An Abstract Characterization of the Determinate/Determinable Distinction

My aim in this paper is to provide an account of what it is for the world to have a determinate/determinable structure. Patches have colors, people have heights, particles have mass. These are all instances of the determinate/determinable structure, with a given state of the world consisting in something's possessing a determinate (be it a given color or height or mass) from within a given determinable (color, height or mass). But what is it for the world as a whole to possess such a structure?

In the *Tractatus*, Wittgenstein took the atomic propositions, by which the world is to be described, to be completely independent of one another. But he later revised his view (Wittgenstein [1929]) and allowed that the atomic propositions might exhibit the kind of dependence that is characteristic of the way in which different determinants of a given determinable are exclusive of one another. Our question might therefore be put in the form: how in the most abstract terms should we conceive of the post-Tractarian world?

There has recently been a revival of interest in the determinate/determinable distinction but the present work is different from the more recent literature in a number of signifcant respects. In the first place, my primary focus has been on determinate or determinable *states* rather than on determinate or determinable *attributes* - on the state of a patch being red, for example, rather than on the attribute of redness itself. With the range of determinate attributes falling under a given determinable may be associated, of course, a corresponding range of states - with each color, for example, may be associated the state of a given patch being of that color. But the converse may not hold. A range of states within a given determinable may not correspond - or may not correspond in any natural way - to a range of attributes within a given determinable. Consider, for example, the state of the world containing exactly such and such individuals. The determinable here is the 'constitution of the universe' and the various determinates are given by the individuals which constitute the universe. But there is no natural counterpart to these determinates and determinable within the realm of attributes. For this and for other reasons, it has been important for me to focus attention on a determinate/determinable distinction among states without presupposing that there must be a corresponding distinction among attributes.

In the second place, my central concern has not been with the determinate/determinable distinction as such but with a characteristic thesis involving the distinction. I have not wanted to say what it is for a determinate to belong to a given determinable but to say what it is for the world to possess the kind of structure that it has when the determinate/determinable distinction is in play. There is some reasonable hope that my account of the one will ultimately bear on the other, but this has not been my central concern.

In the third place, the intended range of applications and the underlying motivation is somewhat different. The most recent literature has tended to focus on the questions of multiple realizability and causal over-determination. It has been supposed, for example, that apparent cases of causal over-determination might simply be ones in which a determinate and a determinable or a more general determinate both serve as a cause of a given effect. But my own interest has been in issues of a more logical or semantical nature. Within a 'truth-maker' semantics, for example, it may be supposed that an atomic statement is made true by certain facts or states within a possible world or false by certain facts or states within the world. But what is the relation between the truth-makers and the falsity- makers? When the world possesses a determinate/determinable structure, a natural answer is immediately forthcoming. For when the statement is made true by the states *s* and *t*, say, then the statement will be made false by the fusion of any states $s'$and t' which are respectively incompatible with *s* and *t* and yet belong to the same determinable. Or again, in providing a semantical account of counterfactuals, we may wish to consider a world that is just like a given world but for the presence of a state s'. But if the world possesses a determinate/determinable structure, then we can make perfectly good sense of what this might mean. For the state s' will be incompatible with a state *s* within the world that belongs to the same determinable; and so we may simply replace *s* with s' to obtain the new world. Although I cannot go into details, my own approach has been engineered to provide a rigorous account of how various applications along these lines might be made.

Any analysis must proceed on the basis of certain primitives. In the present case, I have been able to rest content with a single primitive, that of part to whole. Indeed, part of the interest of the account is that it seems capable of yielding so much on the basis of so little. Simply by taking for granted the relation of part-whole, we have been able to capture - or, at least, to capture an important aspect of - what it is for the world to have a determinate/determinable structure.

However, it has been essential to the account to presuppose an ontology of possible states and implicit within this ontology are two further notions, which would also have had to be taken as primitive within the framework of a broader ontology. For within the space of propositions as a whole, we need to restrict ourselves to those which correspond to the occurrence of a state. We might in a given context allow the proposition or state of a given patch being red, for example, but not of its being green or round. And within the realm of logically possible states, we need to restrict ourselves to those that are genuinely possible. We may allow the state of a given patch being red and round, for example, but not of its being red and green.

These further presuppositions raise deep philosophical questions, with which I shall not be concerned. How exactly are we to understand the modality involved in the determinants of a determinable being exclusive of one another - is it conceptual or metaphysical or perhaps even ‘physical’? Is the determinable/determinate distinction intrinsic to the world or is it somehow imposed upon the world by us? And should we expect to find the determinate/determinable distinction at the most fundamental level of reality or is its presence in the world merely some kind of epiphenomenon? These questions are of general interest to metaphysics but the broad outline of our account can remain the same however they are answered.

Let me give a brief overview of the rest of the paper. I begin the introducing the idea of a *state space*, a space of possible states ordered by part-whole (§1). I then characterize the spaces - the so called *D-spaces* - which correspond to the world having a determinate/determinable structure (§2). Indeed, most of the other sections are devoted to the task of articulating and establishing the exact sense in which this is so. It can be shown that a D-space will contain a full array of 'world-states', corresponding to the possible worlds (§3). We then consider the interaction between modal and mereological notions and are thereby are able to characterize a number of different ways in which one state may be a part of another (§4). The next section (§5) introduces the important notion of *coincidence*. This corresponds, intuitively speaking, to two states being determinates of the same determinable; and it may be shown on the basis of the underlying conditions on a D-space that the relation is an equivalence relation. We then show that each possible world will have a determinate/determinable structure, which is the same from world to world (§§6-7). We also establish a kind of converse result: given that the worlds have an invariant determinable/determinable structure, the underlying space of states must be a D-space (§8). We conclude with an outline of different directions in which the present work might be extended and which I hope may serve as a starting point for further enquiry (§9).[1]

§1 State Space

We define the basic notion of a state space and establish some of the basic properties of state spaces.

Suppose that $\sqsubseteq$ is a binary relation on the set S. Intuitively, we shall think of S as a set of possible states and of $\sqsubseteq$ as the part-whole relation on those states. There is a natural sense in which one state may be composed of other states - the state of a patch being red and round, for example, is composed of the state of its being red and the state of its being round; and we take one state to be a part of another if it either is or helps compose the other. Thus the state of the patch being red, in the above example, will be a part of the state of its being red and round. It should be noted that $t \sqsubseteq s$ requires that s's obtaining entails t's entailing and not that t's obtaining entails s's obtaining.

Recall that $\sqsubseteq$ is a *partial order* (po) *on S* if it is reflexive, transitive and anti-symmetric relation on S. Given the po $\sqsubseteq$ on S, we shall make use of the following standard definitions (with $s, t, u \in S$ and $T \subseteq S$):

s is an *upper bound of T* if $t \sqsubseteq s$ for each $t \in T$;

s is a *least upper bound* (lub) *of T* if s is an upper bound of T and $s \sqsubseteq s'$ for any upper bound s' of T;

s is *null* if $s \sqsubseteq s'$ for each $s' \in S$ and is otherwise *non-null*;

$s \sqsubset t$ (s is a *proper part of t*) if $s \sqsubseteq t$ but not $t \sqsubseteq s$;

s *overlaps t* if for some non-null u, $u \sqsubseteq s$ and $u \sqsubseteq t$;

s is *disjoint from t* if s does not overlap t.

The least upper bound of $T \subseteq S$ is unique if it exists (since if s and s' are least upper bounds, then $s \sqsubseteq s'$ and $s' \sqsubseteq s$ and so, by anti-symmetry, $s = s'$). We denote it by $\bigsqcup T$ and call it the *fusion of T*

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[1] I should like to thank audiences at the Institute Jean Nicod in Paris, at the University of London, at the University of Leeds, at a Conference on Aristotelian Themes in Contemporary Metaphysics in Boise (2011), and at a workshop on metaphysics in Oxford (2011)for many helpful comments; and I am especially grateful for comments from Ralf Bader, Per Martin-Lof, Ian Rumfitt, Robbie Williams and Jessica Wilson.

(or of the members of T). When $T = \{t_1, t_2, ...\}$, we shall sometimes write $\bigsqcup T$ more perspicuously as $t_1 \sqcup t_2 \sqcup \ldots$.

A *state space* $\mathcal{S}$ is now taken to be a non-empty structure $(S, \sqsubseteq)$ subject to the following two conditions:

Partial Order (PO) $\sqsubseteq$ is a po on S;

Bounded Completeness (BC) Any subset of S with an upper bound has a least upper bound.

The state space $\mathcal{S}$ is said to be *total* when every subset of S has an upper bound (and so, by BC, has a least upper bound).

Suppose that s and t are two terms that may or may not designate an object. We shall then find it convenient to use '$s \simeq t$' to indicate that the object designated on the left exists iff the object designated on the right exists and that the designated objects, when they both exist, are the same (we also assume that a term will fail to designate an object if any of its subterms fails to designate an object). We then have:

Lemma 1 (i) Nullity $\bigsqcup\varnothing$ exists.

(ii) Associativity $\bigsqcup(\bigcup\{S_i: i \in I\}) \simeq \bigsqcup(\{\bigsqcup S_i: i \in I\})$.

Proof (i) S is non-empty and so has a member s. Vacuously, s is an upper bound of $\varnothing$ and so, by BC, $\varnothing$ has the lub $\bigsqcup\varnothing$.

(ii) Suppose that $\bigsqcup(\bigcup\{S_i: i \in I\})$ exists. Then $S = \bigcup\{S_i: i \in I\}$ has a lub s. Now s is an upper bound for each S_i, $i \in I$, since $S_i \subseteq S$. So by BC, each $s_i = \bigsqcup S_i$ exists. But s is also an upper bound for $\{s_i: i \in I\}$, since s is an upper bound for each S_i and s_i a lub for S_i. So again by BC, $\bigsqcup(\{\bigsqcup S_i: i \in I\})$ exists. Moreover, s is also a lub for $\{s_i: i \in I\}$, since any upper bound for $\{s_i: i \in I\}$ is an upper bound for $\bigcup\{S_i: i \in I\}$.

Suppose now that $\bigsqcup(\{\bigsqcup S_i: i \in I\})$ exists. Then each S_i has a lub s_i and $\{s_i: i \in I\}$ has a lub s. But s is then an upper bound for $\bigcup\{S_i: i \in I\}$ and so, by BC, $\bigsqcup(\bigcup\{S_i: i \in I\})$ exists. Moreover, any upper bound of $\bigcup\{S_i: i \in I\}$ is an upper bound of $\{s_i: i \in I\}$; and so s is a lub of $\bigcup\{S_i: i \in I\}$.

The state $\bigsqcup\varnothing$ that exists according to Nullity is called the *null* state and is designated by $\wedge$. Note that the null state will indeed be null, since any state is vacuously an upper bound of $\varnothing$. In standard formulations of mereology, the null fusion is not normally taken to exist but we shall find it helpful to suppose that it does exist. The associativity condition tells us that the operation of fusing a number of states is equivalent to fusing some of these states, fusing others, fusing yet others, ... and then fusing the results.

We might take the fusion operation $\bigsqcup$ as primitive in place of the relation $\sqsubseteq$ of part-whole. The two conditions above then provide an alternative characterization of a state space.

The state space $\mathcal{S} = (S, \sqsubseteq)$ is said to be a *regular* (or R-) space if it subject to the following two further conditions:

Supplementation If s is a proper part of t then some non-null part of t is disjoint from s.

Overlap If $\bigsqcup T$ exists and overlaps with t then some member of T overlaps with t.

Supplementation and Overlap are familiar mereological conditions but they are perhaps especially plausible for 'determinate' states of the kind we have in mind. Thus if s and t are states, with s a proper part of t, then there must be a part of t that is left over once one 'subtracts' s; and the state t will be incapable of overlapping with the fusion of the states t_1, t_2, ... unless it overlaps with one of those states. The Overlap condition is often taken to follow from the definition of fusion in terms of *overlap* but under our own approach, in which part-whole is taken to be primitive, it needs to be adopted as a separate assumption.

In the rest of the paper, we shall assume without explicit mention that all of the state spaces under consideration are regular. Let φ be any property of states. The following two general methods of proof will then be useful:

Lemma 2 (i) (Fusion Test) Suppose that every non-null part of the state s contains a non-null φ-part. Then s is the fusion of its φ-parts.

(ii) (Overlap Test) If every non-null part of s overlaps with t, then s is a part of t.

Proof (i) Given a non-null state s, let $S' = \{t: t$ is a φ-state that is a part of $s\}$ and let $s' = \bigsqcup S'$ (which exists by BC). If $s' = s$, we are done. Otherwise, s' is a proper part of s and so, by Supplementation, there is a non-null part s^* of s that is disjoint from s'. By supposition, s^* contains a non-null φ-state s^{**} as a part. But s^{**} is then disjoint from s', since s^* is disjoint from s', and it is also a non-null part of s' by the definition of s'. A contradiction.

(ii) Let φ in the Fusion Test be the property of being a part of t. Then the condition of the test is satisfied and so s is the fusion of its t-parts and hence is itself a part of t.

Given a non-empty set of states T, let ${}^{\sqsubseteq}T = \{u: u \sqsubseteq t$ for each $t \in T\}$. $\bigsqcup{}^{\sqsubseteq}T$ exists by BC and we denote it by $\bigsqcap T$ (or by $t_1 \sqcap t_2 \sqcap ...$ when $T = \{t_1, t_2, ...\}$). The 'intersection' $t_1 \sqcap t_2 \sqcap ...$ is the 'greatest lower bound':

Lemma 3 (GLB) For any non-empty set $T = \{t_1, t_2, ...\}$:

(i) $t_1 \sqcap t_2 \sqcap ...$ exists and is a part of each t_i; and

(ii) if s is a part of each t_i then s is a part of $t_1 \sqcap t_2 \sqcap ....$

Proof (i) $t_1 \sqcap t_2 \sqcap ... = \bigsqcup\{u: u \sqsubseteq t_i$ for each $i \in I\}$. Choose a member t_i of T. Then each u in $\{u: u \sqsubseteq t_i$ for each $i \in I\}$ is a part of t_i and so $t_1 \sqcap t_2 \sqcap ...$ exists by BC and is a part of each t_i.

(ii) if s is a part of each t_i then $s \in {}^{\sqsubseteq}T$ and so s is a part of $\bigsqcup{}^{\sqsubseteq}T = t_1 \sqcap t_2 \sqcap ....$

Lemma 4 (Distribution) Suppose $t_1 \sqcup t_2 \sqcup ...$ exists. Then:

$$s \sqcap (t_1 \sqcup t_2 \sqcup ...) = (s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ....$$

Proof Suppose $t_1 \sqcup t_2 \sqcup ...$ exists. By GLB(i), $s \sqcap (t_1 \sqcup t_2 \sqcup ...)$ exists. Now each $s \sqcap t_i$ is a part of s by GLB(i) and so $(s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ...$ exists by BC. Since each t_i is a part of $(t_1 \sqcup t_2 \sqcup ...)$, it is readily shown that each $s \sqcap t_i \sqsubseteq s \sqcap (t_1 \sqcup t_2 \sqcup ...)$; and so $(s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ... \sqsubseteq s \sqcap (t_1 \sqcup t_2 \sqcup ...)$. To establish that $s \sqcap (t_1 \sqcup t_2 \sqcup ...) \sqsubseteq (s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ...$, we use the Overlap Test. Suppose that u is a non-null part of $s \sqcap (t_1 \sqcup t_2 \sqcup ...)$. Then u is a part of s and of $(t_1 \sqcup t_2 \sqcup ...)$ and, by Overlap, overlaps with some t_i. So u and t_i have a common non-null part u'. But then u' is a common part of s and t_i and hence a part of $(s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ...$; and so u overlaps with $(s \sqcap t_1) \sqcup (s \sqcap t_2) \sqcup ....$

One state may be 'subtracted' from another:

Lemma 5 (Subtraction) Given any states s and t there is a unique state t' disjoint from s for which $t = (t \sqcap s) \sqcup t'$.

Proof Let $T' = \{u: u$ disjoint from s and $u \sqsubseteq t\}$. Then each member u of T' is a part of t and so, by BC, $t' = \bigsqcup T'$ exists.

(1) t' is disjoint from s.

Pf. For suppose s overlaps with t'. By Overlap, s overlaps with some u in T', contrary to the definition of T'.

(2) $(s \sqcap t) \sqcup t'$ is a part of t.

Pf. t' is the fusion of parts of t and so is a part of t; $(s \sqcap t)$ is a part of t by GLB; and so $(s \sqcap t) \sqcup t'$ is a part of t.

(3) $(s \sqcap t) \sqcup t' = t$.

Pf. Given (2), we need to show that t is a part of $(s \sqcap t) \sqcup t'$. We use the Overlap Test. Suppose that u is a non-null part of t. If it is disjoint from s then it is a member of T' and so a part of t' and hence of $(s \sqcap t) \sqcup t'$. If it overlaps with s, then it overlaps with $(s \sqcap t)$ and hence with $(s \sqcap t) \sqcup t'$.

(4) t' is the unique u disjoint from t for which $(s \sqcap t) \sqcup u = t$.

Pf. Suppose that t' and t'' are both states u disjoint from s for which $t = (s \sqcap t) \sqcup u$. We use the Overlap Test to show that t' is a part of t''. Take any non-null part v of t'. Then it is a part of t and so also a part of $(s \sqcap t) \sqcup t''$. By Overlap, u overlaps with $s \sqcap t$ or with t''. But since v is disjoint from s, it does not overlap with $s \sqcap t$ and so it overlaps with t''. Similarly any part v of t'' overlaps with t'; and it follows that $t' = t''$.

§2 D-Space

We define the central notion of a determinate state space (D-space) by imposing certain 'logical' conditions on a state space.

To this end, we first define certain logical features of states. Given a set T of states: we

say T is *jointly compatible* if $\sqcup T$ exists (i.e. if T has an upper bound) and that otherwise it is *jointly incompatible*; and we say that T is *pairwise compatible* if any two states in T are compatible. Alternatively, where $T = \{t_1, t_2, ...\}$, we may talk instead of the member states t_1, t_2, ... themselves being *jointly* or *pairwise compatible*. This terminology is in keeping with our view that the states are *possible* states and that it is only the incompatiblity of two states (as with the state of a given patch being red all over and the state of its being green all over) that can prevent their fusion from existing. It is this definition which provides the fundamental connection between the mereological and modal aspects of a state space.

We say that s (*totally*) *excludes* t if each non-null part of t is incompatible with s and that s and t are (*totally*) *incompatible* if s excludes t and t excludes s (i.e. if no non-null part of either is compatible with the other). Thus a patch being red all over will totally exclude its left half being green all over though not the state of its upper left quarter being green all over and its lower left quarter being red all over, and the patch being red all over will be totally incompatible with the patch being green all over. Note that, as a degenerate case, any state will exclude the null state.

Exclusion is a kind of thorough-going incompatibility. It is not merely sufficient for s to exclude t that they should be incompatible; s should also be incompatible with any (non-null) part of t. For propositions, no such distinction can be drawn. Propositions are either compatible or incompatible; and there is no plausible sense in which the incompatibility can be more or less thorough. But states have a mereological structure that enables them to be incompatible in a more or less thorough-going way.

We now define a *determinate* (D-) *space* to be an R-space (S, $\sqsubseteq$) satisfying the following two 'logical' conditions:

Directed Completeness Any states that are pairwise compatible are jointly compatible

Independence If one state is disjoint from and compatible with another then any state excluded by the first is also compatible with the second.[2]

Directed Completeness gives partial expression to the determinate or non-disjunctive nature of states. Consider the following putative counter-example to Directed Completeness. There are three 'states' s, t and u: s is the state of a given patch being red or green; t is the state of the patch being red or blue; and u is the state of the patch being green or blue. Then any two of the states are compatible since they are compatible with the patch having a single color, while the three

[2]The conditions of Bounded and Directed Completeness are also used in the definition of a Scott domain (Abramsky, Jung [2007]). However, the connection between the two kinds of structure is not something that I have explored.

states are not jointly compatible, since that would require the patch to have two different colors. The assumption of Directed Completeness is meant to exclude disjunctive or non-determinate states of this sort.

We may say that states *s* and *t* are *independent* if they are disjoint from and compatible with one another; there is no logical or mereological interaction between them. The assumption of Independence then states that, when two states are independent, any state excluded by the one will be compatible with the other. It is as if independent states occupy difference 'regions of logical space'. For suppose that *s* is independent of *t* (*s*, for example, might be the left half of a patch being green and *t* might be the right half of the patch being red). Then intuitively, *s* and *t* have 'nothing to do' with one another, they occupy different regions of logical space. Suppose now that *s* excludes s' (as when s' is the state of the upper left half of the patch being blue). Then s' must occupy the region or part of the region of logical space occupied by *s*; and so s' should also be compatible with *t*. (This kind of reasoning in terms of regions of logical space will later be made more precise).

Note that any total space will be a D-space, since any set of states will be jointly compatible and so Directed Completeness and Independence will be vacuously satisfied.

We have the following two special cases of Directed Completeness:

Three-Directed Completeness Any three states that are pairwise compatible are jointly compatible.

Ascent If $s_1 \sqsubseteq s_2 \sqsubseteq \ldots$ then $s_1, s_2, \ldots$ are compatible.

Somewhat surprisingly, we also have a converse result:

Lemma 6 Ascent and Threefold Directed Completeness imply Directed Completeness.

Proof Let *T* be a set of pairwise compatible states of cardinality λ. We show by induction on the cardinal κ, $2 \leq \kappa \leq \lambda$, that the fusion of κ or fewer states from *T* always exists. The case $\kappa = 2$ is trivial. Now suppose that κ is a finite cardinal $m + 1 > 2$ and that $s_1, s_2, \ldots, s_{m+1}$ are m+1 states from *T*. Then $s' = s_1 \sqcup s_2 \sqcup \ldots \sqcup s_{m-1}$ exists by inductive hypothesis. But $s_1 \sqcup s_2 \sqcup \ldots \sqcup s_{m-1} \sqcup s_{m+1}$ also exists by IH and so s' is compatible with s_{m+1}, and $s_1 \sqcup s_2 \sqcup \ldots \sqcup s_{m-1} \sqcup s_m$ also exists by IH and so s' is compatible with s_m. But s_m and s_{m+1} are compatible by supposition; and so, given that s', s_m and s_{m+1} are pairwise compatible, it follows by Three-Directed Completeness that s', s_m and s_{m+1} are jointly compatible; and so $s_1 \sqcup s_2 \sqcup \ldots \sqcup s_m \sqcup s_{m+1}$ exists and $s_1, s_2, \ldots, s_{m+1}$ are jointly compatible.

Now suppose that κ is an infinite cardinal and let $T = \{s_\zeta: \zeta < \kappa\}$ be a set of states of *S* of cardinality κ. For each ζ, let $t_\zeta = \bigsqcup\{s_{\zeta'}: \zeta' < \zeta\}$. Then each t_ζ for $\zeta < \kappa$ exists by IH and $t_{\zeta'} \sqsubseteq t_{\zeta''}$ for $0 \leq \zeta' \leq \zeta'' < \kappa$; and so, by Ascent, $\bigsqcup\{t_\zeta: \zeta < \kappa\} = \bigsqcup\{s_\zeta: \zeta < \kappa\}$ exists.

§3 World-states

A state s of a state space is said to be a *world-state* if a state is a part of s whenever it is compatible with s. World-states are so 'big' that compatibility with them can only achieved through containment.

Let us use w, v, and the like as variables for world-states. Somewhat surprisingly, the conditions imposed upon a D-space enable us to establish the general existence of world-states:

Lemma 7 (i) Any state is a part of some world-state;

(ii) no two world-states are compatible;

(iii) Take any set W of states that conforms to (i) and (ii) above (i.e., any state is a part of some member of W and no two members of W are compatible). Then W is the set of world-states.

Proof (i) Take any state s. Consider the partial order $(T, \leq)$, where $T = \{t: s \sqsubseteq t\}$ and $\leq$ is the restriction of $\sqsubseteq$ to T. Then every totally ordered chain of T has an upper bound by Ascent. So by Zorn's Lemma, there is a maximal element t of T. By definition, $s \sqsubseteq t$. Suppose t' is compatible with t. Then $t \sqcup t'$ exists and belongs to T. But then $t' \sqsubseteq t$ by the maximality of t; and so t is a world-state.

(ii) Suppose w is compatible with v. Then w is a part of v and v a part of w by the defining characteristic of world-states; and so $w = v$.

(iii) Take any set W of states conforming to (i) and (ii) above and suppose w is a world-state. Then it is a part of a member v of W. Since v is compatible with w, v is a part of w by condition (i), and so $w = v \in W$. Suppose now that $w \in W$ but is not a world-state. Then for some state s, s is compatible with w (i.e., $s \sqcup w$ exists) even though it is not a part of w. So $s \sqcup w$ is a part of some member w' of W. Now $w \neq w'$ since s is a part of w' but not of w. But w is a part of w' and so the two are compatible, contrary to the characterization of W.

Note that it follows from (i) and Nullity that some world-state exists. For the null state exists by Nullity and so a world-state exists by (i).

We introduce some familiar modal notions. A state is *necessary* if it is compatible with every state and otherwise is *contingent*. We should note that the null state $\wedge$ is necessary since $s \sqcup \wedge = s$ will exist for every state s. However, there may be other necessary states as well and it should not even be assumed that all necessary non-null states are the same. There is no reason, in principle, for example, why that state of x being identical to x should not be distinct for each object x. For this reason alone, we cannot necessarily identify each state with the set of world-states within which it is contained.

The state s *entails* state t if any state compatible with s is compatible with t and two states are *necessarily equivalent* if they entail one another. We may also say that the state s entails a set of states T (or that it entails $t_1, t_2, \ldots$, where $t_1, t_2, \ldots$ are the members of T) if any state compatible with s is compatible with some member of T. Intuitively, the entailment of $t_1, t_2, \ldots$ corresponds to the entailment of their disjunction though there may be no single state corresponding to the disjunction of these states.

We have the following basic results on these various notions:

Lemma 8 (i) Any part of a necessary state is necessary

(ii) The fusion of any necessary states exists and is necessary

(iii) Any state entails a necessary state

(iv) Any two necessary states are necessarily equivalent

(v) A state is necessary iff it is a part of every world-state (and, consequently, a state is contingent iff it is not a part of some world-state)

(vi) s entails T iff every world-state containing s as a part contains a member of T as a part.

Proof (i) Suppose that s is necessary, i.e. compatible with every state. Then clearly any part of s is also compatible with every state.

(ii) Let $s \simeq s_1 \sqcup s_2 \sqcup ...$ for necessary states $s_1, s_2, ...$ Then $s_1, s_2, ...$ are pairwise compatible and hence jointly compatible, and so $s_1 \sqcup s_2 \sqcup ...$ exists. Now take any state t. Then $t, s_1, s_2, ...$ are also pairwise compatible and hence jointly compatible, and so $s = s_1 \sqcup s_2 \sqcup ...$ is compatible with t.

(iii) & (iv) Trivial from the definitions.

(v) Suppose that s is necessary. Then it is compatible with any given world-state and hence a part of any given world-state. Now suppose that s is not necessary. Then it is not compatible with some state t. By Lemma 7(i), t is part of a world-state w. But then s is not a part of w, since otherwise it would be compatible with t.

(vi) Suppose s entails T and that s is a part of the world-state w. Then s is compatible with w; so some member t of T is compatible with w; and so t is a part of w. Suppose now that s does not entail T. Then s is compatible with some state u even though u is incompatible with any member of T. By Lemma y(i), $s \sqcup u$ and hence s is a part of a world-state w. But no member t of T can be a part of w, since then u, which is a part of w, would be compatible with t.

We may also introduce some less familiar modal notions. A state is said to be *thoroughly contingent* if every non-null part of the state is contingent and a state space is said to be *contingent* if every non-null state in the space is contingent (and hence thoroughly contingent). We should note that a space will be contingent if every non-null state contains a contingent part since then, by the Fusion Test, every non-null state will be the fusion of contingent states and so will itself be contingent. We should also note that the null state will be degenerately thoroughly contingent even though not itself contingent, but that any *non-null* state that is thoroughly contingent will also be contingent.

We have the following decomposition result:

Lemma 9 Any state is the fusion of a thoroughly contingent state and a necessary state.

Proof Take any state s and let $S' = \{u \sqsubseteq s: u \text{ is necessary}\}$. By lemma 8(ii) above, $s' = \bigsqcup S'$ exists and is necessary, and it is a part of s. Let $t = s - s'$. Since $s = s' \sqcup t$, it suffices to show that t is thoroughly contingent. So suppose that u is a non-null part of t but that u is not contingent and hence necessary. Then u is a part of s' contrary to the fact that t and s' are disjoint.

Given any state s, we may denote its thoroughly contingent part by s_∇ and its necessary part by $s_\Box$. Thus $s = s_\nabla \sqcup s_\Box$, where s_∇ and $s_\Box$ are disjoint. We may also use S_∇ for the set of thoroughly contingent states of S and $S_\Box$ for the set of necessary states of S.

§4 The Naming of Parts

From henceforth, all of our results will relate to D-spaces. We first show how the modal-mereological structure of D-spaces enables us to give a fine-grained account of the division of a state into parts.

Say that s (*totally*) *occludes* t if any non-null part of t overlaps with s or is incompatible with s. Clearly, if s excludes t then s will occlude t. But we may have occlusion without exclusion. Suppose, for example, that s is the state of a patch being red and that t is the state of the upper left half being green and the lower left half being red. Then s will occlude t without excluding t. We might think of s excluding t when it competes for the region in logical space occupied by t and we might think of s occluding t when it either competes for or shares in the region in logical space occupied by t. Given our definition of independence, the state s will occlude the state t if every non-null part of t fails to be independent of s.

We shall be interested in maximal states of a certain sort and, in particular, with maximal parts. Given a property φ of states, we say that s is a *maximal* φ-state if s is a φ-state and any φ-state is a part of s. Suppose that φ is closed under fusion, i.e. the fusion of any φ-states exists and is also a φ-state. Then the fusion of all φ-states is a φ-state and so will be a maximal φ-state. A maximal φ-state if it exists is unique. For if s and s' are both maximal φ-states, then each is a part of the other and so they are the same.

We show that certain kinds of part are closed under fusion and hence that there will be unique maximal parts of the given kind.

Lemma 10 Given any states s and t, the following classes of t-parts are closed under fusion:

(i) t-parts compatible with s;
(ii) t-parts occluded by s;
(iii) t-parts disjoint from s;
(iv) t-parts that are s-parts;
(v) t-parts independent of s;
(vi) t-parts excluded by s.

Proof (i) Let $T' = \{u: u \text{ is a part of } t \text{ compatible with } s\}$ and $t' = \bigsqcup T'$. Then t' is compatible with s. For otherwise $T' \cup \{s\}$ would be incompatible; and so by Directed Completeness, two members of $T' \cup \{s\}$ would be incompatible. But clearly, any two members of T' are compatible since they are both parts of t; and any member of T' is compatible with s by definition.

(ii) Let $T' = \{u: u \text{ is a part of } t \text{ occluded by } s\}$ and $t' = \bigsqcup T'$. Then t' is occluded by s. For suppose otherwise. Then some non-null part t'' of t' is disjoint from and compatible with s. By Overlap, t'' overlaps with some member u of T', i.e. they contain a common non-null part t^*. But then t^* is a non-null part of u that is disjoint from and compatible with s and so u is not occluded by s after all.

(iii) By Lemma 5.

(iv) By Bounded Completeness.

(v) Let $T' = \{u$: u is a t-part independent of $s\}$ and $t' = \bigsqcup T'$. Then t' is a fusion of t-parts disjoint from s and so by (iii) is itself disjoint from s; and t' is a fusion of t-parts compatible with s and so by (i) is itself compatible with s. It follows that t' is independent of s.

(vi) Let $T' = \{u$: u is a t-part excluded by $s\}$ and $t' = \bigsqcup T'$. Then t' is excluded by s. For suppose otherwise. Then some non-null part t'' of t' is compatible with s. By Overlap, t'' overlaps with some member u of T', i.e. they contain a common non-null part t^*. But then t^* is a non-null part of u that is compatible with s and so u is not excluded by s after all.

We adopt the following notation for the various kinds of maximal part:
$t \sqcap s$ for the maximal t-part that is a part of s (in conformity with our previous usage)
$t \sqcap_c s$ for the maximal t-part compatible with s
$t \sqcap_e s$ for the maximal t-part excluded by s
$t \sqcap_d s$ for the maximal t-part disjoint from s (also designated by $t - s$)
$t \sqcap_o s$ for the maximal t-part occluded by s
$t \sqcap_i s$ for the maximal t-part independent of s.
We now characterize the connections between the various kinds of part:
<u>Lemma 11</u> Let s and t be any two states. Then $t \sqcap_e s$, $t \sqcap s$ and $t \sqcap_i s$ are disjoint from one another and the following identities hold:
(i) $t = (t \sqcap_e s) \sqcup (t \sqcap s) \sqcup (t \sqcap_i s)$
(ii) $t \sqcap_c s = (t \sqcap s) \sqcup (t \sqcap_i s)$
(iii) $t \sqcap_o s = (t \sqcap s) \sqcup (t \sqcap_e s)$
(iv) $t \sqcap_d s = (t \sqcap_e s) \sqcup (t \sqcap_i s)$
<u>Proof</u> Suppose that $t \sqcap_e s$ and $t \sqcap s$ overlap. Then some non-null u is a common part of both $t \sqcap_e s$ and $t \sqcap s$. Since u is a part of $t \sqcap_e s$, u is incompatible with s; and since u is a part of $t \sqcap s$, u is a part of s. A contradiction.

Suppose now that $t \sqcap_e s$ and $t \sqcap_i s$ overlap. Then some non-null u is a common part of both $t \sqcap_e s$ and $t \sqcap_i s$. Since u is a part of $t \sqcap_e s$, u is incompatible with s; and since u is a part of $t \sqcap_i s$, u is compatible with s. A contradiction.

Suppose finally that $t \sqcap s$ and $t \sqcap_i s$ overlap. Then some non-null u is a common part of both $t \sqcap s$ and $t \sqcap_i s$. Since u is a part of $t \sqcap s$, it is a part of s; and since u is a part of $t \sqcap_i s$, it is disjoint from s. A contradiction.

We prove the identities in turn. It is evident in each case that the state designated on the right is a part of the state designated on the left and so it suffices to show that the state designated on the left is a part of the state designated on the right; and we do this by application of the Overlap Test.

(i) Suppose that u is a non-null part of t but does not overlap with $(t \sqcap_e s) \sqcup (t \sqcap s) \sqcup (t \sqcap_i s)$. By Overlap, u does not overlap with $(t \sqcap_e s)$ or with $(t \sqcap s)$ or with $(t \sqcap_i s)$. Since u does not overlap with ($t \sqcap s$), it is disjoint from s; and since u does not overlap with ($t \sqcap_i s$) it is incompatible with s. For if u were compatible with s then, given that it is disjoint from s, it would be independent of s and hence a part of ($t \sqcap_i s$).

We now show, given that u is incompatible with s, that it contains a non-null part u′ excluded by s and hence that u overlaps with ($t \sqcap_e s$) after all. For consider $u \sqcap_c s$. If $u \sqcap_c s$ = u, then u is compatible with s; and so $u \sqcap_c s$ must be a proper part of u. By Supplementation, u contains a non-null part u^* that is disjoint from $u \sqcap_c s$. But then u* is excluded by s. For take any non-null part u^{**} of u^*. If u^{**} were compatible with s, then it would be a part of $u \sqcap_c s$ and so u* would not be disjoint from $u \sqcap_c s$.

(ii) Suppose u is a non-null part of ($t \sqcap_c s$). Then by (i) and Overlap, it overlaps with ($t \sqcap_e s$) or with ($t \sqcap s$) or with ($t \sqcap_i s$). But it cannot overlap with ($t \sqcap_e s$). For then u and ($t \sqcap_e s$) would have a common non-null part u* and since ($t \sqcap_e s$) is excluded by s, u* would be incompatible with s, contrary to the supposition that it is a part of ($t \sqcap_c s$).

(iii) Suppose u is a part of ($t \sqcap_o s$). Then by (i) and Overlap, it overlaps with ($t \sqcap_e s$) or with ($t \sqcap s$) or with ($t \sqcap_i s$). But it cannot overlap with ($t \sqcap_i s$). For then u and ($t \sqcap_i s$) would have a common non-null part u* and since ($t \sqcap_i s$) is independent of s, u* would be both disjoint from and compatible with s, contrary to the supposition that it is a part of ($t \sqcap_o s$).

(iv) Suppose u is a part of ($t \sqcap_d s$). Then by (i) and Overlap, it overlaps with ($t \sqcap_e s$) or with ($t \sqcap s$) or with ($t \sqcap_i s$). But it cannot overlap with ($t \sqcap s$) since then it would not be disjoint from s.

From this lemma, we obtain three other decompositions of a state t into disjoint parts:

$t = (t \sqcap_c s) \sqcup (t \sqcap_e s)$

$t = (t \sqcap_o s) \sqcup (t \sqcap_i s)$

$t = (t \sqcap_d s) \sqcup (t \sqcap s)$ (= $(t - s) \sqcup (t \sqcap s)$);

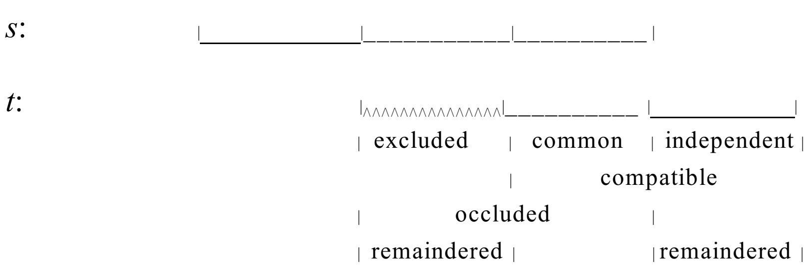
and we also have the following decompositions into overlapping parts:

$t = (t \sqcap_c s) \sqcup (t \sqcap_o s)$

$t = (t \sqcap_c s) \sqcup (t \sqcap_d s)$

$t = (t \sqcap_o s) \sqcup (t \sqcap_d s)$

We may summarize these various connections in the following diagram:



Thus all of the excluded parts will lie within the excluded region, all of the common parts within the common region, and so on. Standard mereology only recognizes two parts: the common part, or intersection, and the remaindered part, or difference. The remaindered part, for us, splits into two other component parts - the excluded part and the independent part; and these can then combine to form two new composite parts - the occluded part and the compatible part.

§5 Coincidence

We make precise the sense in which a D-space divides into regions of logical space. But first we need some preliminary results.

Lemma 12 (i) each state occludes the null state $\wedge$;

(ii) the null state $\wedge$ occludes s only if $s = \wedge$.

Proof (i) Since $\wedge$ has no non-null parts.

(ii) Suppose s is non-null. Then s is a non-null part of s that is both compatible with and disjoint from $\wedge$ and so is not occluded by $\wedge$.

We next establish the transitivity of occlusion (a not altogether trivial result):

Lemma 13 If s occludes t and t occludes u then s occludes u.

Proof Note that the result is trivial if t is null. For if t occludes u it follows by lemma 12(ii) that u is null and hence follows by lemma 12(i) that s occludes u.

Now suppose for reductio that s occludes t and t occludes u but that s does not occlude u. Then some non-null part u' of u is both compatible with and disjoint from s. Since t occludes u, either (a) u' overlaps with t or (b) u' is incompatible with t. We deal with each case in turn.

Case (a). Then u' and t have a common non-null part u^*. Since u is both compatible with and disjoint from s, u^* is both compatible with and disjoint from s, contrary to the supposition that s occludes t.

Case (b). From lemma11, $t = (t \sqcap_c u') \sqcup (t \sqcap_e u')$. We have supposed that t is non-null. So $(t \sqcap_e u')$ must be non-null and hence u' must exclude some non-null part t' of t. Since u' is compatible with s, it follows by the assumption of Independence that t' is compatible with s. Since s occludes t, t' overlaps with s and so they have a common non-null part t^*. Since u' excludes t', u' is incompatible with t^* and hence incompatible with s, contrary to what we had supposed.

Let us say that s and t *coincide* - in symbols, $s \approx t$ - if each occludes the other.

Theorem 1 Coincidence is an equivalence relation and if $s_1 \approx s_1'$, $s_2 \approx s_2'$, ... and $s_1 \sqcup s_2 \sqcup ...$ and $s_1' \sqcup s_2' \sqcup ...$ both exist, then:

$$s_1 \sqcup s_2 \sqcup ... \approx s_1' \sqcup s_2' \sqcup ...$$

Proof It is easy to verify that coincidence is reflexive and symmetric and it follows from the previous result that it is transitive. Let us now suppose that $s_1 \approx s_1'$, $s_2 \approx s_2'$, ..., that $s_1 \sqcup s_2 \sqcup ...$ and $s_1' \sqcup s_2' \sqcup ...$ both exist, and that t is a non-null part of $s_1 \sqcup s_2 \sqcup ....$ By Overlap, t overlaps with some s_i and so t and s_i have a common non-null part t^*. Since $s_i \approx s_i'$, t^* overlaps with or is incompatible with s_i' and so t^* overlaps with or is incompatible with $s_1' \sqcup s_2' \sqcup ....$ This establishes that $s_1' \sqcup s_2' \sqcup ...$ occludes $s_1 \sqcup s_2 \sqcup ...$; and the other direction is established similarly.

Note that we cannot expect to have the result that if $s_1 \approx s_1'$ and $s_2 \approx s_2'$, then $s_1 \sqcup s_2$ exists iff $s_1' \sqcup s_2'$ exists (and so $\approx$ is not a congruence in the full sense of the term). For $s_1 = s_2 = s_1'$ might be the state of a patch being red, for example, while s_2' is the state of the patch being green.

We have the following elementary results on coincidence:

Lemma 14 (i) Compatible coincidents are the same;

(ii) No two distinct parts of the same state are coincident;

(iii) Any state occluded by s is coincident with a part of s;

(iv) If t is coincident with $s_1 \sqcup s_2 \sqcup ...$ then t is of the form $t_1 \sqcup t_2 \sqcup ...$, with each t_i coincident with s_i;

(v) If s is coincident with t and if the s-part s' is coincident with the t-part t' then $s - s'$ is coincident with $t - t'$.

Proof (i) Let s and t be two coincident and compatible states. Then any non-null part u of s overlaps with or is incompatible with t. But u cannot be incompatible with t since s and t are compatible; and so any non-null part of s overlaps with t. Similarly, any non-null part of t overlaps with s; and so s and t are the same.

(ii) From (i), given that parts of the same state must be compatible.

(iii) Suppose t is occluded by s. Now $s = (s \sqcap_o t) \sqcup (s \sqcap_i t)$. We show that t is coincident with $(s \sqcap_o t)$. By definition, t occludes $(s \sqcap_o t)$; and so it suffices to show that $(s \sqcap_o t)$ occludes t. Take a non-null part t' of t. There are two cases (given that t is occluded by s):

(a) t' overlaps with s. So t' and s have a common non-null part u. But u is then a part of $(s \sqcap t) \sqsubseteq (s \sqcap_o t)$ and hence t' overlaps with $(s \sqcap_o t)$.

(b) t' is incompatible with s. Then t' is also incompatible with $(s \sqcap_o t)$. For suppose t' is compatible with $(s \sqcap_o t)$. Now clearly, $(s \sqcap_o t)$ is compatible with $(s \sqcap_i t)$; and t, and hence its part t', is compatible with $(s \sqcap_i t)$. But then by Directed Completeness, t' is compatible with $(s \sqcap_o t) \sqcup (s \sqcap_i t) = s$. A contradiction.

(iv) Suppose t is coincident with $s_1 \sqcup s_2 \sqcup ....$ For each part s_i of s there is, by (iii), a part t_i of t coincident with s_i. Let $t' = t_1 \sqcup t_2 \sqcup ...$ If $t' = t$, we are done. So suppose $t' \sqsubset t$ and let $t^* = t - t'$. By (iii) again, there is a part s^* of s which is coincident with t^* and which, by lemma 12, is non-null given that t^* is non-null. By Overlap, s^* overlaps with some s_i and so s^* and s_i have a common non-null part s^{**}. By (iii) again, s^{**} is coincident with some non-null part t^{**} of t_i and also coincident with some non-null part t^{***} of t^*. But then t^{**} and t^{***} are coincident parts of t and so, by (ii), are identical - which is impossible given that t^* and t_i, and hence t^{***} and t^{**}, are disjoint.

(v) $s = s' \sqcup (s - s')$. By (iv), t is of the form $t_1 \sqcup t_2$ with t_1 coincident with s' and t_2 coincident with $(s - s')$. Since t' is also coincident with s, it follows from (ii) that t_1 and t' are the same. It therefore suffices to show that t' and t_2 are disjoint. But suppose t^* is a common non-null part of t' and t_2. It is then readily shown from lemma 12 and (iv) above that t^* is coincident with a common non-null part s^* of s' and $(s - s')$, which is impossible.

§6 World Space

Given an arbitrary state space $\mathcal{S} = (S, \sqsubseteq)$ and a subset T of S, let the *restriction* $\mathcal{S} \upharpoonright T$ *of* $\mathcal{S}$ *to* T be the structure $(S \cap T, \sqsubseteq \cap T^2)$. We are interested in when $\mathcal{S} \upharpoonright T$ will also be a state space. To this end, we define a set of states F from $\mathcal{S}$ to be a *filter* if it satisfies the following three conditions:

(a) F is non-empty;

(b) (upward closure) any fusion of members of F belongs to F if it exists; and

(c) (downward closure) any part of a member of F is a member of F.

(Note that this is not quite the usual definition of a filter). We then have:

Lemma 15 Let $\mathcal{S} = (S, \sqsubseteq)$ be an arbitrary state space and F a filter from $\mathcal{S}$. Then:

(i) $\mathcal{S}\restriction F$ is a state space;

(ii) $\mathcal{S}\restriction F$ is a R-space if $\mathcal{S}$ is a R-space;

(iii) $\mathcal{S}\restriction F$ is a D-space if $\mathcal{S}$ is a D-space

Proof We establish some preliminary results:

(1) if s is non-null in $\mathcal{S}$ and belongs to F then it is non-null in $\mathcal{S}\restriction F$

(2) if s and t are disjoint in $\mathcal{S}\restriction F$ then they are also disjoint in $\mathcal{S}$;

(3) for $s_1, s_2, ... \in F$, s is a lub of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$ iff s is a lub of $s_1, s_2, ...$ in $\mathcal{S}$;

(4) if $s_1, s_2, ...$ are jointly compatible in $\mathcal{S}$ and belong to F then they are jointly compatible in $\mathcal{S}\restriction F$.

Pf (1) If s is non-null in $\mathcal{S}$ and belongs to F, then $\wedge \in F$ by (c) in the definition of *filter*; so $\wedge \sqsubseteq s$ even though $s \neq \wedge$; and so s is also non-null in $\mathcal{S}\restriction F$.

(2) If u were a common non-null part of s and t in $\mathcal{S}$, then $u \in F$ by (c) and is non-null in $\mathcal{S}\restriction F$ by (1) above and so u would be a common non-null part of t in $\mathcal{S}\restriction F$.

(3) If s is a lub of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$ then s is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}$. But if s were not the lub t of $s_1, s_2, ...$ in $\mathcal{S}$ then $t \sqsubseteq s$; and so $t \in F$ by (c) and s would not be a lub of s_1, s_2, ... in $\mathcal{S}\restriction F$ after all. If, on the other hand, s is a lub of $s_1, s_2, ...$ in $\mathcal{S}$ then $s \in F$ by (b) and is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$. But any upper bound of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$ is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}$; and so if s is a lub of $s_1, s_2, ...$ in $\mathcal{S}$ then s is also a lub of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$.

(4) if $s_1, s_2, ...$ are jointly compatible in $\mathcal{S}$ and belong to F, then they have a lub s by Bounded Completeness and so $s \in F$ by (b) and hence is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$.

We now turn to (i) - (iii) of the lemma:

(i) By condition (a) in the definition of *filter*, $\mathcal{S}\restriction F$ is non-empty and, clearly, $\sqsubseteq \cap T^2$ will be a po given that $\sqsubseteq$ is a po. To verify Bounded Completeness, suppose that s is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$. Then s is an upper bound of $s_1, s_2, ...$ in $\mathcal{S}$. By Bounded Completeness, s_1, s_2, ... has a lub $s' \sqsubseteq s$ in $\mathcal{S}$. But then by (3), s' is a lub of $s_1, s_2, ...$ in $\mathcal{S}\restriction F$.

(ii) Suppose that $\mathcal{S} = (S, \sqsubseteq)$ is an R-space, i.e. that it satisfies Supplementation and Overlap. We need to show that $\mathcal{S}\restriction F$ also satisfies Supplementation and Overlap. Suppose first, in order to verify Supplementation, that s is a proper part of t in $\mathcal{S}\upharpoonright F$. Then clearly s is a part, and hence also a proper part, of t in $\mathcal{S}$. So by Supplementation for $\mathcal{S}$, some non-null part u of t is disjoint from s in $\mathcal{S}$. But then u non-null in $\mathcal{S}\upharpoonright F$ by (1) and t is disjoint from s in $\mathcal{S}\upharpoonright F$ by (2); and so some non-null part u of t is disjoint from s in $\mathcal{S}\upharpoonright F$.

Now suppose, in order to verify Overlap, that s is the lub of $s_1, s_2, \ldots$ in $\mathcal{S}\upharpoonright F$ and overlaps with $t \in F$. By (3), s is the lub of $s_1, s_2, \ldots$ in $\mathcal{S}$ and overlaps with t in $\mathcal{S}$. So by Overlap for $\mathcal{S}$, some s_i overlaps with t in $\mathcal{S}$. But then, by (2), s_i overlaps with t in $\mathcal{S}\upharpoonright F$.

(iii) Suppose that $\mathcal{S} = (S, \sqsubseteq)$ is a D-space, i.e. that it satisfies Directed Completeness and Independence. We need to show that $\mathcal{S}\upharpoonright F$ also satisfies Directed Completeness and Independence. Suppose first, in order to verify Directed Completeness, that the states $s_1, s_2, \ldots$ are pairwise compatible in $\mathcal{S}\upharpoonright F$. Then clearly they are pairwise compatible in $\mathcal{S}$. So by Directed Completeness and BC, they have a lub s. By (3), s is a lub for $s_1, s_2, \ldots$ in $\mathcal{S}\upharpoonright F$ and so $s_1, s_2, \ldots$ are also jointly compatible in $\mathcal{S}\upharpoonright F$.

Now suppose, in order to verify Independence, that, in $\mathcal{S}\upharpoonright F$, s is disjoint from and compatible with t and that t' is excluded by t. Then s is disjoint from t in $\mathcal{S}$ by (2), s is compatible with t in $\mathcal{S}$, since any upper bound of s and t in $\mathcal{S}\upharpoonright F$ is an upper bound in $\mathcal{S}$, and also t' is excluded by t in $\mathcal{S}\upharpoonright F$. For if u were a non-null part of t' in $\mathcal{S}$ that is compatible with t then, by (1) and (4), it would also be a non-null part of t' in $\mathcal{S}\upharpoonright F$ that was compatible with t. It therefore follows by Independence for $\mathcal{S}$ that s is compatible with t' in $\mathcal{S}$ and so, by (4), s is also compatible with t' in $\mathcal{S}\upharpoonright F$.

Given a world-state w from a state space S, let $S_w = \{s \in S: s \sqsubseteq w\}$. Then in the particular case in which $T = S_w$, we denote $\mathcal{S}\upharpoonright T$ as $\mathcal{S}_w$ and call it the *world-space for w*. We might think of $\mathcal{S}_w$ as the space of *actual* states (relative to w).

As a special case of the above result, we then have:

Lemma 16 If $\mathcal{S}$ is a D-space, then $\mathcal{S}_w$, for each world-state w of $\mathcal{S}$, is a total D-space.

Proof It should be evident that S_w is a filter; and it should also be evident that any states of s will be compatible and hence will have a lub.

We turn next to the relationship between occlusion and world state-hood:

Lemma 17 (i) Any world-state occludes any state;

(ii) For any world-state v and state s there is a unique part s_v of v coincident with s;

(iii) Any two worlds are coincident;

(iv) For any states $s_1, s_2, \ldots$, there exist states $r_1, r_2, \ldots$ which are respectively coincident with $s_1, s_2, \ldots$ and jointly compatible;

(v) Two necessary states coincide iff they are the same.

Proof (i) Let w be a world-state and s a state. Take any non-null part s' of s. By the definition of world-state, s' is either incompatible with w or is a part of w and hence overlaps with w.

(ii) s_v exists by (i) above and lemma 14(iii). Suppose that two states t and t' satisfy the defining conditions of s_v. Then t and t' are compatible, since they are both parts of v, and they are coincident, since they are both coincident with s. But then by lemma 14(i), they are the same.

(iii) Directly from (i).

(iv) Pick a world w. By (ii), there are parts $r_1, r_2, \ldots$ of w which are coincident with $s_1, s_2, \ldots$ and which, being parts of w, are jointly compatible.

(v) Suppose that the two necessary states s and t coincide. Then any non-null part of s overlaps with or is incompatible with t. But it cannot be incompatible with t; and so every non-null part of s overlaps with t. Similarly, every non-null part of t overlaps with s; and so, by the Overlaps Test, s and t are the same. The other direction is trivial.

According to this result, world-states and necessary states represent the extremes of coincidence, with world-states always coinciding and with necessary states never coinciding except when they are the same.

§7 Regions of Logical Space

For each state s of a D-space $\langle S, \sqsubseteq\rangle$: let $[s]/\approx$ (or, more simply, $[s]$) be the equivalence class $\{t \in S: t \approx s\}$; and let $[S]/\approx$ (or, more simply, $[S]$) be $\{[s]: s \in S\}$. We call $[s]$ the (*logical*) *region occupied by* s and $[S]$ the (*logical*) *space* of regions. Intuitively, we might think of s as a determinate and of $[s]$ as the determinable to which it belongs. Thus it is the operation [ ] which most explicitly provides the state space $\mathcal{S}$ with a determinate/determinable structure.

We define $\sqsubseteq$ on regions by:

$\boldsymbol{s} \sqsubseteq \boldsymbol{t}$ if for some $s \in \boldsymbol{s}$ and for some $t \in \boldsymbol{t}$, $s \sqsubseteq t$.

It should be noted that $\boldsymbol{s} \sqsubseteq \boldsymbol{t}$ does *not* imply that $s \sqsubseteq t$ for any $s \in \boldsymbol{s}$ and $t \in \boldsymbol{t}$. A patch being red, for example, will be a part of itself but the patch being red will not be a part of the patch being green.

Given a state space $\mathcal{S} = (S, \sqsubseteq)$, we let the *region space* $\mathcal{S}/\approx$ be the corresponding structure $(S/\approx, \sqsubseteq)$. Now let π_w be the map for which $\pi_w(s) = [s]$ for each $s \in S_w$. Then:

Theorem 2 (World/Region Isomorphism) For each world-state w, π_w is an isomorphism from the world space $\mathcal{S}_w$ onto the region space $\mathcal{S}/\approx$.

Proof π_w is onto. For take any region $[s]$. By lemma 17(ii), $s \approx s_w$ for $s_w \sqsubseteq w$; and so $\pi_w(s_w) = [s]$.

π_w is one-one. For suppose $\pi_w(s) = \pi_w(t)$ for $s, t \in S_w$. Then $[s] = [t]$ and so $s \approx t$. But s and t are compatible since they are both parts of w; and so, by lemma 14(i), they are the same. Now suppose $s \sqsubseteq t$ for $s, t \in S_w$. Then $[s] \sqsubseteq [t]$ by the definition of $\sqsubseteq$ on $S/\approx$.

Finally suppose $[s] \sqsubseteq [t]$ for $s, t \in S_w$. Then for some $s' \approx s$ and $t' \approx t$, $s' \sqsubseteq t'$. So $s' \sqcup t'$ exists and is identical to t'. Since s and t are both parts of w, they are compatible; and so $s \sqcup t$

also exists. But $s \approx s'$ and $t \approx t'$; and so, by theorem 1, $s \sqcup t \approx s' \sqcup t'$. Since $s' \sqcup t' = t'$, $s \sqcup t = t$; and so $s \sqsubseteq t$.

We can think of the map π_w as specifying the logical form of the world w. Thus, from this point of view, the theorem tells us that each world will have the same logical form.

Corollary 2.1 $\mathcal{S}/\approx$ is a D-space.

Proof From the theorem and lemma 16.

Given any two world-states w and v, let $\pi_{w,v}$ be the map for which $\pi_{w,v}(s) = s_v$ for each $s \in S_w$.

Corollary 2.2 (World/World Isomorphism) For any two world-states w and v, the map $\pi_{w,v}$ is an isomorphism from $\mathcal{S}_w$ onto $\mathcal{S}_v$.

Proof By the theorem, π_w is an isomorphism from $\mathcal{S}_w$ to $\mathcal{S}/\approx$ and π_v an isomorphism from $\mathcal{S}_v$ to $\mathcal{S}/\approx$. So the composition map $\pi_w \circ \pi_v^{-1}$ is an isomorphism from $\mathcal{S}_w$ to $\mathcal{S}_v$. It therefore remains to show that $\pi_{w,v} = \pi_w \circ \pi_v^{-1}$. But for $s \in S_w$, $\pi_w(s) = [s]$ and $\pi_v^{-1}([s]) = t$, where $[t] = [s]$, i.e. where $t \approx s$, for $t \in S_v$; and so $\pi_{w,v}(s) = t$, as required.

The corollary provides us with a natural isomorphism between worlds, as mediated through their logical form. It is readily shown that the map $\pi_{w,w}$ is the identity map on S_w, $\pi_{w,v}$ the inverse, in the usual sense, of $\pi_{v,w}$, and $\pi_{w,u}$ the composition of the maps $\pi_{w,v}$ and $\pi_{v,u}$. Thus the world-states W and the maps $\pi_{w,u}$ will constitute a 'category' in the sense of category theory.

Corollary 2.3 (State/Region Projection) The map $[\ ]: S \rightarrow S/\approx$ preserves fusions in the sense that:

(i) if s' is the fusion of $s_1, s_2, \ldots$ in $\mathcal{S}$ then $[s']$ is the fusion of $[s_1], [s_2], \ldots$ in $\mathcal{S}/\approx$; and

(ii) if S' is the fusion of $S_1, S_2, \ldots$ in $\mathcal{S}/\approx$ then, for some representatives $s', s_1, s_2, \ldots$ of S', $S_1, S_2, \ldots$, s' is the fusion of $s_1, s_2, \ldots$ in $\mathcal{S}$.

Proof (i) Suppose s' is the fusion of $s_1, s_2, \ldots$ in $\mathcal{S}$. Choose a world-state w of which s' is a part. Then $s', s_1, s_2, \ldots \in S_w$ and, since s' is the fusion of $s_1, s_2, \ldots$ in $\mathcal{S}_w$ it follows by the theorem that $[s']$ is the fusion of $[s_1], [s_2], \ldots$ in $\mathcal{S}/\approx$.

(ii) Suppose S' is the fusion of $S_1, S_2, \ldots$ in $\mathcal{S}/\approx$. Pick a world-state w. Let $s' = \pi_w^{-1}(S')$, $s_1 = \pi_w^{-1}(S_1)$, $s_2 = \pi_w^{-1}(S_2)$, ... Then $s', s_1, s_2, \ldots$ are representatives of $S', S_1, S_2, \ldots$ and, by the theorem, s' is the fusion of $s_1, s_2, \ldots$ in $\mathcal{S}$.

§8 From Regions to States

In the previous result, we have derived the fundamental conditions on regions from those on states. But it is also possible to prove a kind of converse result in which, assuming the existence of regions in logical space, we may derive the fundamental conditions on states from those on regions. This converse result has some philosophic interest. For the conditions on regions have some independent plausibility and thereby provide an alternative justification for accepting the conditions on states. Indeed, the conditions on the regions can simply be seen as a natural way of cashing out the metaphor of 'logical space' and so we see how this metaphor gives rise to the definition of D-space that we have in fact given.

A *state-region space* will be a quintuple (S, $\boldsymbol{R}$, [ ], $\sqsubseteq$, $\sqsubseteq$). Intuitively, we may think of S as a set of possible states, $\boldsymbol{R}$ as a set of logical regions, and $[s]$ as the location of the state s. For the quintuple to be a state-region space, it is required that (S, $\sqsubseteq$) be a state space, that ($\boldsymbol{R}$, $\sqsubseteq$) be a total regular space, and that the following additional conditions be satisfied:

Location if $\bigsqcup T$ exists for $T \sqsubseteq S$ then $[\bigsqcup T] = \bigsqcup\{[t]: t \in T\}$
Incompatibility if $[s] = [t]$ then s and t are identical or else incompatible
Compatibility if $[s_1]$, $[s_2]$, ... are pairwise disjoint, then s_1, s_2, ... are jointly compatible
Counterpart if $R \sqsubseteq [s]$ then, for some $r \sqsubseteq s$, $[r] \sqsubseteq R$
Occupancy for each $R \in \boldsymbol{R}$ there is a state s for which $[s] = R$.

We say *s occupies* $\boldsymbol{r}$ if $[s] = \boldsymbol{r}$ and call $\boldsymbol{r}$ the *location of s*. Thus Location tells us that the location of a fusion (if it exists) is the fusion of the location of its components, Incompatibility tells us that distinct occupants of the same region are incompatible, Compatibility tells us that occupants of disjoint regions are compatible, Counterpart tells us that any part of an occupied region is occupied by a part of its occupant, and Occupancy tells us that each region is occupied.

Lemma 19 The following hold in any state-region space:

(i) $[\wedge] = \wedge$
(ii) if $[s] = \wedge$ then $s = \wedge$
(iii) if $s \sqsubseteq t$ then $[s] \sqsubseteq [t]$
(iv) if $s \sqsubset t$ then $[s] \sqsubset [t]$
(v) if s is independent of t then $[s]$ is disjoint from $[t]$.

Proof (i) $\wedge = \bigsqcup\varnothing$ exists and so $[\wedge] = [\bigsqcup\varnothing]$

$$= \bigsqcup\{[s]: s \in \varnothing\} \text{ by Location}$$

$$= \bigsqcup\varnothing$$

$$= \wedge$$

(ii) Suppose that $[s] = \wedge$ but that s is distinct from $\wedge$. Since $[\wedge] = \wedge$ by (i), it follows from Incompatibility that s and $\wedge$ are incompatible, which is impossible.

(iii) Suppose $s \sqsubseteq t$. Then $s \sqcup t = t$. So $[s \sqcup t] = [t]$. By Location, $[s \sqcup t] = [s] \sqcup [t]$. But then $[s] \sqcup [t] = [t]$ and hence $[s] \sqsubseteq [t]$.

(iv) Suppose $s \sqsubset t$. Then $s \sqsubseteq t$ and so $[s] \sqsubseteq [t]$ by (iii) above. Also, $[s] \sqcup [t] = [s \sqcup t] \neq [s]$ and hence $[s] \sqsubset [t]$. For suppose $[s \sqcup t] = [s]$. Given that $s \sqcup t$ and t are distinct, they are then incompatible by Incompatibility,which is impossible.

(v) Suppose that s is compatible with and disjoint from t. Then $[s]$ and $[t]$ are disjoint. For suppose that $\boldsymbol{r}$ is a common non-null part of $[s]$ and $[t]$. By Counterpart, there is a non-null part s' of s whose location is $\mathbf{r}$ and a non-null part t' of t whose location is $\mathbf{r}$. Now s' and t'

cannot be identical, since s is disjoint from t. But nor can they be distinct since then, by Incompatibility, s' and t' and, hence also s and t, are incompatible.

It is readily shown that any D-space (S, $\sqsubseteq$) will give rise to a state-region space (S, $\boldsymbol{R}$, [ ], $\sqsubseteq$, $\sqsubseteq$), where $\boldsymbol{R}$, [ ], and $\sqsubseteq$ are defined in the natural way. But we also have the converse result:
Theorem 3 If (S, $\boldsymbol{R}$, [ ], $\sqsubseteq$, $\sqsubseteq$) is a state-region space then (S, $\sqsubseteq$) is a D-space. Moreover, t is occluded by s in the state-region space iff $[t] \sqsubseteq [s]$.
Proof We already know that (S, $\sqsubseteq$) is a state space; and so we need to go through the additional conditions required for it to be a D-space:

Overlap Suppose $t_1 \sqcup t_2 \sqcup ...$ exists and overlaps with s. Then some non-null part s' of s is a part of $t_1 \sqcup t_2 \sqcup ....$ So by (iii) from the lemma above, $[s'] \sqsubseteq [s]$ and $[s'] \sqsubseteq [t_1 \sqcup t_2 \sqcup ....] = [t_1] \sqcup [t_2] \sqcup ...$ (by Location). By Overlap for regions, $[s']$ overlaps with a $[t_i]$. So for some non-null region $\boldsymbol{r}$, $\boldsymbol{r} \sqsubseteq [s']$ and $\boldsymbol{r} \sqsubseteq [t_i]$. By Counterpart, there is a part s^* of s' for which $[s^*] = \boldsymbol{r}$ and a part t_i^* of t_i for which $[t_i^*] = \boldsymbol{r}$. Since $\boldsymbol{r}$ is non-null, it follows by (i) above that s^* and t_i^* are non-null. Now suppose for reductio that $s^* \neq t_i^*$. By Incompatibility, they are incompatible. But s^* and t_i^* are both parts of $t_1 \sqcup t_2 \sqcup ....$ A contradiction. So $s^* = t_i^*$ and s and t_i do indeed overlap.

Supplementation Suppose s is a proper part of t. By (iv) above, $[s]$ is a proper part of $[t]$. By Supplementation for regions, there is a non-null region $\boldsymbol{t'}$ that is a part of $[t]$ and disjoint from $[s]$. By Counterpart, there is a part t' of t whose location is $\boldsymbol{t'}$. Since $\boldsymbol{t'}$ is non-null, it follows by (ii) above that t' is non-null. t' is also disjoint from s. For suppose that u is a common non-null part of t' and s. Then by (ii) and (iii) above, $[u]$ is a common non-null part of $[t']$ and $[s]$, contrary to supposition.

Directed Completeness Let S' be a set $\{s_\zeta: \zeta < \kappa\}$ of pairwise compatible states. We define a related set of states $\{t_\zeta: \zeta < \kappa\}$ by ordinal recursion. Suppose that t_ζ for $\zeta < \alpha$, $0 \leq \alpha < \kappa$, are already defined. Let $u_\alpha = \bigsqcup\{t_\zeta: \zeta < \alpha\}$ if $\bigsqcup\{t_\zeta: \zeta < \alpha\}$ exists and otherwise we arbitrarily set $u_\alpha = \wedge$. We then set $t_\alpha = s_\alpha - u_\alpha$. Thus the t_ζ are formed by successively taking what is new from the s_ζ.

We show by a simultaneous transfinite recursion that for each $\alpha \leq \kappa$:

(1) the t_ζ are disjoint for distinct $\zeta < \alpha$;

(2) $\bigsqcup\{t_\zeta: \zeta < \alpha\}$ exists;

(3) $\bigsqcup\{t_\zeta: \zeta < \alpha\} = \bigsqcup\{s_\zeta: \zeta < \alpha\}$.

Pf (1) Clearly (1) holds when α is 0 or, given the IH, when α is a limit ordinal. Now suppose $\alpha = \beta + 1$. By IH, the t_ζ are disjoint for distinct $\zeta < \beta$. So we need to establish that t_β is disjoint from each t_ζ for $\zeta < \beta$. Now $t_\beta = s_\beta - u_\beta$. But $\bigsqcup\{t_\zeta: \zeta < \beta\}$ exists by IH for (2); and so $t_\beta = s_\beta - \bigsqcup\{t_\zeta: \zeta < \beta\}$ and t_β is disjoint from $\bigsqcup\{t_\zeta: \zeta < \beta\}$ and hence from each t_ζ for $\zeta < \beta$.

(2) It is evident from the construction that each t_ζ, $\zeta < \alpha$, is a part of s_ζ and so, since the s_ζ are pairwise compatible, the t_ζ are pairwise compatible. By (1), which we have just established,

the t_ζ are disjoint and hence also independent. By lemma 18(v), the regions $[t_\zeta]$, for $\zeta < \alpha$, are pairwise disjoint; and so, by Compatibility, the ζ are compatible and $\bigsqcup\{t_\zeta: \zeta < \alpha\}$ exists.

(3) Clearly (3) holds when α is 0 or, given the IH, when α is a limit ordinal. Now suppose $\alpha = \beta + 1$. Then:

$$\begin{aligned}\bigsqcup\{t_\zeta: \zeta < \beta + 1\} &= t_\beta \sqcup \bigsqcup\{t_\zeta: \zeta < \beta\} \\ &= (s_\beta - u_\beta) \sqcup \bigsqcup\{t_\zeta: \zeta < \beta\} \\ &= (s_\beta - \bigsqcup\{t_\zeta: \zeta < \beta\}) \sqcup \bigsqcup\{t_\zeta: \zeta < \beta\} \text{ since } \bigsqcup\{t_\zeta: \zeta < \beta\} \text{ exists by IH for (2)} \\ &= (s_\beta - \bigsqcup\{s_\zeta: \zeta < \beta\}) \sqcup \bigsqcup\{s_\zeta: \zeta < \beta\} \text{ by IH for (3)} \\ &= \bigsqcup\{s_\zeta: \zeta < \beta + 1\}.\end{aligned}$$

Setting $\alpha = \kappa$ in (3),we get that $\bigsqcup\{s_\zeta: \zeta < \kappa\}$ exists and hence that the s_ζ are compatible, as required.

Independence Before deriving Independence, we establish the equivalence (following 'moreover' in the statement of the theorem):

t is occluded by s in the state-region space iff $[t] \sqsubseteq [s]$.

Pf Suppose not $[t] \sqsubseteq [s]$. Then $[t] - [s]$ is non-null. By Counterpart, there is a non-null part $t'$of t that occupies $[t] - [s]$. Since $[t] - [s]$ and $[s]$ are disjoint, t' is compatible with s by Compatibility. But t' is also disjoint from s since otherwise they have a common non-null part and so it would follow from (ii) and (iii) of lemma 18 that $[t] - [s]$ and $[s]$ have a common non-null part. So t is not occluded by s.

Now suppose that t is not occluded by s, so that some non-null part $t'$of t is independent of s. By lemma 18(v), the regions occupied by t' and s are disjoint and, since $[t'] \sqsubseteq [t]$ and $[t']$ is non-null, not $[t] \sqsubseteq [s]$.

We return to Independence. Suppose that s is independent of t. Then $[s]$ and $[t]$ are disjoint by lemma 18(v). Now suppose s excludes u. Then s occludes u and so, by the previous equivalence, $[u]$ is a part of $[s]$. But then $[u]$ is also disjoint from $[t]$ and so it follows by Compatibility that u and t are compatible.

§9 Further Questions

There are many different directions in which the present work may be extended or elaborated; and I shall here give the briefest sketch of how some of them might go.

(1) We might try to relate the present discussion to the more traditional task of defining the determinate/determinable distinction (as surveyed in Sanford [2011]). To this end, it may be

helpful to distinguish the task of defining the relation of determinate to determinable from the task of defining the co-determinable relation, which holds between two states (or attributes) when they belong to the same determinable. The second task is somewhat easier and avoids the tricky question of saying what exactly a determinable is. It may also be helpful to distinguish the task of defining the coincidence relation within an unrestricted space of states or attributes (one closed under arbitrary Boolean combinations) from the task of defining the relation within a space that is restricted to relatively determinate states or attributes (which will not in general be closed under negation or disjunction). Again, the second task is somewhat easier; and it avoids the tricky question of saying what the candidate determinates should be. Our relation of coincidence may perhaps be taken to provide a necessary condition for two states or attributes to belong to the same co-determinable within a suitably restricted space. But I doubt that it can be regarded as both a necessary and sufficient condition; and so further work would be required to convert it into a full definition.

(2) Relaxation of the conditions. Some of the conditions on a state space and on a D-space are quite strong and it would be of interest to determine to what extent the present results could be reproduced under weaker conditions. Consider, for example, the case in which states are composed of 'atomic' states of a given particle being at a given location (where spatial coincidence of different particles is not allowed). Then particle a being at location L will presumably be mutually exclusive *and* hence coincident with particle b being at location L, for b distinct from a, and b being at location L will presumably be mutually exclusive and hence coincident with b being at location L', for L' distinct from L, but a being at location L will be independent from and hence not coincident with b being at location L'. Thus coincidence is not transitive, even though the states concerning each particular particle will constitute a D-space. Or again, the colors of different objects may be taken to constitute a D-space. But we may also wish to include the states of the objects not existing and these cannot plausibly be taken to belong to a more comprehensive D-space.

In order to take account of such examples, we should weaken the conditions on a D-space and we should define a region of logical space, not as an equivalence class of coincident states, but as a maximal class of states, any two of which are coincident.

(3) Refinements. We may 'wrap' coincident states (the determinants of a given determinable) into a single region of logical space (the determinable). But we may also wrap them into subregions of logical space (less specific determinates), which may then themselves be wrapped into the more comprehensive region (the original determinable). A patch being a specific shade of a particular color (green, red, blue etc.), for example, may be identified with its being any other shade of that color and the new states or determinates will then consist in the patch being a particular color, rather than a particular shade of color. From a mathematical point of view, what we will have is a 'congruence relation' on the states (with coincidence being the most coarse-grained congruence) and states in the new state space will be identified with the equivalence classes under the congruence relation.

However, not every way of identifying states will lead to a D-space, one with a determinate/determinable structure. Suppose, for example, that we allowed the states of a patch being green or red, of its being red or blue, and of its being green or blue. Directed Completeness would then be violated since any two of these states would be compatible even though all three of them were not. On the other hand, identifying 'neighboring' colors would not lead to difficulties of this sort. We should therefore look at the general conditions under which states can be identified so as to form a new D-space and of the various ways in which these general conditions can be satisfied.

(4) Component Analysis. Determinables may often be decomposed into other determinables (and similarly for the corresponding determinates). The location of a given object, for example, may be decomposed into its coordinates and the color of an object into its hue, saturation and brilliance. From a mathematical point of view, decomposition is a matter of one state space being a 'product' of others; and so the analysis of a determinable into components should go hand in hand with the study of products.

(5) Extensions of the Framework. The conceptual basis of the existing framework is very austere, the only primitive being that of part to whole; and there are a number of ways in which it might be extended. Topological considerations are obviously relevant in the large number of cases; and talk of 'regions' in logical space suggests as much. Thus in the case of color, there is an obvious sense in which the different colors are 'convex', shades in between shades of a color are also shades of the color; and so it is natural to bring topological considerations explicitly into play in the formal definition of a state space and to work out what consequences they might have for the underlying structure. It seems clear, for example, that insistence on the requirement of convexity provides a means by which the determinate/determinable structure may be preserved under the identification of one state with another.[3]

A somewhat different consideration concerns the 'uniformity' of a state. Some states naturally belong together. All the states concerning a particular determinate attribute belong together, for example, as do all the states concerning a particular object. By taking into accounts various considerations of uniformity, we may be in a position to determine from the states themselves what their underlying structure in terms of objects or attributes should be taken to be.

(6) Representation. A state space may not itself be a D-space and yet it may be 'embeddable' within a D-space. In other words, the facts as they present themselves may not have a determinate /determinable structure and yet they may seen to be the reflection of an underlying determinate/determinable structure. Suppose, for example, that the states are composed of states of one person being at least as tall as another. Then we do not have a determinate/determinable structure in this case. But suppose, instead, that the states are composed of states of a given person being of a particular height. We do then have a determinate/determinable structure; and there is an obvious sense in which the one state space is embeddable in the other, since the state of one person being at least as tall as another will be a matter of the one person having a given height h and the other person having a height h', where $h' \geq h$. This raises the general question of when one state space is representable within a D-space or within a D-space of a certain kind. This question, in its turn, is very relevant to the application of science, since the success of science has largely depended upon its ability to

[3] A convexity requirement on concepts is considered at length in Gardenfors [2000].

represent the world by means of the 'local' independence provided by D-spaces; and the question may also be seen as a more general way of formulating the problem of when a qualitative phenomenon may be represented in quantitative fashion, with the abstract idea of a D-space now replacing the more specific idea of a quantitative representation.

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