to a lower level of explanation and, for this reason, a truth can never be strict ground for itself. Weak grounds, on the other hand, may also move us sideways in the explanatory hierarchy. They may take us to a truth at the same level as what is grounded and, for this reason, we may always allow a given truth to be a weak ground for itself.[3]

[3] We might compare truths which are weak grounds for one another with 'equations' in systems of term rewriting (Terese, 2003).

As we have seen, weak ground can be explained in terms of strict ground: P, Q, ... will weakly ground R if P, Q, ..., ___ grounds S whenever R, ___ grounds S. But strict ground can also be explained in terms of weak. For P, Q, ... will be a strict ground for T if they are a weak ground for T and if T is not a weak partial ground for any of P, Q, ... In other words, a weak ground is a strict ground in which the grounding relationship cannot be 'reversed'.

Certainly, the notion of strict ground is the more familiar of the two notions. But my suspicion is that it is the weak notion that is most naturally adopted as a primitive. As we shall see, its semantics is simpler and its inferential behavior more straightforward. And given that this is so, we may naturally see the complexities in the semantics and inferential behavior of the strict notion as arising from its definition in terms of the weak notion.

Once we introduce the notion of weak ground, it is possible to make further distinctions in the notion of strict/partial ground. There is first of all the notion that is the natural counterpart of strict full ground and that we naturally think of as *the* notion of partial ground. P will be a partial ground of Q in this sense if P, on its own or with some other truths, is a strict full ground for Q. We call this notion *partial strict* ground and symbolize it by $\prec^*$.

There is then the notion of strict/partial ground that is the natural counterpart to the notion of weak partial ground. P is weak partial ground for Q if P, possibly with some other truths, is a weak full ground for Q; and P will be a corresponding strict/partial ground for Q if P is a weak partial ground for Q but Q is not a weak partial ground for P. We call this notion *strict partial* ground and symbolize it by $\preceq$.

Finally, there is the result of chaining partial strict ground with weak partial ground. P will be a partial/strict ground for Q in this sense if, for some truth R, P is a partial strict ground for R and R is a weak partial ground for Q. We call this notion *part strict* ground and symbolize it by $\prec'$. There is also a corresponding notion that results from chaining weak partial ground with partial strict ground. But it gives nothing new—the result is coincident with partial strict ground.

It is readily shown, under plausible assumptions, that if P is partially strict ground for Q then P is a part strict ground for Q and that if P is a part strict ground for Q then P is a strictly partial ground for Q. However, there is no obvious way to establish the reverse implications and it may plausibly be argued—either by reference to the semantics or by appeal to counterexamples —that they fail to hold.

Although it is the partial strict notion of ground that naturally springs to mind, it turns out to be the strict partial notion that is most suitable for developing a system of rules and a semantics; and it is only by investigating the semantical and logical features of these various notions and the interrelations between them that this became clear. For the most part, therefore, our focus will be on the strict partial notion, although we shall occasionally make reference to the other strict/partial notions of ground.

§2. Logic. We suppose given a nonempty set At of *atomic truths* or *atoms*. For the purposes of the logic, the internal structure of the atoms is irrelevant and we may allow At to be of any nonzero cardinality, finite or infinite.

The language contains four primitive operators:

$\leq$ weak full ground
$\preceq$ weak partial ground
$<$ strict full ground
$\prec$ strict partial ground.

We may set these out in a chart as follows:

	Strict	Weak
Full	$<$	$\leq$
Partial	$\prec$	$\preceq$

There are four corresponding types of sequent:

$\Delta \leq C$ weak full sequent
$A \preceq C$ weak partial sequent
$\Delta < C$ strict full sequent
$A \prec C$ strict partial sequent

Here A and C are atoms from At and Δ is a set of atoms (possibly infinite and possibly empty). For formal purposes, we might identify the sequent $\Delta \leq C$ with the triple $< \Delta, \leq, C>$ (and similarly for the other cases). We adopt standard conventions for the abbreviation of sequents; $\Delta, A, B \leq C$, for example, may be used in place of $\Delta \cup \{A, B\} \leq C$. We also use a nested notation, such as $A \prec B \preceq C$, to indicate the pairing of sequents, $A \prec B$, $B \preceq C$. The atoms to the left of the operator in a sequent are its *antecedents* and the atom to the right of the operator is its *consequent*.

We write $\Delta \leq_1 C$ for $\Delta \leq C$ and $\Delta \leq_0 C$ for $\Delta < C$; and similarly for $A \preceq_1 C$ and $A \preceq_0 C$. We use π as an index for 0 or 1. Thus $\Delta \leq_\pi C$ will be $\Delta \leq C$ or $\Delta < C$, depending upon whether π is 1 or 0. We call $\Delta < C$ and $A \prec C$, respectively, the *strict counterparts of* $\Delta \leq C$ and $A \preceq C$. A sequent of the form $A \leq A$, $A \preceq A$, $A < A$, or $A \prec A$ is said to be an *identity* (which may be *weak* or *strict* and *partial* or *full*).

The system PLG (the *pure logic of ground*) is a system for deriving sequents from sequents and is constituted by the following rules:

Subsumption

$(</\leq)$: $$\frac{\Delta < C}{\Delta \leq C}$$ $(\prec/\preceq)$: $$\frac{A \prec C}{A \preceq C}$$

$(</\prec)$: $$\frac{\Delta, A < C}{A \prec C}$$ $(\leq/\preceq)$: $$\frac{\Delta, A \leq C}{A \preceq C}$$

Cut$(\leq/\leq)$: $$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

Transitivity $(\preceq/\preceq)$: $$\frac{A \preceq B \quad B \preceq C}{A \preceq C}$$ $(\preceq/\prec)$: $$\frac{A \preceq B \quad B \prec C}{A \prec C}$$ $(\prec/\preceq)$: $$\frac{A \prec B \quad B \preceq C}{A \prec C}$$

Identity$(\leq)$: $$\frac{}{A \leq A}$$ Non-Circularity$(\prec)$: $$\frac{A \prec A}{\perp}$$

Reverse Subsumption$(\leq/\prec)$: $$\frac{A_1, A_2, \ldots \leq C \quad A_1 \prec C \quad A_2 \prec C \ldots}{A_1, A_2, \ldots < C}$$

It is these rules, I believe, which constitute the means implicitly at our disposal for reasoning about the concept of ground.

There are two kinds of subsumption rule: the first going from strict ground (either full or partial) to weak ground; and the second going from full ground (either strict or weak) to partial ground. The cut rule allows us to chain weak sequents with a weak sequent to obtain a weak sequent. The transitivity rules for $\preceq/\preceq$, $\preceq/\prec$, and $\prec/\preceq$ allow us to chain two partial sequents to obtain a partial sequent, which is strict as long as the one of the two given partial sequents is strict. According to Identity, the weak identity sequent $A \leq A$ is an axiom, derivable from zero premises. In the formulation of Non-Circularity, $\perp$ is not a separate symbol of the language but is used to indicate that any sequent whatever can be derived from the premise sequent. Reverse Subsumption tells us that we can go from a statement $A_1, A_2, \ldots \leq C$ of weak ground to a statement $A_1, A_2, \ldots < C$ of strict ground as long as all of the antecedents $A_1, A_2, \ldots$ are at a 'lower level' than the consequent C.

Suppose that S is a set of sequents and s is a single sequent. We then say that s is *derivable from* S *within* PLG—or, in symbols, $S \mid\!-_{PLG} s$—if s can be obtained from the members of S by means of the above rules. That is to say, s should belong to any set of sequents that contains S and is closed under the rules (with the conclusion of any rule belonging to the set as long as the premises of the rule belong to the set). We may set out derivations in the form of a well-ordered sequence of atoms $A_1, A_2, \ldots$, with any member of the sequence being an assumption (i.e., a member of S) or derivable by one of the rules from earlier members of the sequence; and we shall often suppose that this has been done. We may derive the following general cut and transitivity rules:

<u>Cut</u>($\leq_\pi/\leq_\pi$):
$$\frac{\Delta_1 \leq_\pi A_1 \quad \Delta_2 \leq_\pi A_2 \ldots \; A_1, A_2, \ldots \leq_\pi C}{\Delta_1, \Delta_2, \ldots \leq_{\pi\pi'} C}$$

<u>Transitivity</u>($\preceq_\pi/\preceq_\pi$):
$$\frac{A \preceq_\pi B \quad B \preceq_{\pi'} C}{A \prec_{\pi\pi'} C}$$

For suppose given the premises $A \prec B$ and $B \prec C$ for an application of the second rule in which $\pi = \pi' = 0$ (the one case not covered by the existing rules). Then $B \preceq C$ can be derived by Subsumption($\prec/\preceq$); and so $A \prec C$ can be derived by Transitivity($\prec/\preceq$). Now suppose given the premises $\Delta_1 \leq_\pi A_1 \Delta_2 \leq_\pi A_2, \ldots$ and $A_1, A_2, \ldots \leq_{\pi'} C$ for an application of the first rule in which at least one of π or π' is 0. Then $\Delta_1 \leq A_1, \Delta_2 \leq A_2, \ldots$ and $A_1, A_2, \ldots \leq C$ can be derived by the appropriate application of Subsumption($</\leq$) and so $\Delta_1, \Delta_2, \ldots \leq C$ can be derived by Cut($\leq/\leq$). We wish to show $\Delta_1, \Delta_2, \ldots < C$. By Subsumption in ($\leq/\preceq$) and ($</\prec$), $B \preceq_\pi A_k$ is derivable for each B in an Δ_k and $A_k \preceq_{\pi'} C$; and so by Transitivity($\preceq_\pi/\preceq_{\pi'}$), $B \prec C$ is derivable for each B in an Δ_k. But then $\Delta_1, \Delta_2, \ldots < C$ by Reverse Subsumption.

It should be noted that Transitivity($\prec/\preceq$) is no longer valid when partial strict ground is substituted for strict partial ground, as in:

($\prec^*/\preceq$)
$$\frac{A \prec^* B \quad B \preceq C}{A \prec^* C}$$

For suppose $A \prec^* B$ holds via $A, \Delta < B$ and $B \preceq C$ holds via $B, \Gamma \leq C$. Then $A, \Delta, \Gamma \leq C$ holds. But there is no assurance that the truths of Γ are at a lower level than C and that $A, \Delta, \Gamma \leq C$ must therefore hold.

The following two rules are also derivable:

Amalgamation($\leq$): $$\frac{\Delta_1 \leq C \quad \Delta_2 \leq C \quad \ldots}{\Delta_1, \Delta_2 \ldots \leq C}$$ Amalgamation($<$): $$\frac{\Delta_1 < C \; \Delta_2 < C \; \ldots}{\Delta_1, \Delta_2 \ldots < C}$$

In other words, the weak or strict grounds for a given truth can be amalgamated, or combined, into a single ground. The derivation of Amalgamation($\leq$) is as follows (using Identity on the first line and Cut on the second line):

$$\dfrac{\Delta_1 \leq C \; \Delta_2 \leq C \; \ldots \qquad \dfrac{}{C, C, \ldots . \leq C}}{\Delta_1, \Delta_2 \ldots \leq C}$$

The derivation of Amalgamation($<$) is somewhat more complicated. Let $A_{k1}, A_{k2}, \ldots$ be the members of Δ_k. We then have the following derivation (using Subsumption($</\leq$) and Identity on the second lines, Subsumption($</\prec$) and Cut on the third lines, and Reverse Subsumption on the last line):

$$\dfrac{\dfrac{\Delta_1 < C}{A_{11} \prec C} \; \dfrac{\Delta_1 < C}{A_{12} \prec C} \; \ldots \; \dfrac{\Delta_2 < C}{A_{21} \prec C} \; \dfrac{\Delta_2 < C}{A_{22} \prec C} \; \ldots \; \ldots \; \dfrac{\dfrac{\Delta_1 < C}{\Delta_1 \leq C} \; \dfrac{\Delta_2 < C}{\Delta_2 \leq C} \; \ldots \; C, C, \ldots . \leq C}{\Delta_1, \Delta_2 \ldots \leq C}}{\Delta_1, \Delta_2, \ldots < C}$$

These consequences may appear surprising but they are forced upon us given certain very plausible assumptions concerning ground.

§3. Semantics and soundness. We can provide a natural semantics for ground in terms of the idea of a fact making a statement true. Under this semantics, each truth A will be associated with a *verification-set* [A], the set of facts that make it true. We should think of the facts of [A] not as *possible* worlds but as *parts* of the actual world; and we should think of 'making true' not as *necessitating* but as *relevantly* verifying. The whole fact, so to speak, should be relevant to the verification of the truth.

Given the facts $f, g, h, \ldots$, we take there to be a *composite* fact or *fusion* $f.g.h.\ldots$ that is the 'factual conjunction' of the component facts $f, g, h, \ldots$, obtaining just in case all of the components facts obtain; and we shall suppose that whenever the facts $f, g, h, \ldots$ verify the truth A, their fusion $f.g.h.\ldots$ also verifies A (this is the semantic counterpart to the Amalgamation rule). On our understanding of verification as relevant verification, it should not be supposed that if f verifies a truth A then any 'larger' fact $f.g$ must also verify A.

Under the standard possible worlds semantics, we have the following account of 'logical' consequence:

C is a consequence of $A_1, A_2, \ldots$ iff whenever a world w verifies each of $A_1, A_2, \ldots$ then w verifies C.

Under the present factual semantics, by contrast, we adopt the following account of 'ground-theoretic' consequence:

C is a consequence of $A_1, A_2, \ldots$ iff whenever f_1 verifies A_1, f_2 verifies $A_2, \ldots$ then $f_1.f_2.f_3\ldots$ verifies C.

The verifiers for the antecedent truths $A_1, A_2, A_3, \ldots$ will cooperate, so to speak, in verifying the consequent C. It is this one change in the classical conception of consequence that will enable us to give a satisfactory semantics for ground-theoretic consequence.

Let us make these ideas precise. A *fact frame* $\mathcal{F}$ is an ordered pair $<F, \prod>$, where F (facts) is a nonempty set and $\prod$ (factual fusion) is a function taking each subset E of F into a member $\prod E$ of F subject to the following associativity condition on an indexed family E_i, $i \in I$, of subsets of F:

$$\prod\{\prod E_i\colon i \in I\} = \prod \bigcup\{E_i\colon i \in I\}.$$

In other words, the fusion of the fusions $e_1 = \prod\{e_{1_1}, e_{1_2}, \ldots\}$, $e_2 = \prod\{e_{2_1}, e_{2_2}, \ldots\}$, $\ldots$ is identical to the fusion of the component facts $e_{1_1}, e_{1_2}, \ldots, e_{2_1}, e_{2_2}, \ldots, \ldots$. We shall also denote the fusion $e = \prod\{e_1, e_2, \ldots\}$ by $e_1.e_2.\ldots$ (and similarly for other cases of this sort). Note that $\prod E$ is defined for the cases in which $E = \varnothing$ and $E = F$, and we denote these respective facts by $n_{\mathcal{F}}$ and $w_{\mathcal{F}}$ (dropping the subscript if it is obvious from the context). $n_{\mathcal{F}}$ is the *null* fact, while w_F is the *full* fact, or the *world.*

Given a fact frame $\mathcal{F} = <F, \prod>$, we say that a subset E of F is (*upward*) *closed* if $\prod D \in E$ for any nonempty subset D of E. We assume (at the level of informal interpretation) that whenever f_i verifies a given truth for each $i \in I$ then the fusion $\prod\{f_i\colon i \in I\}$ also verifies the truth. Thus the set of verifiers for a given truth—its verification-set—will be upward closed. Note the restriction to nonempty subsets in the definition of closure. Thus we cannot in general assume that the null fact $n_{\mathcal{F}}$ will be a member of any closed subset of F (or that it will verify any given truth).

A *generalized fact frame* is an ordered triple $<F, V, \prod>$, where $<F, \prod>$ is a fact frame and V (the verification space) is a nonempty collection of nonempty closed subsets of F. Intuitively, V consists of the subsets of F that are capable of being the verification set for some truth. A generalized fact frame $<F, V, \prod>$ is said to be *full* if V consists of all of the nonempty closed subsets of F.

A *fact model* $\mathcal{M}$ is an ordered triple $<F, \prod, []>$, where $<F, \prod>$ is a fact frame and [] (verification valuation) is a function taking each atom A of the language into a nonempty closed subset [A] of F. Intuitively, [A] is the verification set for A. Since A is a truth, it will be verified by some fact and [A] will consequently be nonempty. Similarly, a *generalized fact model* will be an ordered quadruple $<F, V, \prod, []>$, where $<F, V, \prod>$ is a generalized fact frame and [] is a function taking each atom A into a member of V.

Given a sequence $F_1, F_2, \ldots$ of subsets of F (in a fact model $\mathcal{F} = <F, \prod, []>$ or in V of a generalized fact model $<F, V, \prod, []>$), we let $F_1.F_2.\ldots = \{f_1.f_2.\ldots\colon f_1{\in}F_1, f_2{\in}F_2, \ldots\}$ (when the sequence $F_1, F_2, \ldots$ is empty, we may take $F_1.F_2.\ldots$ to be $n_{\mathcal{F}}$). Thus the facts of $F_1, F_2, \ldots$ are fused componentwise. Likewise, given a sequence $A_1, A_2, \ldots$ of atoms from the language At, we let $[A_1, A_2, \ldots] = [A_1].[A_2].\ldots$. We may show that the fusion of a collection of verification sets is the same regardless of how they might be enumerated:

LEMMA 3.1. (i) If $\{F_1, F_2, \ldots\} = \{G_1, G_2, \ldots\}$ then $F_1.F_2.\ldots = G_1.G_2.\ldots$;
(ii) if $\{A_1, A_2, \ldots\} = \{B_1, B_2, \ldots\}$ then $[A_1, A_2, \ldots] = [B_1, B_2, \ldots]$.

Proof. (ii) follows directly from (i). (i) is obvious when the set $\{F_1, F_2, \ldots\} = \{G_1, G_2, \ldots\}$ is empty and so assume that it is nonempty. It suffices to show that $F_1.F_2.\ldots \subseteq G_1.G_2.\ldots$ (since, by symmetry, it follows that $G_1.G_2.\ldots \subseteq F_1.F_2.\ldots$). To this end, suppose that $f = f_1.f_2.\ldots$, where $f_1 \in F_1$, $f_2 \in F_2$, ... For each j (where j is an index to one of the $G_1, G_2, \ldots$), let $g_j = \prod\{f_i\colon F_i = G_j\}$. Since $\{G_1, G_2, \ldots\} \subseteq \{F_1, F_2, \ldots\}$, the set $\{f_i\colon F_i = G_j\}$ is nonempty and so $g_j \in G_j$ by Upward Closure. Hence $g_1.g_2.\ldots \in G_1.G_2.\ldots$; and given that $\{F_1, F_2, \ldots\} \subseteq \{G_1, G_2, \ldots\}$, it follows by Associativity that $f = g$. □

In the light of this lemma, we may, without ambiguity, define the fusion $\prod\{F_1, F_2, \ldots\}$ of the collection of verification- sets $F_1, F_2, \ldots$ to be $F_1.F_2$. ...; and use of the lemma in this way will often be implicit.

We are now in a position to define various relationships between verification- sets corresponding to the various ground-theoretic operators. Let $\mathscr{M}$ be a generalized model $<F, V, \prod, []>$ and let $F_1, F_2, \ldots, G$, and H be arbitrary members of V. We then say:

(i) $\{F_1, F_2, \ldots\} \leq_{\mathscr{M}} G$ ($F_1, F_2, \ldots$ is a *weak full ground for G in* $\mathscr{M}$) if $F_1.F_2.\ldots \subseteq G$;

(ii) $H \preceq_{\mathscr{M}}$ G (H is a *weak partial ground for G in* $\mathscr{M}$) if, for some $F_1, F_2, \ldots$ from V, $\{H, F_1, F_2, \ldots\} \leq_{\mathscr{M}} H$, (i.e., $H.F_1.F_2.\ldots \subseteq G$);

(iii) $\{F_1, F_2, \ldots\} <_{\mathscr{M}} G$ ($F_1, F_2, \ldots$ is a *strict full ground for G in* $\mathscr{M}$) if $\{F_1, F_2, \ldots\} \leq_{\mathscr{M}} G$ and for no F_i does $G \preceq_{\mathscr{M}} F_i$;

(iv) $H \prec^*_{\mathscr{M}} G$ (H is a *partial strict ground for G in* $\mathscr{M}$) if, for some $F_1, F_2, \ldots$ from V, $\{H, F_1, F_2, \ldots\} <_{\mathscr{M}} G$;

(v) $H \prec'_{\mathscr{M}} G$ (H is a *part strict ground for G in* $\mathscr{M}$) if, for some E from V, H $\prec^*_{\mathscr{M}} E$ and $E \preceq_{\mathscr{M}} G$;

(vi) $H \prec_{\mathscr{M}} G$ (H is a *strict partial ground for G in* $\mathscr{M}$) if $H \preceq_{\mathscr{M}} G$ but not $G \preceq_{\mathscr{M}} H$.

By the previous lemma, nothing turns on the choice of $F_1, F_2, \ldots$ in specifying the set $\{F_1, F_2, \ldots\}$ in clauses (i) and (iii); and often, in specifying a grounding relationship such as $\{F_1, F_2, \ldots\} \leq_{\mathscr{M}} G$, we shall drop the set-braces { and }. Note that the clause for strict ground under (iii) involves a *non-reversibility* requirement. Not only must $F_1, F_2, \ldots$ be a weak ground for G but the weak ground must be nonreversible in the sense that G is not a weak partial ground for any one of $F_1, F_2, \ldots$. Also note that the definitions under clauses (ii)–(iv) have all be given in terms of the relation $\leq_{\mathscr{M}}$ of weak ground that was defined under clause (i)

We may extend these definitions to a simple model $\mathscr{M} = <F, \prod, []>$ by taking the relationship '$F_1, F_2, \ldots \leq_{\mathscr{M}} G$' etc. to hold if the relationship holds in the corresponding full model $<F, V, \prod, []>$. What this means, in effect, is that the existential quantifiers implicit in definitions (ii)–(v) are taken to range over *any* nonempty closed subsets of F. We may also extend these definitions in the obvious way to say when a sequent is true in a given model $\mathscr{M}$. Let [Δ] = {[A]: A ∈Δ}. Then:

(i) $\mathscr{M} \models \Delta \leq$ C if $[\Delta] \leq_{\mathscr{M}}$[C];

(ii) $\mathscr{M} \models$ A $\preceq$ C if [A] $\preceq_{\mathscr{M}}$[C];

(iii) $\mathscr{M} \models \Delta <$ C if $[\Delta] <_{\mathscr{M}}$[C];

(iv) $\mathscr{M} \models$ A $\prec^*$ C if [A] $\prec^*_{\mathscr{M}}$[C];

(v) $\mathscr{M} \models$ A $\prec'$ C if [A] $\prec'_{\mathscr{M}}$[C];

(vi) $\mathscr{M} \models$ A $\prec$ C if [A] $\prec_{\mathscr{M}}$[C].

Spelling out the meaning of clause (i) gives us the following equivalences:

$\mathcal{M} \models \Delta \leq C$ iff $[\Delta] \leq_{\mathcal{M}} [C]$
iff $\prod[\Delta] \subseteq [C]$
iff whenever $f_1 \in [A_1]$, $f_2 \in [A_2], \ldots$ then $f_1.f_2.\ldots \in [C]$ (given that $\Delta = \{A_1, A_2, \ldots\}$).

And similarly for the other cases.

Where S is a set of sequents, $\mathcal{M} \models$ S (S is *true in* $\mathcal{M}$) if each sequent of S is true in $\mathcal{M}$. We say S $\models_{\text{PLG}}$ s (s is a *consequence of* S *within* PLG) if s is true in any generalized model in which S is true.

THEOREM 3.2. (Soundness). S $\vdash_{\text{PLG}}$ s implies S $\models_{\text{PLG}}$ s.

Proof. We go through each of the rules (but altering the order for convenience of proof).

Cut Suppose $\mathcal{M} \models \Delta_1 \leq A_1$, $\mathcal{M} \models \Delta_2 \leq A_2, \ldots$, and $\mathcal{M} \models A_1, A_2, \ldots \leq C$. Then $[\Delta_1] \leq_{\mathcal{M}} [A_1]$, $[\Delta_2] \leq_{\mathcal{M}} [A_2], \ldots$, and $[A_1], [A_2], \ldots \leq_{\mathcal{M}} [C]$. So $\prod[\Delta_1] \subseteq [A_1]$, $\prod[\Delta_2] \subseteq [A_2], \ldots$ and $[A_1].[A_2].\ldots \subseteq [C]$. But then $\prod[\Delta_1].\prod[\Delta_2].\ldots \subseteq [C]$; so by Associativity, $\prod([\Delta_1] \cup [\Delta_2] \cup \ldots) \subseteq [C]$; and so $\Delta_1 \cup \Delta_2 \cup \ldots \leq_{\mathcal{M}} [C]$, as required.

Subsumption

$(</\leq)$. Suppose $\mathcal{M} \models \Delta < C$. Then $[\Delta] <_{\mathcal{M}} [C]$; so $[\Delta] \leq_{\mathcal{M}} [C]$ by definition of $<_{\mathcal{M}}$ and so $\mathcal{M} \models \Delta \leq C$.

$(</\prec)$. Suppose $\mathcal{M} \models \Delta, A < C$. Then $[\Delta], [A] \leq_{\mathcal{M}} [C]$; and so $[A] \preceq_{\mathcal{M}} [C]$. Also, not $[C] \preceq_{\mathcal{M}} [A]$; and so $[A] \prec_{\mathcal{M}} [C]$ by definition of $\prec_{\mathcal{M}}$

$(\prec/\preceq)$. Suppose $\mathcal{M} \models A \prec C$. Then $[A] \prec_{\mathcal{M}} [C]$; and so $[A] \preceq_{\mathcal{M}} [C]$ by definition of $\prec_{\mathcal{M}}$

$(\leq/\preceq)$. Suppose $\mathcal{M} \models \Delta, A \preceq C$. Then $[\Delta],[A] \leq_{\mathcal{M}} [C]$; and so $[A] \preceq_{\mathcal{M}} [C]$ by definition of $\preceq_{\mathcal{M}}$

Transitivity

$(\preceq/\preceq)$. Suppose $\mathcal{M} \models A \preceq B$ and $\mathcal{M} \models B \prec C$. Then for some $F_1, F_2, \ldots$ and for some $G_1, G_2, \ldots$ from V, $[A], F_1, F_2, \ldots \leq_{\mathcal{M}} B$ and $[B], G_1, G_2, \ldots \leq_{\mathcal{M}} C$. So by the validity of Cut, $[A], F_1, F_2, \ldots, G_1, G_2, \ldots \leq_{\mathcal{M}} C$; and so $A \preceq_{\mathcal{M}} C$.

$(\preceq/\prec)$. Suppose $\mathcal{M} \models A \preceq B$ and $\mathcal{M} \models B \prec C$. Then $\mathcal{M} \models B \preceq C$ and so $\mathcal{M} \models A \preceq C$ by the previous case. Suppose, for reductio, that $\mathcal{M} \models C \preceq A$. Given $\mathcal{M} \models A \preceq B$, it follows by the previous case that $\mathcal{M} \models C \preceq B$, contrary to the fact that $\mathcal{M} \models B \prec C$.

$(\prec/\preceq)$. Similar to the case $(\preceq/\prec)$.

Identity $[A] \subseteq [A]$; and so $[A] \leq_{\mathcal{M}} [A]$.

Non-Circularity Not $\mathcal{M} \models A \prec A$, since otherwise not $[A] \leq_{\mathcal{M}} [A]$, contrary to Identity.

Reverse Subsumption Suppose $\mathcal{M} \models A_1, A_2, \ldots \leq C$ and $\mathcal{M} \models A_1 \prec C$, $\mathcal{M} \models A_2 \prec C, \ldots$. Then $A_1, A_2, \ldots \leq_{\mathcal{M}} C$ and not $[C] <_{\mathcal{M}} [A_k]$ for k = 1, 2, …; and so $A_1, A_2, \ldots <_{\mathcal{M}} C$. □

§4. The partial system. For the purposes of the completeness result, it will be helpful to set up a system for the partial operators $\preceq$ and $\prec$. We might label it PLPG (the pure logic of partial ground). It is constituted by the following rules: Subsumption $(\prec/\preceq)$, Transitivity $((\preceq/\preceq), (\preceq/\prec), (\prec/\preceq))$, Identity$(\preceq)$ and Non-Circularity $(\prec)$. The only new rule is Identity$(\preceq)$, which is the partial counterpart:

$$\frac{}{A \preceq A}$$

of Identity($\leq$). Clearly, any partial sequent derivable from a set S of partial sequents within PLPG will be derivable from S within PLG.

Derivations in PLPG have a simple structure, which we now describe. Let us note, in the first place, that Identity($\preceq$) can be confined to a single initial application and is otherwise redundant:

LEMMA 4.1. Suppose that there is a derivation D of s from S within PLPG and that s is not a weak identity (of the form $A \preceq A$). Then s can be derived from S within PLPG without the use of Identity($\preceq$) and using only the application of rules made within D.

Proof. By induction on the length of the derivation. If s appears on the first line of a derivation and is not a weak identity, then obviously it cannot have been obtained with the help of Identity($\preceq$). Suppose now that s appears at a later line of a derivation and is not a weak identity. Then it must have been obtained by means of an application of Subsumption or Transitivity or Non-Circularity. In the case of Subsumption and Non-Circularity, the premises cannot be weak identities. So by IH, there will be a derivation of the premises of the required sort and hence a derivation of s of the required sort. In the case of Transitivity, there will be four ways in which one or more of the premises might take the form of a weak identity, which are displayed below:

$$\frac{A \preceq A \preceq B}{A \preceq B} \qquad \frac{A \preceq B \preceq B}{A \preceq B} \qquad \frac{A \preceq A \prec B}{A \prec B} \qquad \frac{A \prec B \preceq B}{A \prec B}$$

But in each case the conclusion is one of the premises and so, by IH, there will again be a derivation of the conclusion of the required sort. □

A related result can be established for the system PLG as long as Cut is so formulated as to allow some of the antecedents A_i to be left alone and not ‘replaced’ by a corresponding Δ_i.

Say that a set S of partial sequents is *inconsistent* if a strict identity $A \prec A$ can be derived from S and that otherwise S is *consistent*. Inconsistency can be established without the use of Non-Circularity or Identity:

LEMMA 4.2. Suppose S is inconsistent in PLPG. Then a strict identity $A \prec A$ can be derived from S in PLPG without the use of Non-Circularity or Identity.

Proof. Suppose S is inconsistent. Then there is a derivation of a strict identity s from S. If the derivation does not make use of Non-Circularity then, by the previous lemma, there is a derivation of s that does not make use of Non-Circularity or Identity; and we are done. So suppose that the derivation does make use of Non-Circularity and take the first application of Non-Circularity to appear in the derivation. This will take the form of an inference of a sequent s from $C \prec C$. But the subderivation of $C \prec C$ will then be a derivation of a strict identity that makes no use of Non-Circularity; and so again by the previous lemma, there will be a derivation of $C \prec C$ that makes no use of Non-Circularity or Identity. □

We might note that an analogous result can be established for PLG (with the modified form of Cut).

Finally, we show that all of the remaining derivations can be put in a certain canonical form. Call a sequence of partial sequents $s_1, s_2, \ldots, s_{n-1}, s_n$, $n > 1$, a *chain* if it is of the form $A_1 \preceq_\pm A_2, A_2 \preceq_\pm A_3, \ldots, A_{n-1} \preceq_\pm A_n$, $n > 1$, where each of the sequents $s_1, s_2, \ldots, s_{n-1}, s_n$ is either weak or strict. We may write a chain more simply in the form $A_1 \preceq_\pm A_2 \preceq_\pm A_3 \preceq_\pm \ldots \preceq_\pm A_{n-2} \preceq_\pm A_{n-1} \preceq_\pm A_n$. We say that the chain $A_1 \preceq_\pm A_2 \preceq_\pm A_3 \preceq_\pm \ldots \preceq_\pm A_{n-2} \preceq_\pm A_{n-1} \preceq_\pm A_n$ is *strict* if one of the component sequents $A_k \preceq_\pm A_{k+1}$ is strict and that otherwise it is *weak*; and we say that the chain *establishes* $A \preceq_\pm C$ if $A_1 = A$, $A_n = C$ and the sequent $A \preceq_\pm C$ is weak or strict according as to whether the chain is weak or strict. And given a set of sequents S, we say that the chain $s_1, s_2, \ldots, s_n$ is an S-*chain*, or a chain *from* S if, for each $k = 1, 2, \ldots, n$, either s_k or its strict counterpart is a member of S.

We have the following straightforward result:

LEMMA 4.3. (PLPG). Suppose some S-chain establishes $A \preceq_\pm C$. There is then a derivation of $A \preceq_\pm C$ from S which makes no use of Identity or Non-Circularity.

Proof. Suppose the S-chain establishing $A \preceq_\pm C$ is of the form $A = A_1 \preceq_\pm A_2, A_2 \preceq_\pm A_3, \ldots, A_{n-1} \preceq_\pm A_n = C$. Then each of $A_1 \preceq_\pm A_2, A_2 \preceq_\pm A_3, \ldots, A_{n-1} \preceq_\pm A_n$ can be derived from S either directly as an assumption or by means of Subsumption($\prec/\preceq$). $A_1 \preceq_\pm A_n$ can then be obtained by repeated applications of the transitivity rules. □

We also have the converse result:

LEMMA 4.4. (PLPG). Suppose that $A \preceq_\pm C$ can be derived from S without the use of Identity($\preceq$) or Non-Circularity. Then some S-chain establishes $A \preceq_\pm C$.

Proof. By induction on the length of the derivation of $A \preceq_\pm C$ from S. If $A \preceq_\pm C$ is an assumption (i.e., a member of S), then the result is immediate.

Suppose now that $A \preceq_\pm C$ is obtained by Subsumption. Then $A \preceq_\pm C$ is of the form $A \preceq C$ and is obtained from $A \prec C$. By IH, there is a strict S-chain $A = A_1 \preceq_\pm A_2 \preceq_\pm A_3 \preceq_\pm \ldots \preceq_\pm A_{n-2} \preceq_\pm A_{n-1} \prec_\pm A_n = C$. But $A = A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{n-2} \preceq A_{n-1} \preceq A_n = C$ is then an S-chain that establishes $A \preceq C$.

Suppose next that $A \preceq_\pm C$ is obtained by Transitivity ($\preceq/\preceq$). Thus $A \preceq_\pm C$ is of the form $A \preceq C$ and is obtained from premises of the form $A \preceq B$ and $B \preceq C$. By IH, there are S-chains $A = A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{m-1} \preceq A_m = B$ and $B = B_1 \preceq B_2 \preceq B_3 \preceq \ldots \preceq B_{n-1} \preceq B_n = C$. But $A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{m-1} \preceq A_m = B_1 \preceq B_2 \preceq B_3 \preceq \ldots \preceq B_{n-1} \preceq B_n = C$ is then an S-chain that establishes $A \preceq C$.

Now suppose that $A \preceq_\pm C$ is obtained by Transitivity ($\preceq/\prec$). Then $A \preceq_\pm C$ is of the form $A \prec C$ and is obtained from premises of the form $A \preceq B$ and $B \prec C$. By IH, there is a weak S-chain $A = A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{m-1} \preceq A_m = B$ and a strict S-chain $B = B_1 \preceq_\pm B_2 \preceq_\pm B_3 \preceq_\pm \ldots \preceq_\pm B_{n-1} \preceq_\pm B_n = C$. But $A = A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{m-1} \preceq A_m = B_1 \preceq_\pm B_2 \preceq_\pm B_3 \preceq_\pm \ldots \preceq_\pm B_{n-1} \preceq_\pm B_n = C$ is then a strict S-chain that establishes $A \prec C$.

The proof for the case in which $A \preceq_\pm C$ is obtained by Transitivity ($\prec/\preceq$) is similar. □

A chain $A_1 \preceq_\pm A_2, A_2 \preceq_\pm A_3, \ldots, A_{n-1} \preceq_\pm A_n$ is said to be *tight* if $A_1, A_2, A_3, \ldots, A_{n-1}, A_n$ are pairwise distinct with the possible exception of A_1 and A_n. We can restrict the S-chains from the previous lemma to those that are tight:

LEMMA 4.5. Suppose that some S-chain establishes $A \preceq_\pm C$. Then either some tight S-chain establishes a strict partial identity (of the form $B \prec B$) or some tight S-chain establishes $A \preceq_\pm C$.

Proof. Let $A = A_1 \preceq_\pm A_2 \preceq_\pm A_3 \preceq_\pm \ldots \preceq_\pm A_{n-2} \preceq_\pm A_{n-1} \preceq_\pm A_n = C$ be an S-chain that establishes $A \preceq_\pm C$. We establish the result by induction on the number p of pairs of numbers j and k for which $j < k$, $A_j = A_k$ and it is not the case that $j = 1$ and $k = n$. If $p = 0$, we are done. So suppose $p > 0$ and let j and k be a pair of numbers for which $j < k$, $A_j = A_k$ and it is not the case that $j = 1$ and $k = n$. We distinguish two subcases:

(a) The chain $A_j \preceq_\pm A_2 \preceq_\pm \ldots \preceq_\pm A_{k-1} \preceq_\pm A_k$ is weak. Then $A = A_1 \preceq_\pm A_2 \preceq_\pm \ldots \preceq_\pm A_j = A_k \preceq_\pm A_{k+1} \preceq_\pm \ldots \preceq_\pm A_{n-1} \prec_\pm A_n = C$ is an S-chain establishing $A \preceq_\pm C$ for which p has a lower value and so, by IH, there is a tight S-chain that establishes $A \preceq_\pm C$.

(b) The chain $A_j \preceq_\pm A_2 \preceq_\pm \ldots \preceq_\pm A_{k-1} \prec_\pm A_k = A_j$ is strict. Then $A_j \preceq_\pm A_2 \preceq_\pm \ldots \preceq_\pm A_{k-1} \prec_\pm A_k = A_j$ is an S-chain establishing $A_j \prec A_j$ for which p has a lower value and so, by IH, there is a tight S-chain that establishes $A_j \prec A_j$. □

We now obtain:

LEMMA 4.6. (PLPG).

(i) Suppose that $A \preceq_\pm C$ can be derived from S without the use of Identity or Non-Circularity. Then either some tight S-chain establishes a strict partial identity or some tight S-chain establishes $A \preceq_\pm C$.

(ii) If S is inconsistent then some tight S-chain establishes a strict identity.

Proof.

(i) From the previous two lemmas.

(ii) Suppose S is inconsistent. By Lemma 4.2, a strict identity $A \prec A$ can be derived from S within PLPG without the use of Non-Circularity or Identity. So by part (i), either some tight S-chain establishes a strict partial identity or some tight S-chain establishes $A \prec A$. In either case, some tight S-chain establishes a strict identity.

□

We now consider the connection between the system PLPG for partial sequents and the full system PLG. Given a set S of sequents (from PLG), let S_p consist of the corresponding partial sequents. Thus $S_p = \{A \preceq C\colon A \preceq C \in S \text{ or } \Delta \leq C \in S \text{ and } A \in \Delta\} \cup \{A \prec C\colon A \prec C \in S \text{ or } \Delta < C \in S \text{ and } A \in \Delta\}$. Since each member of S_p is derivable from S within PLG and the rules of PLPG are derived rules of PLG, we have:

LEMMA 4.7. Any partial sequent derivable from S_p within PLPG is derivable from S within PLG.

We also have the converse result:

LEMMA 4.8. If a partial sequent is derivable from S within PLG, then it is derivable from S_p within PLPG.

Proof. By induction of the length of the derivation. For the purposes of the induction, we establish the following stronger result:

If s is derivable from S within PLG, then either (i) s is of the form $A \preceq_\pm C$ and is also derivable from S_p within PLPG or (ii) s is of the form $\Delta \leq_\pm C$ and $A \leq_\pm C$ is derivable from S_p within PLPG for any $A \in \Delta$.

Suppose first that s is not inferred from any premises. Thus s is either an instance $A \leq A$ of Identity($\leq$) or is a member of S. In the first case, $A \preceq A$ will be derivable from S_p by Identity($\preceq$); and in the second case, s will either be of the form $A \preceq_\pm C$ and hence also a member of S_p or it will be of the form $\Delta \leq_{\pm} C$ and so $A \preceq_{\pm} C$ will be a member of S_p for any $A \in \Delta$ by the definition of S_p.

Now suppose s is inferred from some premises. We then have the following cases:

(a) s is derived from S by an application $\Delta < C$ / $\Delta \leq C$ of Subsumption($</\leq$). Take any $A \in \Delta$. By IH, $A \prec C$ is derivable from S_p within PLPG. So by Subsumption($\prec/\preceq$), $A \preceq C$ is derivable from S_p within PLPG.

(b) s is derived from S by an application $\Delta < C$ / $A \prec C$ of Subsumption($</\prec$) with $A \in \Delta$. By IH, $A \prec C$ is derivable from S_p within PLPG.

(c) s is derived from S by an application $A \prec C$ / $A \preceq C$ of Subsumption($\prec/\preceq$). By IH, $A \prec C$ is derivable from S_p within PLPG and so by, Subsumption($\prec$, $\preceq$), $A \preceq C$ is derivable from S_p within PLPG.

(d) s is derived from S by an application $\Delta \leq C$ / $A \preceq C$ of Subsumption($\leq$, $\preceq$) with $A \in \Delta$. By IH, $A \preceq C$ is derivable from S_p within PLPG.

(e) s is derived from S by an application:

$$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

of Cut within PLG. Take an $A \in \Delta_i$ for some i = 1, 2, ... Then $A \preceq A_i$ is derivable from S_p within PLPG by IH applied to the premise $\Delta_i \preceq A_i$. But $A_i \preceq C$ is derivable from S_p within PLPG by IH applied to the premise $A_1, A_2, \ldots, \Gamma \leq C$; and so by Transitivity($\preceq$, $\preceq$), $A \preceq C$ is derivable from S_p within PLPG.

(f) s is derived from S within PLG by Transitivity in ($\preceq/\preceq$), ($\preceq/\prec$) and ($\prec/\preceq$). Straightforward since these rules are also available in PLPG.

(g) $s = A \leq A$ is derivable from S within PLG by an application of Identity($\leq$). Then $A \preceq A$ is derivable from S_p within PLPG by an application of Identity($\preceq$).

(h) s is derivable from S within PLG by an application $A \prec A$ / s of Non-Circularity. By IH, $A \prec A$ is derivable from S_p within PLPG and so s is derivable from S_p within PLPG by an application of Non-Circularity.

(I) $s = A_1, A_2, \ldots < C$ is derivable from S within PLG by an application:

$$\frac{A_1, A_2, \ldots \leq C \quad A_1 \prec C \quad A_2 \prec C, \ldots}{A_1, A_2, \ldots < C}$$

of Reverse Subsumption. Then for i = 1, 2, ..., $A_i \prec C$ is derivable from S_p within PLPG by IH. □

§5. Conservativity. For each weak partial sequent $A \preceq C$ from the original language At, choose a witnessing constant $\underline{f}$, where the constants are different from one another for different sequents $A \preceq C$ and different from the atoms of At; and, similarly, for each strict partial sequent $A \prec C$ from At, choose a witnessing constant $\underline{g}$, where the constants are different from one another for different sequents $A \prec C$, different from the witnessing constants $\underline{f}$ for the weak partial sequents $A \preceq C$, and different from the atoms of At. It should be evident that the system PLG will be conservative with respect to the extended language; if a sequent from the original language is derivable from a premise set in the original language within the system with added constants then it will be derivable from the premise within the system without added constants, since all of the added constants using in the derivation can be replaced by an atom from the original language, thereby giving us a derivation within the original language.

Given a premise set S from the original language, we let its *witnessed* extension S^+ be the result of adding $A, \underline{f} \leq C$ to S whenever S contains $A \preceq C$ and $\underline{f}$ is the witnessing constant for $A \preceq C$ and of adding $A, \underline{g} \leq C$ to S whenever S contains $A \prec C$ and $\underline{g}$ is the witnessing constant for $A \prec C$. We may indicate the required extension by means of the following 'reverse rules':

$$\frac{A \preceq C}{A, \underline{f} \leq C} \qquad\qquad \frac{A \prec C}{A, \underline{g} \leq C}$$

The partial grounds in S get witnessed, so to speak, as full grounds.

We wish to show that extending S to S^+ does not enable us to derive any new sequents from the original language. We first establish some partial results.

LEMMA 5.1. If S^+ is inconsistent within PLG then so is S.

Proof. Suppose S^+ is inconsistent within PLG. Then a strict identity $B \prec B$ is derivable from S^+ within PLG. By Lemma 4.8, $B \prec B$ is derivable from $(S^+)_p$ within PLPG; and so $(S^+)_p$ is also inconsistent within PLPG. By Lemma 4.2, a strict identity $B \prec B$ is derivable from $(S^+)_p$ within PLPG without the use of Non-Circularity or Identity; and so by Lemma 4.4, there is an $(S^+)_p$-chain $C = A_1 \preceq_\pm A_2 \preceq_\pm A_3 \preceq_\pm \ldots \preceq_\pm A_{n-2} \preceq_\pm A_{n-1} \preceq_\pm A_n = C$ that establishes $C \prec C$. The atoms $A_1, A_2, \ldots, A_{n-1}, A_n$ of the chain occur in pairs on the left and the right of a sequent—either at the beginning and the end, as with $C = A_1 \preceq_\pm A_2$ and $A_{n-1} \preceq_\pm A_n = C$, or in the middle, as with $A_i \preceq_\pm A_{i+1}$ and $A_{i+1} \preceq_\pm A_{i+2}$. Now witnessing constants cannot occur on the right in $(S^+)_p$ and so the chain cannot contain any witnessing constants. Thus each sequent in the chain, or its strict counterpart, comes from S_p. By Lemma 4.3, $C \prec C$ is derivable from S_p within PLPG. But then by Lemma 4.7, $C \prec C$ is derivable from S within PLG and S is inconsistent within PLG. □

We establish by similar methods:

LEMMA 5.2. Suppose that S^+ is consistent and that s is a partial sequent without witnessing constants derivable from S^+ within PLG. Then s is derivable from S within PLG.

Proof. Given that s is derivable from S^+ in PLG, it is derivable from $(S^+)_p$ within PLPG by Lemma 4.8. If $(S^+)_p$ is inconsistent in PLPG, then S^+ is inconsistent in PLG by the previous lemma. So we may assume that s is not an identity and that it is derivable from $(S^+)_p$ without the use of Identity or Non-Circularity. So by Lemma 4.4, there is an $(S^+)_p$-chain $A = A_1 \preceq A_2 \preceq A_3 \preceq \ldots \preceq A_{n-2} \preceq A_{n-1} \preceq_\pm A_n = C$ establishing $s = A \preceq_\pm C$. Since s contains no witnessing constants, neither A nor C is a witnessing constant and so we may argue in a similar way to before that the chain contains no witnessing constants and that the sequent s is derivable from S within PLG. □

We establish by a straightforward induction:

LEMMA 5.3. Suppose that S^+ is consistent and that s is derivable from S^+ and is not an identity ($C \preceq C$ or $C \leq C$). Then a witnessing constant occurs in s only if it occurs on the left.

Proof. A left constant cannot occur on the right of any assumption in S^+. But if a left constant occurs on the right in the conclusion of any inference (but Identity and Non-Circularity) then it must also occur on the right in a premise of the inference. □

We can now establish our main conservativity result:

THEOREM 5.4. (PLG). Suppose the sequent s and the set of sequents S are without witnessing constants. Then s is derivable from S^+ only if it is derivable from S.

Proof. The result is clearly correct when S^+ is inconsistent since, by Lemma 5.1, S is then inconsistent. So we may assume that S^+ is consistent. The result also holds when s is a partial sequent (of the form $A \preceq B$ or $A \prec C$) by Lemma 5.2. The result is also clearly correct when s is an axiom (obtained by Identity) or is an assumption (belonging to S^+). The remaining cases are as follows:

(a) s is obtained via an application $\Delta < C$ / $\Delta \leq C$ of Subsumption($</\leq$). But if the conclusion $\Delta \leq C$ is without witnessing constants, then so is the premise $\Delta < C$. By IH, $\Delta < C$ is derivable from S; and so by Subsumption($</\leq$), $\Delta \leq C$ is derivable from S.

(b) s is obtained via an application:

$$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1,\ \Delta_2, \ldots \leq C}$$

of Cut. Suppose the conclusion is without witnessing constants. If A_k were a witnessing constant for some $k = 1, 2, \ldots$, then either $\Delta_k \leq A_k$ would be the identity $A_k \leq A_k$ and so a witnessing constant would occur in the conclusion after all, or the premise $\Delta_k \leq A_k$ would be a nonidentity containing a witnessing constant A_k on the right, which is impossible by Lemma 5.3.

(c) s is obtained via an application:

$$\frac{A_1, A_2, \ldots \leq C \quad A_1 \prec C \quad A_2 \prec C, \ldots}{A_1, A_2, \ldots < C}$$

of Reverse Subsumption. Since the conclusion is without witnessing constants, so are the premises; and we are done. □

There are two other conservativity-type results we will need in order to establish completeness:

LEMMA 5.5. (PLG).

(i) Suppose $A \prec C$ is not derivable from S. Then $S \cup \{C \preceq A\}$ is consistent.

(ii) Suppose $\Delta \leq C$ is derivable from S but $\Delta < C$ is not. Then for some A in Δ, $S \cup \{C \preceq A\}$ is consistent.

Proof.

(i) Suppose, for reductio, that $A \prec C$ is not derivable from S and that $S \cup \{C \preceq A\}$ is inconsistent. Then $C \prec C$ is derivable from $S \cup \{C \preceq A\}$ within PLG; and so, by Lemma 4.8, $C \prec C$ is derivable from $S_p \cup \{C \preceq A\}$ within PLPG and hence $S_p \cup \{C \preceq A\}$ is inconsistent within PLPG . By Lemma 4.6(ii), there is a strict and tight chain

$B = D_1 \preceq_\pm D_2 \preceq_\pm D_3 \preceq_\pm \ldots \preceq_\pm D_{n-2} \preceq_\pm D_{n-1} \preceq_\pm D_n = B$ for $S_p \cup \{C \preceq A\}$ that establishes a strict identity $B \prec B$. Now S is consistent (since $A \prec C$ is not derivable from S). So one (and exactly one) of the members of the chain $D_1 \preceq_\pm D_2 \preceq_\pm D_3 \preceq_\pm \ldots \preceq_\pm D_{n-2} \preceq_\pm D_{n-1} \preceq_\pm D_n$, say $D_j \preceq_\pm D_{j+1}$ for $j < n$, must be the sequent $C \preceq A$. But then $A = D_{j+1} \preceq_\pm D_{j+2} \preceq_\pm \ldots \preceq_\pm D_n = D_1 \preceq_\pm D_1 \preceq_\pm \ldots \preceq_\pm D_{j-1} \preceq_\pm D_j = C$ is a strict chain for S that establishes $A \prec C$ and $A \prec C$ is therefore derivable from S, contrary to supposition.

(ii) Let $\Delta = \{A_1, A_2, \ldots\}$. Then $A_k \prec C$ will not be derivable from S for some $k = 1, 2, \ldots$, since otherwise we may obtain $\Delta < C$ from $\Delta \leq C$ and $A_1 \prec C$, $A_2 \prec C, \ldots$ by Reverse Subsumption. So by part (i) of the lemma, $S \cup \{C \preceq A_k\}$ is consistent for some A_k in Δ.

□

§6. Completeness. Let S be a set of sequents (in a given set of atoms). We define the *canonical* (*generalized*) *model* $\mathscr{M}_S = < F_S, V_S, \prod_S, []_S >$ *for* S by:

(i) F_S = the collection of all subsets of atoms;

(ii) $V_S = \{E \subseteq F_S$: $E = \{\Delta$: $\Delta \leq C$ is derivable from S$\}$ for some atom C$\}$;

(iii) $\prod_S E = \bigcup E$ for each subset E of F_S;

(iv) $[C]_S = \{\Delta$: $\Delta \leq C$ is derivable from S$\}$.

It should be clear that the resulting structure is indeed a generalized model. For: each member of V_S is nonempty, given that $C \leq C$ is derivable from S; each member of V_S is upward closed, given that $\Delta_1 \leq C$, $\Delta_2 \leq C, \ldots$ derivable from S implies $\Delta_1 \cup \Delta_2 \cup \ldots \leq C$ derivable from S by Amalgamation; and from the definitions of $[C]_S$ and V_S, it follows that $[C]_S$ is always a member of V_S.

It is readily shown that truth in the canonical model coincides with derivability in the case of weak full sequents:

LEMMA 6.1. The sequent $\Delta \leq C$ is true in the canonical model $\mathscr{M}_S$ for S iff it is derivable from S.

Proof. Take Δ to be $\{A_1, A_2, \ldots\}$. First suppose $\Delta \leq C$ is derivable from S. Pick $\Delta_1 \in [A_1]_S$, $\Delta_2 \in [A_2]_S, \ldots$. Then $\Delta_1 \leq A_1$, $\Delta_2 \leq A_2, \ldots$ are derivable from S by the definition of $[A_k]_S$. We wish to show that $\Delta_1 ._S \Delta_2 ._S \ldots = \Delta_1 \cup \Delta_2 \cup \ldots \in [C]_S$, that is, that $\Delta_1 \cup \Delta_2 \cup \ldots \leq C$ is derivable from S. But this follows from the derivable sequents $\Delta_1 \leq A_1$, $\Delta_2 \leq A_2, \ldots$ and $\Delta = \{A_1, A_2, \ldots\} \leq C$ by Cut.

Now suppose that $\Delta \leq C$ is not derivable from S. $\{A_1\} \leq A_1$, $\{A_2\} \leq A_2, \ldots$ are derivable from S by Identity and so $\{A_1\} \in [A_1]_S$, $\{A_2\} \in [A_2]_S, \ldots$. But $A_1, A_2 \ldots \leq C$ is not derivable from S and so $\{A_1, A_2 \ldots\} = \{A_1\}._S\{A_2\}._S \ldots$ $[C]_S$ and $A_1, A_2 \ldots \leq C$ is not true in $\mathscr{M}_S$. □

Unfortunately, this result does not hold for all sequents. But we may show that derivable sequents will be true in the canonical model as long as partial sequents are appropriately witnessed:

LEMMA 6.2. Take any consistent set S of sequents. Then any sequent s of S is true in the canonical model $\mathscr{M}_{S^+}$ for the witnessed extension S^+ of S.

Proof. Given that S is consistent, S^+ will be consistent by Lemma 5.1. We distinguish four cases:

(a) s is of the form $\Delta \leq C$. From the previous result, since $\Delta \leq C$ is derivable from S^+.

(b) s is of the form $\Delta < C$. $\Delta \leq C$ is derivable from S by Subsumption($</\leq$) and so, by (a) above, $\Delta \leq C$ is true in the canonical model $\mathscr{M}_{S+}$. We wish to show that for no $B \in \Delta$ is $C \preceq B$ true in the canonical model. Suppose otherwise. Then for some $[D_1]_{S+}, [D_2]_{S+}, \ldots$ from V_{S+}, $[C]_{S+}, [D_1]_{S+}, [D_2]_{S+}, \ldots \leq_{\mathscr{M}_{S+}} [B]_{S+}$. Since$\{C\} \in [C]_{S+}$, $\{D_1\} \in [D_1]_{S+}$, $\{D_2\} \in [D_2]_{S+}, \ldots$, $\{C\} \cup \{D_1\} \cup \{D_2\} \cup \ldots = \{C, D_1, D_2, \ldots\} \in [B]_{S+}$; and so $C, D_1, D_2, \ldots \leq B$ is derivable from S^+ and consequently, by Subsumption, $C \preceq B$ is derivable from S^+. Since $\Delta < C \in S$ and $B \in \Delta$, $B \prec C$ is derivable from S^+. But then by Transitivity($\preceq/\prec$), $C \prec C$ is derivable from S^+, contrary to the consistency of S^+.

(c) s is of the form $A \prec C$. Given that $A \preceq C \in S$, $A, \underline{f} \leq C \in S^+$ for some witnessing constant $\underline{f}$. So by the part (a), $A, \underline{f} \leq C$ is true in $\mathscr{M}_{S+}$ and hence $A \leq C$ is also true in $\mathscr{M}_{S+}$.

(d) s is of the form $A \prec C$. Given that $A \prec C \in S$, the sequent $A, \underline{g} \leq C \in S^+$ for some witnessing constant $\underline{g}$. So by part (a), $A, \underline{g} \leq C$ is true in $\mathscr{M}_{S+}$ and hence $A \preceq C$ is true in $\mathscr{M}_{S+}$. Suppose, for reductio, that $C \preceq A$ is true in $\mathscr{M}_{S+}$. By the reasoning under (c), $C \preceq A$ is derivable from S^+. Since $A \prec C \in S$, it follows by Transitivity($\preceq/\prec$) that $C \prec C$ is derivable from S^+, contrary to the consistency of S^+. □

We turn to the converse result

LEMMA 6.3. Suppose that the sequent s is not derivable from S within PLG. Then s is not true in the canonical model $\mathscr{M}_{S'+}$ for some witnessed extension S'^+ of some consistent extension S' of S.

Proof. There are four cases:

(a) s is of the form $\Delta \leq C$. Take S' to be S. By Theorem 5.4, $\Delta \leq C$ is not derivable in $S'^+ = S^+$; and so s will fail to be true in $\mathscr{M}_{S+}$ by Lemma 6.2.

(b) s is of the form $A \preceq C$. Again, take S' to be S and suppose, for reductio, that s is true in $\mathscr{M}_{S+}$. Then for some $[B_1]_{S+}, [B_2]_{S+}, \ldots$ from V_{S+}, $[A]_{S+}, [B_1]_{S+}, [B_2]_{S+}, \ldots \leq_{\mathscr{M}_{S+}} [C]_{S+}$. So $A, B_1, B_2, \ldots \leq C$ is true in $\mathscr{M}_{S+}$ and so, by Lemma 6.1, $A, B_1, B_2, \ldots \leq C$ is derivable from S^+. But then $A \preceq C$ is derivable from S^+ by Subsumption($\leq/\preceq$) and hence derivable from S by Theorem 5.4.

(c) s is of the form $A \prec C$. By Lemma 5.5(i), $S \cup \{C \preceq A\}$ is consistent. Let $S' = S \cup \{C \preceq A\}$. Then $C \preceq A$ is true in the canonical model $\mathscr{M}_{S+}$ for S^+ by the previous lemma. But this means that $A \prec C$ is not true in $\mathscr{M}_{S+}$.

(d) s is of the form $\Delta < C$. We consider two subcases:
(d1) $\Delta \leq C$ is not derivable from S. Let $S' = S$. By part (a), $\Delta \leq C$ is not true in the canonical model $\mathscr{M}_{S+}$ for $S'^+ = S^+$. But then $\Delta < C$ is not true in $\mathscr{M}_{S+}$.
(d2) $\Delta \leq C$ is derivable from S. By Lemma 5.5(ii), $S \cup \{C \preceq A\}$ is consistent for some A in Δ. Let $S' = S \cup \{C \preceq A\}$. By the previous lemma, $C \preceq A$ is true in the canonical model $\mathscr{M}_{S'+}$ for S'^+. But this then means that $\Delta < C$ is not true in $\mathscr{M}_{S'+}$. □

THEOREM 6.4. (Completeness for PLG). If s is a consequence of S it is derivable from S.

Proof. Suppose s is not derivable from S. By the previous lemma, s is not true in the canonical model $\mathscr{M}_{S'+}$ for some witnessed extension S'^+ of some consistent extension S' of S. By Lemma 6.2, each member of S (and hence of S) is true in $\mathscr{M}_{S'+}$; and so s is not a consequence of S. □

This result is a loose analogue of the completeness result from Scott (1971) for the classical Gentzen rules: just at those rules correspond to the standard possible worlds account of consequence, our rules correspond to the fact-based account of ground.

§7. Consequences. We note a number of variants and consequences of our completeness result and its proof.

(1) Suppose that in the semantics we interpret $\prec$ as part strict ground. Thus A $\prec$ C will be true in a model $\mathcal{M}$ just in case, for some E from V, [A] $\prec^*_{\mathcal{M}}$ E *and* E $\preceq_{\mathcal{M}}$ [C]. Then it may be shown that the system PLG is still sound under the resulting reinterpretation. Somewhat surprisingly, the system will still be complete under the resulting reinterpretation. Thus strengthening the content of A $\prec$ C in this way makes no difference to the logic.
Completeness for the new semantics may be established by appropriately altering the definition of the witnessed extension $S^+$of S. Whenever S contains a sequent of the form A $\prec$ C, we must add sequents of the form A, $\underline{g} < \underline{h}$ and $\underline{h}$, $\underline{k} \leq$ C to S^+ for new witnessing constants $\underline{g}$, $\underline{h}$, and $\underline{k}$. With somewhat more effort, the required conservativity results may then be established.

(2) In setting up the system PLG, we have allowed the sequents $\Delta \leq$ C and $\Delta <$ C to be infinitary (i.e., for Δ to be infinite). We might impose the restriction that they should be finitary (i.e., that Δ should be finite). The various rules can then all be formulated in finitary form. For example, Cut($\leq/\leq$) might be formulated in the form:

$$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2 \ldots \Delta_n \leq A_n \quad A_1, A_2, \ldots, A_n \leq C}{\Delta_1, \Delta_2, \ldots, \Delta_n \leq C}$$

which, in its turn, can be derived from repeated applications of the singulary form of the rule:

$$\frac{\Delta \leq A \quad A, \Gamma \leq C}{\Delta, \Gamma \leq C}$$

Completeness might then be proved in the same way as before and, since the derivations are all finite, we may thereby establish the following compactness result: where s and the sequents of S are all finitary, s will be a consequence of S only if it is a consequence of a finite subset of S.

(3) We may also establish the finite model property and decidability when only finitary sequents are in question. For suppose that s is not derivable from the finite set S of finitary sequents. Then there will be a canonical model $\mathcal{M}_{S+}$ for some witnessed extension S'^+ of some consistent extension S' of S. But it is readily seen from the construction that S'^+ can be taken to be finite, given that S is a finite, and that the domain F_{S+} of facts of $\mathcal{M}_{S+}$ can be taken to consist of subsets of the atoms of S'^+. Thus $\mathcal{M}_{S+}$ will be a finite model and so the finitary version of PLG will have the finite model property and will be decidable.

(4) In proving completeness, we have also imposed no restriction on the verification space V; and it is natural to wonder whether the completeness result would still hold under various natural conditions on the space. If we examine the above completeness proof, we see that it was not essential to take V_S to be *exactly* $V = \{E \subseteq F_S$:

$E = \{\Delta: \Delta \leq \text{C is derivable from S}\}$. We could have taken it to be any collection of upward-closed subsets of F_S that *contained* the set V and the proof of completeness would still have gone through. It follows, in particular, that completeness will still hold for the simple models in which V is taken to be maximal. It would be of interest to establish other results of this sort.

§8. Fragments. The system PLG contains the full grounding operators, $\leq$ and $<$, and the partial grounding operators, $\preceq$ and $\prec$. We have already dealt with the system PLPG that contains only the partial grounding operators $\preceq$ and $\prec$; and it is natural to consider the complementary system PLFG (the *pure logic of full ground*) that contains only the full grounding operators $\leq$ and $<$. We might also consider the systems that contain the single operator $\leq$ for weak ground or the single operator $<$ for strict ground. For each of these systems, we would like to establish that they are 'fragments' of the original system with all four operators, that is, that they enable us to derive anything in their own language that can be derived within the original system. In principle, such results could be established semantically by showing that each of the systems is sound and complete for the factual semantics, but it will be instructive to give a purely syntactic proof of these results. It might also be of interest to obtain other results of this sort, either by adding new operators, such as the partial strict operator $\prec^*$, or by considering other combinations of the operators.

The system PLFG is comprised of the following rules:

Subsumption ($</\leq$):

$$\frac{\Delta < C}{\Delta \leq C}$$

Cut

$\leq/\leq$:

$$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

$\leq/<$:

$$\frac{\Delta_1 \leq A_1 \quad \Delta_2 \leq A_2, \ldots \quad A_1, A_2, \ldots < C}{\Delta_1, \Delta_2, \ldots < C}$$

Identity

$$\frac{}{A \leq A}$$

Non-Circularity ($<$):

$$\frac{A, \Delta < A}{\perp}$$

Reverse Subsumption($\leq/\leq_\pi$)

$$\frac{A_1, A_2, \ldots \leq C \quad A_1, \Delta_1 < B_1 \quad B_1, \Gamma_1 \leq C \quad A_2, \Delta_2 < B_2 \quad B_2, \Gamma_2 \leq C \ldots}{A_1, A_2, \ldots < C}$$

In Reverse Subsumption($\leq/\leq_\pi$), the side premises $A_k, \Delta_k < B_k$ and $B_k, \Gamma_k \leq C$ are 'long-hand' for $A_k \prec C$, as in the original rule. It is the use of the relation of strict partial ground that enables us to avoid this clumsy formulation.

It is readily established that:

LEMMA 8.1. If the full sequent s is derivable from the set of full sequents S within PLFG, then s is derivable from S within PLG.

Proof. We have already seen that Cut($\leq/<$) is derivable within PLG. Non-Circularity (in $<$) follows from the original rule of Non-Circularity (in $\prec$) and the fact that $A \prec A$ can be derived from $\Delta, A < A$ by Subsumption($</\prec$). Reverse Subsumption($\leq/\leq_\pi$) follows from the original rule of Reverse Subsumption and the fact that $A_k \prec C$ can be derived from $A_k, \Delta_k < B_k$ and $B_k, \Gamma_k \leq C$. For we may first obtain $A_k \prec B_k$ and $B_k \preceq C$ by Subsumption in ($</\prec$) and ($\leq/\preceq$) and then obtain $A_k \prec C$ by Transitivity($\prec/\preceq$). □

We also have the converse result:

LEMMA 8.2. Suppose S is a set of full sequents and that the full sequent s is derivable from S within PLG. Then s is derivable from S within PLFG.

Proof. We establish a somewhat stronger result by induction. Let s be *any* sequent (full or partial) derivable from S within PLG. Then either:

(i) s is a full sequent and is derivable from PLFG; or
(ii) s is the weak partial sequent $A \preceq C$ and $A, \Delta \leq C$ is derivable from S within PLFG for some set of atoms Δ; or
(iii) s is the strict partial sequent $A \prec C$ and $A, \Delta < B$ and $B, \Gamma \leq C$ are derivable from S within PLFG for some atom B and sets of atoms Δ and Γ.

The result clearly holds when s is an assumption of S or is obtained by Identity. The other cases are as follows:

(a) $= \Delta \leq C$ is derived by Subsumption ($</\leq$) from $\Delta < C$. By IH, $\Delta < C$ is derivable from S within PLFG; and so $\Delta \leq C$ is derivable from S within PLFG by ($</\leq$).
(b) $s = A \prec C$ is derived by Subsumption($</\prec$) from $\Delta, A < C$. But then $\Delta, A < C$ is derivable from S within PLFG by IH and $C \leq C$ is derivable from S within PLFG by Identity—as required under (iii).
(c) $s = A \preceq C$ is derived by Subsumption ($\prec/\preceq$) from $A \prec C$. By IH, $A, \Delta < B$ and $B, \Gamma \leq C$ are derivable within PLFG for some Δ, B and Γ. By Subsumption, $A, \Delta \leq B$ is derivable within PLFG; and so, by Cut($\leq/\leq$) and Identity, $A, \Delta, \Gamma \leq C$ is derivable within PLFG—as required.
(d) $s = A \preceq C$ is derived by Subsumption($\leq/\preceq$) from $\Delta, A \leq C$. By IH, $\Delta, A \leq C$ is derivable within PLFG; and the result follows.
(e) $s = A \preceq C$ is derived by Transitivity($\preceq/\preceq$) from $A \preceq B$ and $B \preceq C$. By IH, $A, \Delta \leq B$ and $B, \Gamma \leq C$ are derivable within PLFG for some Δ and Γ. By Cut($\leq/\leq$), $A, \Delta, \Gamma \leq C$ is then derivable—as required.
(f) $s = A \prec C$ is derived from $A \preceq B$ and $B \prec C$ by Transitivity($\preceq,\prec$). By IH, $A, \Delta \leq B$, $B, \Gamma < D$ and $D, \Sigma \leq C$ are derivable within PLFG for some Γ and Σ. By Cut($\leq,<$), $A, \Delta, \Gamma < D$ is derivable within PLFG and so $A, \Delta, \Gamma < D$ and $D, \Sigma \leq C$—as required.
(g) $s = A \prec C$ is derived from $A \prec B$ and $B \preceq C$ by Transitivity($\prec/\preceq$). By IH, $A, \Delta < D$ and $D, \Gamma \leq B$ and $B, \Sigma \leq C$ are derivable within PLFG for some Δ, Γ, D, and Σ. By Cut($\leq/\leq$), $D, \Gamma, \Sigma \leq C$ is derivable; and so $A, \Delta < D$ and $D, \Gamma, \Sigma \leq C$ are derivable—as required.
(h) $s = A \leq A$ is derived by Identity. But then it is derivable by Identity within PLFG.
(i) s is derived from $A \prec A$ by Non-Circularity. By IH, $A, \Delta < B$ and $B, \Gamma \leq A$ are derivable within PLFG for some B, Δ and Γ. So $B, \Gamma, \Delta < B$ is derivable by Cut($\leq/<$); and so s is derivable by Non-Circularity($<$).

(j) $s = A_1, A_2, \ldots < C$ is derived by Reverse Subsumption from $A_1, A_2, \ldots \leq C, A_1 \prec C, A_2 \prec C, \ldots$. By IH, $A_1, A_2, \ldots \leq C$, $A_1, \Delta_1 < B_1$, $B_1, \Gamma_1 \leq C$, $A_2, \Delta_2 < B_2$, $B_2, \Gamma_2 \leq C, \ldots$ are derivable within PLFG; and so by Reverse Subsumption$((\leq/\leq_\pi)$, $A_1, A_2, \ldots < C$ is derivable within PLFG. □

Putting the two lemmas together we obtain:

THEOREM 8.3. Let s be a full sequent and S a set of full sequents. Then s is derivable from S within PLG iff it is derivable from S within PLFG.

We have related results for the PLWFG (the pure logic of weak full ground) and PLSFG (the pure logic of strict full ground). PLSFG is comprised of the following two rules:

Cut
$$\frac{\Delta_1 \leq A_1 \ \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

Identity
$$\frac{}{A \leq A}$$

PLSFG is comprised of the following three rules:

Cut$(</</\Gamma)$
$$\frac{\Delta_1 < A_1 \ \Delta_2 < A_2 \ldots \quad A_1, A_2, \ldots, \Gamma < C}{\Delta_1, \Delta_2, \ldots, \Gamma < C}$$

Non-Circularity$(<)$
$$\frac{A, \Delta < A}{\bot}$$

Amalgamation$(<)$
$$\frac{\Delta_1 < C \quad \Delta_2 < C \ldots}{\Delta_1, \Delta_2 \ldots \leq C}$$

We should note that the logic for weak full ground is considerably more elegant than the logic for strict full ground, which is another indication of the inherent simplicity of the weaker notion.

THEOREM 8.4. (i) For any weak full sequent s, s is derivable from a set S of weak full sequents within PLG iff it is derivable from S within PLWFG;

(ii) For any strict full sequent s, s is derivable from a set S of strict full sequents within PLG iff it is derivable from S within PLSFG.

Proof. The right to left directions are straightforward. By the previous theorem, a full sequent is derivable from S within PLG iff it is derivable from S within PLFG; and so it suffices to prove the left to right directions with PLFG in place of PLG.

(i) If S contains only weak sequents, then it is readily established that no strict sequents are derivable from S within PLFG; and so only the rules Identity and Cut will be applicable. But this then means any weak sequent derivable from S within PLFG will be derivable from S within PLWFG.

(ii) In case S contains only strict sequents, we may show by induction that:

if s is derivable from S within PLFG then either (1) s is a strict sequent and is derivable from S within PLSFG or (2) s is a weak identity $A \leq A$ or (3) s is a weak sequent of the form $\Delta \leq A$ that is not an identity $A \leq A$ and $\Delta - \{A\} < A$ is derivable from S within PLSFG.

We go through each of the cases, dealing with the simpler cases first:

(a) s is a member of S. Then s is a strict identity and falls under (1).

(b) s is derived by Identity. Then s is a weak identity and falls under (2).

(c) s is derived by Non-Circularity(<) from $A, \Delta < A$. By IH, $A, \Delta < A$ is derivable from S within PLSFG and so s is derivable from S by Non-Circularity(<).

(d) $s = \Delta \leq C$ is derived from $\Delta < C$ by Subsumption($</\leq$). By IH, $\Delta < C$ is derivable from S within PLSFG. If $C \notin \Delta$, then $\Delta - \{C\} = \Delta < C$ is derivable from PLSFG in accordance with (3). If $C \in \Delta$, then it follows, by Non-Circularity, from the fact that $\Delta < C$ is derivable from S within PLSFG that any sequent is derivable from S within PLSFG.

(e) $s = \Delta_1, \Delta_2, \ldots \leq C$ is derived by means of an application:

$$\frac{\Delta_1 \leq A_1 \Delta_2 \leq A_2 \ldots \quad A_1, A_2, \ldots \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

of Cut($\leq/\leq$). By IH, we know that the premises fall within the categories (2) or (3) above. We distinguish various cases.

Suppose first that the major premise $A_1, A_2, \ldots \leq C$ is the identity $C \leq C$. The application of Cut($\leq/\leq$) therefore takes the following form:

$$\frac{\Delta_1 \leq C \ \Delta_2 \leq C \ldots \ C \leq C}{\Delta_1, \Delta_2, \ldots \leq C}$$

If each of the minor premises $\Delta_1 \leq C, \Delta_2 \leq C, \ldots$ were the weak identity $C \leq C$, then the conclusion would be the weak identity, in accordance with (2); and so we may assume that one of the minor premises is not the weak identity $C \leq C$. If one of the minor premises were the weak identity $C \leq C$, we could assume without loss of generality that it is the first. The application of Cut may therefore be taken to be of the following form:

$$\frac{(C \leq C) \ \Delta_1 \leq C \ \Delta_2 \leq C \ldots \ C \leq C}{(C), \Delta_1, \Delta_2, \ldots \leq C}$$

where the parentheses in $(C \leq C)$ and (C) indicate that the item $C \leq C$ or C may or may not be present. By IH, $\Delta_1 - \{C\} \leq C$, $\Delta_2 - \{C\} < C, \ldots$ are all derivable from S within PLSFG. So by Amalgamation(<), $\Delta_1 - \{C\}, \Delta_2 - \{C\}, \ldots < C, \ldots$ is derivable from S within PLSFG. But $\{(C) \cup \Delta_1 \cup \Delta_2 \cup \ldots\} - \{C)\} = (\Delta_1 - \{C\}) \cup (\Delta_2 - \{C\}) \cup \ldots$; and so $((C) \cup \Delta_1 \cup \Delta_2 \cup \ldots) - \{C)\} \leq C$ is derivable from S within PLSFG, in accordance with (3).

Suppose next that the major premise $A_1, A_2, \ldots \leq C$ is not the identity $C \leq C$. Let us rewrite the major premise in the form:

$$B_1, B_2, \ldots, D_1, D_2, \ldots, (C) \leq C,$$

where the $B_1, B_2, \ldots$ are those of the original antecedents A_k of the major premise that are distinct from C and for which $\Delta_k \leq A_k$ is not the identity $A_k \leq A_k$, and the $D_1, D_2, \ldots$ are those of the original antecedents that are distinct from C and occur in some Δ_k (there may be some overlap between the B_k's and the D_k's). Let $\Gamma_1 \leq B_1, \Gamma_2 \leq B_2, \ldots$ now be the corresponding minor premises $\Delta_k \leq A_k$ which are not identities and for which A_k is distinct from C; and let $\Sigma_1 \leq C, \Sigma_2 \leq C, \ldots$ be all those minor premises, if there be such, which are not identities and whose consequent is C. By IH, the following sequents will be derivable from S within PLSFG:

(1) $B_1, B_2, \ldots, D_1, D_2, \ldots < C$;
(2) each $\Gamma_k - \{B_k\} < B_k$; and
(3) each $\Sigma_k - \{C\} < C$.

We may assume that $C \notin \Gamma_k - \{B_k\}$. For otherwise from $\Gamma_k - \{B_k\}, C < B_k$ and $B_1, B_2, \ldots, D_1, D_2, \ldots < C$, we may derive a sequent of the form $\ldots C \ldots < C$; and so any sequent whatever will be derivable.

From (1) and (2), $\Gamma_1 - \{B_1\}, \Gamma_2 - \{B_2\}, \ldots, D_1, D_2, \ldots < C$ is derivable from S within PLSFG by Cut($</</\Gamma$); and so $\Gamma_1 - \{B_1\}, \Gamma_2 - \{B_2\}, \ldots, D_1, D_2, \ldots, \Sigma_1 - \{C\}, \Sigma_2 - \{C\}, \ldots < C$ will be derivable by Amalgamation($<$).

It may now be shown that the antecedent of this conclusion is of the desired form. For C does not occur in the antecedent, since C is distinct from the $D_1, D_2, \ldots$ and does not belong to the $\Gamma_1 - \{B_1\}, \Gamma_2 - \{B_2\}, \ldots$, as we have seen. Evidently, any atom in this antecedent occurs in the original antecedent to the conclusion of the inference. Take now any atom in the original antecedent distinct from C. If it is one of $A_1, A_2, \ldots$, it will be one of $D_1, D_2, \ldots$; and if it is not one of $A_1, A_2, \ldots$, then it will belong to one of $\Gamma_1 - \{B_1\}, \Gamma_2 - \{B_2\}, \ldots$ or to one of $\Sigma_1 - \{C\}, \Sigma_2 - \{C\}, \ldots$.

(f) The case in which $s = \Delta_1, \Delta_2, \ldots \leq C$ is obtained by an application of Cut($\leq/<$) is proved by similar means (and without some of the complications of the preceding case).

(g) The final case is one in which s is derived by means of an application:

$$\frac{A_1, A_2, \ldots \leq C \quad A_1, \Delta_1 < B_1 \quad B_1, \Gamma_1 \leq C \quad A_2, \Delta_2 < B_2 \quad B_2, \Gamma_2 \leq C \ldots}{A_1, A_2, \ldots < C}$$

of Reverse Subsumption($\leq/\leq_\pi$). Suppose one of $A_1, A_2, \ldots$, say A_k, is C. Look at the pair of minor premises $C = A_k, \Delta_k < B_k, B_k, \Gamma_k \leq C$. By IH, $A_k, \Delta_k < B_k$ is derivable within PLSFG. If $B_k, \Gamma_k \leq C$ is the weak identity $C \leq C$, then $A_k, \Delta_k < B_k$ is the strict identity $C = A_k, \Delta_k < C$ and hence anything is derivable within PLSFG by Non-Circularity. If $B_k, \Gamma_k \leq C$ is not the weak identity $C \leq C$, then $(B_k \cup \Gamma_k) - \{C\} < C$ is derivable within PLSFG. So by Cut($</<$), a strict identity $C, \Delta_k, B_k, \Gamma_k < C$ is again derivable.

So we may assume that none of $A_1, A_2, \ldots$ is C; and it follows by IH that $A_1, A_2, \ldots < C$ is derivable. □

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