THE PERIPATETIC PROGRAM IN CATEGORICAL LOGIC: LEIBNIZ ON PROPOSITIONAL TERMS

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Abstract. Greek antiquity saw the development of two distinct systems of logic: Aristotle's theory of the categorical syllogism and the Stoic theory of the hypothetical syllogism. Some ancient logicians argued that hypothetical syllogistic is more fundamental than categorical syllogistic on the grounds that the latter relies on modes of propositional reasoning such as *reductio ad absurdum*. Peripatetic logicians, by contrast, sought to establish the priority of categorical over hypothetical syllogistic by reducing various modes of propositional reasoning to categorical form. In the 17th century, this Peripatetic program of reducing hypothetical to categorical logic was championed by Gottfried Wilhelm Leibniz. In an essay titled *Specimina calculi rationalis*, Leibniz develops a theory of propositional terms that allows him to derive the rule of *reductio ad absurdum* in a purely categorical calculus in which every proposition is of the form A *is* B. We reconstruct Leibniz's categorical calculus and show that it is strong enough to establish not only the rule of *reductio ad absurdum*, but all the laws of classical propositional logic. Moreover, we show that the propositional logic generated by the nonmonotonic variant of Leibniz's categorical calculus is a natural system of relevance logic known as $\mathrm{RMI}_{\rightarrow}^{\neg}$.

From its beginnings in antiquity up until the late 19th century, the study of formal logic was shaped by two distinct approaches to the subject. The first approach was primarily concerned with simple propositions expressing predicative relations between terms. The second focused on compound propositions and the operations by which they are constructed from simple ones. The former approach treated logic fundamentally as a logic of terms, the latter as a logic of propositions.

While the term-based approach has its origins in Aristotle's theory of the categorical syllogism, the proposition-based approach was exemplified in antiquity by the Stoic theory of the hypothetical syllogism. During the Hellenistic period, various arguments were put forward in favor of the relative priority of one of these two approaches over the other. For instance, some ancient logicians argued that hypothetical logic is prior to categorical logic on the grounds that Aristotle's assertoric syllogistic relies on modes of propositional reasoning such as *reductio ad absurdum*. Peripatetic logicians, by contrast, sought to establish the priority of categorical over hypothetical logic by reducing various modes of propositional reasoning to categorical syllogisms. This latter ambition of reducing hypothetical to categorical logic is known in the literature as the 'Peripatetic program'.

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While traces of the Peripatetic program appear intermittently throughout the history of logic, it received renewed attention by logicians in the 17th century. The champion of this early modern movement in the Peripatetic program was Gottfried Wilhelm Leibniz. Leibniz proposed a comprehensive implementation of the program based on the overarching idea that every proposition can be conceived of as a term. Thus, according to Leibniz, a categorical proposition such as *Peter is wise* can be transformed into an abstract propositional term, *Peter's being wise*, which can then serve as the subject or predicate of a further categorical proposition, as in *Peter's being wise is Peter's being just*.

Leibniz employed this device of propositional terms in the analysis of hypothetical syllogisms such as:

If Peter is wise, then Peter is just
If Peter is just, then Peter is noble
Therefore: *If Peter is wise, then Peter is noble*.

By transforming the constituent propositions into propositional terms, this hypothetical syllogism is reduced to the following categorical one:

Peter's being wise is Peter's being just
Peter's being just is Peter's being noble
Therefore: *Peter's being wise is Peter's being noble*.

On the model of this example, Leibniz developed a theory of propositional terms that allowed him to reduce a wide class of hypothetical syllogisms to categorical ones. By enabling such a wide-ranging reduction, Leibniz's theory of propositional terms lies at the heart of one of the most systematic and successful implementations of the Peripatetic program in the history of logic.

The plan of this article is as follows. We begin by situating Leibniz within the broader tradition of the Peripatetic program dating back to Theophrastus' treatment of wholly hypothetical syllogisms (§1). We next discuss the problem of how to give a satisfactory formulation of the rule of *reductio ad absurdum* in categorical logic (§2). We proceed to explain how Leibniz solves this problem by means of his theory of propositional terms. In particular, we consider how Leibniz utilizes this theory to derive the rule of *reductio ad absurdum* in an essay titled *Specimina calculi rationalis*. We reconstruct the proof of *reductio* given by Leibniz in the *Specimina*, and show that this proof can be formulated in a purely categorical language in which every proposition is of the form *A is B* (§3 and §4). As it turns out, the categorical calculus employed by Leibniz in this proof is strong enough to establish not only the rule of *reductio* but all the laws of classical propositional logic (§5). Moreover, the nonmonotonic variant of this calculus gives rise to a natural system of relevance logic known as $\mathrm{RMI}_{\overrightarrow{\neg}}$ (§6). These results serve to illustrate how Leibniz's theory of propositional terms provides a fruitful framework for exploring the ways in which various systems of propositional logic can be built up from purely categorical foundations.

In what follows, we refer to the works of Aristotle by means of the standard Bekker pagination. The works of ancient commentators such as Alexander of Aphrodisias, Ammonius, and Philoponus are referred to by means of the standard pagination employed in the Berlin Academy edition of the *Commentaria in Aristotelem Graeca* (1882–1909). In addition, we adopt the following abbreviations for editions of Leibniz's writings:

A *Gottfried Wilhelm Leibniz*: *Sämtliche Schriften und Briefe*. Ed. by the Deutsche Akademie der Wissenschaften, Berlin: Akademie Verlag. Cited by series, volume, and page.

C *Opuscules et fragments inédits de Leibniz*. Ed. by L. Couturat, Paris: Félix Alcan, 1903.

The two essays by Leibniz that we will have most frequent occasion to reference are the *Generales inquisitiones de analysi notionum et veritatum* (A VI.4 739–88) and the *Specimina calculi rationalis* (A VI.4 807–14). Each of these essays consists of serially numbered sections, which we designate by references such as '§16' and '§72'.[1] We sometimes use the abbreviation '*GI*' for the *Generales inquisitiones*. Translations from the Greek and Latin are our own.

§1. The Peripatetic program. The history of formal logic begins with Aristotle's theory of the categorical syllogism. The core of this theory consists of the assertoric syllogistic developed in the opening chapters of the *Prior Analytics*. There, Aristotle investigates the inferential relations that obtain between the following four kinds of proposition:

AaB	*A belongs to all B*	(or: *Every B is A*)
AeB	*A belongs to no B*	(or: *No B is A*)
AiB	*A belongs to some B*	(or: *Some B is A*)
AoB	*A does not belong to some B*	(or: *Some B is not A*).

In these propositional forms, the letters 'A' and 'B' stand for terms such as *man*, *animal*, and *walking*. Propositions instantiating these forms are traditionally referred to as categorical propositions, and valid inferences consisting of such propositions are referred to as categorical syllogisms.

Aristotle's assertoric syllogistic deals with a class of simple categorical syllogisms which are divided into three figures based on the internal arrangement of their terms. In chapters 1.2 and 1.4–6 of the *Prior Analytics*, Aristotle develops a deductive system in which some of these syllogisms can be derived from others. The principles of this system include the following 'perfect' syllogistic moods in the first figure:

$$\begin{array}{ll} AaB,\ BaC \vdash AaC & \text{(Barbara)} \\ AeB,\ BaC \vdash AeC & \text{(Celarent)} \\ AaB,\ BiC \vdash AiC & \text{(Darii)} \\ AeB,\ BiC \vdash AoC & \text{(Ferio).} \end{array}$$

In addition, the principles of Aristotle's deductive system include the following three conversion rules:

$$\begin{array}{ll} AeB \vdash BeA & \text{(e-conversion)} \\ AiB \vdash BiA & \text{(i-conversion)} \\ AaB \vdash BiA & \text{(a-conversion).} \end{array}$$

By means of these perfect first-figure moods and conversion rules, Aristotle is able to establish the validity of syllogistic moods in the second and third figures. In most

[1] The *Specimina calculi rationalis* consists of two parts, each of which comprises its own series of numbered sections. Unless otherwise specified, our references to the *Specimina* are exclusively to the first part (A VI.4 807–10).

cases, he does so by a direct deduction, as in the following proof of the mood Camestres (27a9–14):

1	BaA	major premise
2	BeC	minor premise
3	CeB	e-conversion: 2
4	CeA	Celarent: 1, 3
5	AeC	e-conversion: 4

There are, however, some valid syllogistic moods that cannot be established by means of a direct deduction in Aristotle's system. In these cases, Aristotle establishes the validity of the mood in question by an indirect deduction using the method of *reductio ad absurdum*. This method is based on the assumption that an a-proposition is the contradictory of the corresponding o-proposition, and an e-proposition the contradictory of the corresponding i-proposition. Thus, for example, Aristotle provides the following proof by *reductio* of the second-figure mood Baroco (27a36–27b1):

1	BaA		major premise
2	BoC		minor premise
3		AaC	assumption for *reductio*
4		BaA	reiteration: 1
5		BaC	Barbara: 3, 4
6	AoC		*reductio*: 2, 3–5

Aristotle provides a similar proof by *reductio* of the third-figure mood Bocardo, which likewise cannot be established by means of a direct deduction in his system (28b17–20).[2] Given the method of *reductio ad absurdum*, the perfect moods and conversion rules suffice to derive all the syllogisms which Aristotle identifies as valid in his assertoric syllogistic.

Now, Aristotle's theory of the categorical syllogism was not the only system of logic developed in Greek antiquity. One of the most prominent alternatives was the theory of the hypothetical syllogism developed by the Stoics. While Aristotle's logic is concerned exclusively with simple propositions expressing predicative relations between terms, Stoic logic deals with compound propositions such as those of the form:

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[2] Neither Baroco nor Bocardo can be established by means of a direct deduction in Aristotle's system because it does not include a conversion rule for o-propositions (see Alexander *In Pr. An.* 83.12–25, Łukasiewicz 1957: 54). In the case of Bocardo, Aristotle mentions an alternative proof that employs the method of *ecthesis* instead of *reductio* (*Pr. An.* 1.6 28b20–1). However, he does not mention a proof by *ecthesis* for Baroco, and it is not clear whether such a proof is available. For example, the formulation of the rule of *ecthesis* given by Parsons (2014: 36–7) allows for a proof of Bocardo but not Baroco (similarly, Ebert & Nortmann 2007: 333–7). Without appealing to *ecthesis*, the only way for Aristotle to establish the validity of Baroco and Bocardo in his system is by means of *reductio*.

$$\begin{array}{l} \textit{If } \varphi \textit{ then } \psi \\ \textit{Either } \varphi \textit{ or } \psi \\ \textit{Both } \varphi \textit{ and } \psi. \end{array}$$

In these propositional forms, the letters 'φ' and 'ψ' do not stand for terms but for propositions. Compound propositions instantiating these forms are traditionally referred to as hypothetical propositions, and valid inferences containing at least one hypothetical proposition are referred to as hypothetical syllogisms.[3]

Paradigmatic examples of hypothetical syllogisms include those of the form *modus ponens*, which the Stoics called first indemonstrables:

$$\textit{If } \varphi \textit{ then } \psi, \;\; \varphi \;\vdash\; \psi.$$

The Stoics also identified a number of other indemonstrable hypothetical syllogisms including, for example, those of the form *modus tollens*. In addition, they posited a number of reduction rules, or *themata*, by which various hypothetical syllogisms can be reduced to indemonstrable ones. The first of these *themata*, which pertains to the logic of propositional negation, is:

$$\frac{\varphi, \psi \vdash \chi}{\varphi, \textsc{con}(\chi) \vdash \textsc{con}(\psi)}$$

In this formulation of the first *thema*, 'CON' is used to indicate the contradictory of a proposition. Thus, φ and $\textsc{con}(\varphi)$ are contradictory propositions, i.e., either $\textsc{con}(\varphi)$ is the propositional negation of φ or vice versa. Based on the *themata* and indemonstrables, the Stoics developed a system of logic by means of which they were able to validate a wide variety of hypothetical syllogisms.[4]

It was a topic of discussion among some ancient logicians how exactly the theory of the categorical syllogism relates to that of the hypothetical syllogism. In particular, the question was raised as to whether one of these two theories ought to be regarded as, in some sense, prior to the other. Thus, Alexander of Aphrodisias reports in his commentary on Aristotle's *Topics*:

> Even in logic there are some matters that are inquired into comparatively, such as: 'Which is more convincing, induction or deduction?' and 'Which kind of syllogism is primary, the categorical or the hypothetical?'. (Alexander of Aphrodisias, *On Aristotle's Topics* 218.3–5)

The second question cited by Alexander in this passage concerns the relative priority of categorical and hypothetical syllogisms. Apart from Alexander's testimony that there was some discussion of this question among ancient logicians, we do not have much further evidence regarding the exact nature and scope of the debate.[5] For one thing, it is not clear whether the participants in the debate had in mind the Stoic theory of the hypothetical syllogism or some other system of hypothetical logic developed in antiquity.[6] Nor is it clear

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[3] See Barnes 1983: 284, 1985: 563, Bobzien 1997: 299–301.

[4] See the exposition of Stoic hypothetical syllogistic in Bobzien 1996.

[5] See Frede 1974: 30–2.

[6] It is conceivable, for example, that the debate was addressed to one of the systems of hypothetical syllogistic developed by Peripatetic logicians working in the tradition of Theophrastus. For a discussion of these Peripatetic systems of hypothetical logic, see Bobzien 2002a: 380–93, 2002b, 2002c, 2004: 80–100.

whether Aristotle or the Stoics themselves entertained such questions of relative priority. In any case, Alexander's report confirms that these questions were a topic of interest to some ancient logicians, with various considerations being put forward in support of one or the other side of the matter.[7]

One consideration that might speak in favor of the priority of hypothetical over categorical logic concerns Aristotle's appeal to the method of *reductio ad absurdum* in his assertoric syllogistic. Thus, for example, the 2nd-century author Apuleius (or whoever wrote the Latin *De interpretatione*) claims that, in employing this method, Aristotle relies on the first *thema* of the Stoics:

> Common to all [categorical syllogisms] ... is another proof, which is called *per impossibile* and named by the Stoics the *first law* or *first thema*. They define it in this way: if from two propositions a third follows, then from either one of them together with the contradictory of the conclusion, the contradictory of the remaining one follows. (Apuleius, *De interpretatione* XII 209.10–14)

According to Apuleius, the law underwriting Aristotle's proofs by *reductio* in the assertoric syllogistic is the first *thema* of the Stoics.[8] For example, given that $\textsc{con}(AaB)$ is the proposition AoB, Aristotle's indirect proof of Baroco by means of Barbara relies on the following instance of the first *thema*:

$$\frac{BaA,\, AaC \vdash BaC}{BaA,\, \textsc{con}(BaC) \vdash \textsc{con}(AaC)}$$

In the same vein, Neoplatonic authors such as Philoponus, Pseudo-Ammonius, and Proclus argue that Aristotle's proofs by *reductio* rely on principles of Stoic hypothetical syllogistic (though not on the first *thema*, but on *modus tollens* and other indemonstrables).[9] A similar claim to the effect that these proofs rely on principles of propositional logic was endorsed several centuries later by Avicenna and his followers.[10] In modern times, the same point has been made by commentators such as Łukasiewicz and Patzig.[11] If the point stands,

[7] Barnes describes the ancient debate over the relative priority of categorical and hypothetical syllogistic as follows: 'It was often supposed that the two syllogistics were partners, each adequate in its own area. This irenic view is misleading. The Peripatetics thought that their categorical syllogistic embraced the whole of logic: any argument which submitted to formal treatment at all submitted to categorical syllogistic. And the Stoics held the same for their hypothetical syllogisms. The two systems regarded themselves as rivals; and behind the texts there is a genuine and philosophical rivalry, between 'term' logic which supposes that relations among terms are the fundamental logical relations and 'sentence' logic which assumes that it is sentences on which logic must ground itself' (Barnes 1999: 77–8).

[8] See Londey & Johanson 1987: 37.

[9] See Philoponus *In Pr. An.* 246.15–247.19, *In Post. An.* 292.7–16, 293.26–30, Pseudo-Ammonius *In Pr. An.* 69.11–25, Proclus *In Eucl. Elem.* 254.21–256.8; similarly, Pseudo-Themistius *In Pr. An.* 94.32–95.8. While their analyses differ in some details, all of these commentators agree that Aristotle's proofs by *reductio* rely on the second Stoic indemonstrable, i.e., *modus tollens*; see Ierodiakonou 2016: 352–7.

[10] Avicenna *al-Shifā': al-Qiyās* 8.3 408.4–411.7 (transl. Hodges 2017: 597–9), *al-Najāt: al-Manṭiq* 91 (transl. Ahmed 2011: 79–80); cf. Shehaby 1973: 277–8, Street 2004: 545–7 and 576–7.

[11] See Łukasiewicz 1935: 120–1, 1957: 49 and 56–9, 1963: 14 and 106–7, Patzig 1968: 151–6. Similarly, Barnes (2007: 510–11) argues that Aristotle's proofs by *reductio* rely on a certain kind of hypothetical syllogism, namely, the disjunctive syllogism.

then Aristotle's use of *reductio* in his indirect proofs of Baroco and Bocardo relies upon the Stoic theory of the hypothetical syllogism or some other comparable system of propositional logic. In this event, hypothetical logic would be presupposed by categorical logic and would thus have a claim to priority over the latter.

In fact, some ancient logicians working in the Stoic tradition argued that the dependence of categorical on hypothetical logic extends well beyond Aristotle's use of *reductio ad absurdum*. As Alexander reports, these logicians claimed that even instances of the mood Barbara should be rendered as hypothetical syllogisms by adding a further conditional premise asserting that the conclusion of the syllogism is implied by its two categorical premises.[12] On this account, even the perfect first-figure moods posited by Aristotle as principles of his system are underwritten by hypothetical syllogisms.

By contrast, some Peripatetic logicians adopted the view that certain hypothetical syllogisms can be reduced to categorical ones. This Peripatetic ambition of reducing hypothetical to categorical syllogisms can be traced back, in part, to an elliptical remark made by Aristotle in *Prior Analytics* 1.32:

> If it is necessary for animal to be if man is, and substance to be if animal is, then it is necessary for substance to be if man is; but this has not yet been syllogized, for the premises are not disposed in the manner that we have said.[13] (Aristotle, *Prior Analytics* 1.32 47a28–31)

In this passage, Aristotle discusses an argument which consists of three conditionals, each of the form *It is necessary for A to be if B is*. Abbreviating these conditionals by the phrase '*If B is, then A is*', this argument reads as follows:

	If man is, then animal is
	If animal is, then substance is
Therefore:	*If man is, then substance is.*

The propositions that appear as antecedents and consequents in these conditionals are *Man is*, *Animal is*, and *Substance is*. While Aristotle does not explain the precise meaning of these propositions, and there is some room for debate over their interpretation, most commentators agree that these propositions signify existential statements such as *Man exists* and *Animal exists* (or: *There is a man* and *There is an animal*).[14] For present purposes,

[12] Alexander *In Pr. An.* 262.28–9, 345.13–18, 390.16–18; see Mueller 1969: 179–80, Frede 1974: 2–5 and 10, Barnes 1990: 73–5 and 114–16.

[13] πάλιν εἰ ἀνθρώπου ὄντος ἀνάγκη ζῷον εἶναι καὶ ζῴου οὐσίαν, ἀνθρώπου ὄντος ἀνάγκη οὐσίαν εἶναι· ἀλλ' οὔπω συλλελόγισται· οὐ γὰρ ἔχουσιν αἱ προτάσεις ὡς εἴπομεν. Our translation of the phrase ἀνθρώπου ὄντος ἀνάγκη ζῷον εἶναι as 'it is necessary for animal to be if man is' follows Smith (1989: 51). By contrast, Striker (2009: 52) translates this phrase as 'what is a man is necessarily an animal'. However, the relative clause 'what is a man' is not an adequate translation of Aristotle's genitive-absolute phrase ἀνθρώπου ὄντος (see Ebrey 2015: 190 n. 12). For, the latter phrase does not contain a relative pronoun, and it would be unusual for Aristotle to use a genitive-absolute participle when the subject of the participle is co-referential with the accusative subject of the superordinate clause (see Kühner & Gerth 1904: 78–9). Striker (2009: 214–15) claims that her translation is supported by Alexander's interpretation of the passage, but this is not the case (see n. 25 below).

[14] This existential interpretation is adopted, e.g., by von Kirchmann 1877: 75, Owen 1889: 157, Jenkinson 1928, Tredennick 1938: 365–7, Ross 1949: 399, Smith 1989: 51, Bolton 1994: 117, Primavesi 1996: 163–4 and 178 n. 26, Ebert & Nortmann 2007: 78 and 805. There are some commentators who reject this existential interpretation and instead take the propositions in

however, all that is important is that the argument discussed by Aristotle is of the following form (where 'φ', 'ψ', and 'χ' stand for any propositions):[15]

$$\textit{If } \varphi \textit{ then } \psi, \ \textit{If } \psi \textit{ then } \chi \ \vdash \textit{If } \varphi \textit{ then } \chi.$$

Arguments of this form, and similar arguments consisting exclusively of conditionals, are traditionally referred to as 'wholly hypothetical syllogisms'.

In the passage just quoted, Aristotle acknowledges that the conclusion of a wholly hypothetical syllogism follows necessarily from its premises.[16] He insists, however, that the argument is not a syllogism because, as he puts it, 'the premises are not disposed in the manner that we have said'. This claim admits of multiple interpretations. Alexander, for example, takes Aristotle to mean that the validity of a wholly hypothetical syllogism derives from a missing categorical premise that somehow asserts the transitivity of the conditional connective '*If* $\cdots$ *then* —'. He argues that, by supplying this missing premise, a wholly hypothetical syllogism can be transformed into a categorical one.[17]

An alternative proposal for how to transform wholly hypothetical into categorical syllogisms was put forward by Aristotle's pupil Theophrastus. In parallel with the three figures of categorical syllogisms, Theophrastus classified wholly hypothetical syllogisms into three figures depending on the arrangement of their constituent propositions.[18] His classification is based on the following analogy between conditionals and categorical propositions:

Aristotle's argument to be of the form *What is A is B* or *If something is A, then it is B* (Slomkowski 1997: 106–7 n. 42, Bobzien 2000: 92–6 and 107–8, Barnes 2007: 434–5, 2012: 442–9, Striker 2009: 52, Crubellier 2014: 141). In our view, however, the existential interpretation is more plausible because it is supported by a number of parallel passages in Aristotle's writings. For example, when Aristotle writes 'if fish is, then animal is, but it is not necessary for fish to be if animal is' (ἐνύδρου μὲν ὄντος ἔστι ζῷον, ζῴου δὲ ὄντος οὐκ ἀνάγκη ἔνυδρον εἶναι, *Cat.* 13 15a6–7), he means that the existence of fish implies the existence of animals but not vice versa (Ackrill 1963: 40). Similarly, Aristotle writes: 'if two are, it follows at once that one is, but if one is, it is not necessary that two are' (δυεῖν μὲν γὰρ ὄντων ἀκολουθεῖ εὐθὺς τὸ ἓν εἶναι, ἑνὸς δὲ ὄντος οὐκ ἀναγκαῖον δύο εἶναι, *Cat.* 12 14a31–2). By this, Aristotle means that the existence of two units implies the existence of one unit but not vice versa (Panayides 1999: 329, Coope 2005: 91 n. 14, Menn 2009: 217 n. 6, Peramatzis 2011: 235; see also the pseudo-Aristotelian *Divisiones Aristotelis* 65 64.15–24). Since, presumably, being two is incompatible with being one, the conditional expressed by Aristotle in this passage cannot be taken to be of the form *What is A is B* or *If something is A, then it is B*. For, it is neither the case that *What is two is one*, nor that *If something is two, then it is one*. For further passages that support the existential interpretation of Aristotle's conditionals, see *Cat.* 7 7b19–35, *EE* 7.1 1235a27–8, *De caelo* 1.7 276a10.

[15] Against this, Bobzien (2000: 92–6 and 107–8) contends that the argument discussed by Aristotle does not consist of conditionals of the form *If* φ *then* ψ, but instead of universally quantified statements of the form *For any x, if x is A, then x is B*. This interpretation, however, is not a natural way of reading Aristotle's Greek, for the reasons given in nn. 13 and 14 (see Ebrey 2015: 190–1; cf. Crivelli 2011: 148 n. 120). Bobzien does not offer direct textual evidence for her interpretation but adopts it primarily because, in her view, this is the only plausible reading of the argument on which it is reducible to a categorical syllogism of the form Barbara (see Bobzien 2000: 93–5 and 107–8). In our view, this consideration is not conclusive since, as we shall see below, there exist alternative ways of carrying out such a reduction.

[16] *Pr. An.* 1.32 47a22–4 and 47a31–5; cf. Alexander *In Pr. An.* 346.10–348.39.

[17] Alexander *In Pr. An.* 347.23–348.9; similarly, Philoponus *In Pr. An.* 323.5–14. See Barnes 1990: 114–16, 2007: 435.

[18] See Alexander *In Pr. An.* 326.8–328.5, Philoponus *In Pr. An.* 302.6–19; cf. Bobzien 2000: 102–11.

> To be a consequent, or to follow from something, is analogous to being a predicate, and to be an antecedent is analogous to being a subject. For, in a way, an antecedent is a subject for what is inferred from it. (Alexander of Aphrodisias, *On Aristotle's Prior Analytics* 326.31–2)

As Alexander reports in this passage, Theophrastus held that the antecedent of a conditional is analogous to the subject term of a categorical proposition, and the consequent to its predicate term. Based on this analogy, Theophrastus observed an isomorphism between wholly hypothetical and categorical syllogisms. This isomorphism is described by Philoponus as follows:

> Theophrastus says that [wholly hypothetical syllogisms], too, can be reduced to the three figures [of categorical syllogisms]. For when we say *If A, then also B*; *If B, then also C*, and then conclude *hence*: *If A, then also C*, then *A* is analogous to the minor term, i.e., the subject, *B* to the middle, which is predicated of *A* and subject of *C*, which is itself analogous to the major term. Hence, in this way, there will be the first figure. (Philoponus, *On Aristotle's Prior Analytics* 302.14–19)

Now, it is conceivable that Theophrastus' analogy was meant to do nothing more than highlight a superficial structural similarity between wholly hypothetical and categorical syllogisms. Both Alexander and Philoponus, however, state that Theophrastus intended this analogy to do something more, namely, to provide the basis for 'reducing' (ἀνάγεσθαι) wholly hypothetical syllogisms to the three figures of categorical syllogisms.[19] Alexander's testimony, in particular, implies that Theophrastus sought to explain the validity of wholly hypothetical syllogisms by reducing them to categorical form.[20] Thus, Theophrastus aimed at a more substantial reduction according to which, as Bobzien puts it, 'a wholly hypothetical syllogism would reduce to a categorical syllogism ..., if its validity was derived from the latter; that is, if one can logically transform it into a categorical syllogism'.[21]

The categorical reduction of wholly hypothetical syllogisms envisioned by Theophrastus proceeds by transforming each of the three conditionals involved in the syllogism into a categorical proposition. Specifically, a conditional *If φ then ψ* is transformed into an equivalent categorical proposition whose subject and predicate terms are the reductive

[19] Alexander *In Pr. An.* 326.20–2, Philoponus *In Pr. An.* 302.14–15.

[20] Alexander reports that Theophrastus showed how wholly hypothetical syllogisms are 'reduced (ἀνάγονται) to the three figures' (*In Pr. An.* 326.20–1). Aristotle uses the phrase 'reduce to the three figures' in *Prior Analytics* 1.32 to describe a general procedure whereby arguments that are valid but do not have the form of a categorical syllogism can be reformulated as categorical syllogisms (1.32 46b40–47a1; see Ebert & Nortmann 2007: 798, Striker 2009: 212). The aim of this procedure is to resolve, or analyze (ἀναλύειν, 47a4), valid noncategorical arguments into categorical syllogisms (Pacius 1597: 184, Byrne 1997: 62–4). In particular, this procedure is meant to be applied to the wholly hypothetical syllogism discussed by Aristotle in chapter 1.32 (47a28–31). In his commentary on chapter 1.32, Alexander notes that this procedure was discussed by Theophrastus in his *On the Analysis of Syllogisms*, and that Theophrastus listed various applications of the procedure in a treatise titled *Arguments Reduced to Figures* (*In Pr. An.* 340.13–21). Thus, when Alexander states that Theophrastus reduced wholly hypothetical syllogisms to the three figures (326.20–2), it seems clear that this reduction is a particular application of the same general procedure of rendering valid noncategorical arguments as categorical syllogisms.

[21] Bobzien 2000: 107.

analogues of the propositions φ and ψ, respectively. To give a general representation of such a reduction, we will use '$[\varphi]$' and '$[\psi]$' as abstract placeholders for the term-analogues of these propositions. In this notation, the categorical reduct of the conditional *If φ then ψ* is a categorical proposition in which $[\varphi]$ and $[\psi]$ figure as subject and predicate terms. In order to validate wholly hypothetical syllogisms, this categorical reduct must be the a-proposition $[\psi]\,a\,[\varphi]$.[22] For, among a-, e-, i-, and o-propositions, only a-propositions express a transitive relation between terms. Thus, a wholly hypothetical syllogism of the form:

$$\textit{If } \varphi \textit{ then } \psi, \ \textit{If } \psi \textit{ then } \chi \ \vdash \ \textit{If } \varphi \textit{ then } \chi$$

is reduced to the following categorical syllogism in the mood Barbara:

$$[\chi]\,a\,[\psi],\ [\psi]\,a\,[\varphi] \vdash [\chi]\,a\,[\varphi].$$

Of course, this representation of Theophrastus' reduction does not settle the question of how to understand the terms $[\varphi]$ and $[\psi]$ which serve as the reductive analogues of propositions. Unfortunately, the ancient sources offer little guidance on this point. With respect to the hypothetical premise *If man is, then animal is* mentioned by Aristotle in *Prior Analytics* 1.32, it has been suggested by some that its categorical reduct is the a-proposition *Animal belongs to all man* (or: *Every man is an animal*).[23] On this account, the wholly hypothetical syllogism discussed by Aristotle reduces to the following categorical syllogism:

	Substance belongs to all animal
	Animal belongs to all man
Therefore:	*Substance belongs to all man.*

In the above notation, this would mean that the terms [*man is*], [*animal is*], and [*substance is*] that figure in Theophrastus' proposed reduction are just the terms *man*, *animal*, and *substance*.

As natural as this may seem, there is little evidence to suggest that this is what Theophrastus had in mind. On the contrary, there is reason for doubt. In order for this proposed reduction to be successful, the putative categorical reduct *Animal belongs to all man* must be, in some sense, equivalent to the conditional *If man is, then animal is*. For otherwise it would be unclear how the validity of the wholly hypothetical syllogism could be derived from the validity of the categorical syllogism to which it is reduced. But given the usual meaning of conditional constructions of the form *If A is, then B is* in Aristotle's writings, the conditional *If man is, then animal is* is not equivalent to *Animal belongs to all man*. For example, Aristotle accepts as true the conditional *If two are, then one is*, by which he means: if there exists a plurality consisting of two items, then there exists a single item.[24] Presumably, however, he would not accept the categorical proposition *One belongs to all two*, since, by definition, no plurality consisting of two items is a single item. So, *One belongs to all two* is not equivalent to the original conditional *If two are, then one is*. Thus, as this example serves to illustrate, Aristotle's intended meaning of a conditional of the

[22] See Barnes 1983: 315.

[23] Pacius 1597: 185–6, Bobzien 2000: 92–3; similarly, Ebert & Nortmann 2007: 805–6, Striker 2009: 214–15.

[24] *Cat.* 12 14a31–2; see n. 14 above.

form *If A is, then B is* precludes its straightforward reduction to the categorical proposition *B belongs to all A*.[25]

The same conclusion is reached by Jonathan Barnes, who argues that the simplistic reduction described above misrepresents Theophrastus' intended analogy between hypothetical and categorical propositions. Barnes argues that Theophrastus and other Peripatetic logicians aimed at a more comprehensive reduction which would apply not only to the specific sort of argument discussed by Aristotle in *Prior Analytics* 1.32, but to a much wider class of wholly hypothetical syllogisms. According to Barnes, they took the desired reduction to be based on the general postulate that any proposition can be transformed into a term, and that any conditional is equivalent to the a-proposition relating the terms that result from transforming its antecedent and consequent.[26] This kind of reduction is naturally viewed as part of a broader theoretical program aimed at a systematic reduction of hypothetical syllogisms to categorical ones. Following Barnes, we will refer to this program as the 'Peripatetic program' in categorical logic.[27]

It is not clear to what extent ancient logicians made progress towards the fulfillment of this Peripatetic program. Nor is it clear to what extent they were committed to the comprehensive aim of reducing all hypothetical syllogisms to categorical ones. What we do know is that Theophrastus sought to extend his reductive analysis to a wide variety of wholly hypothetical syllogisms including those he grouped into the second and third figures. This latter group includes arguments of the form:

$$\textit{If } \varphi \textit{ then } \psi,\ \textit{If } \chi \textit{ then } \mathrm{CON}(\psi) \ \vdash\ \textit{If } \varphi \textit{ then } \mathrm{CON}(\chi).$$

The validity of such wholly hypothetical syllogisms depends not only on the conditional connective '*If* $\cdots$ *then* —' but also on the operation $\mathrm{CON}(\cdots)$, which maps a proposition to its contradictory. The same is true for all wholly hypothetical syllogisms in the second and third figures.[28] Thus, insofar as Theophrastus and his followers sought a categorical reduction of these wholly hypothetical syllogisms, they must have reckoned with the problem of how to represent $\mathrm{CON}(\cdots)$ in the language of categorical logic. In doing so, they might even have been led to consider modes of propositional reasoning involving $\mathrm{CON}(\cdots)$ that fall outside the scope of the wholly hypothetical syllogistic, such as the rule of *reductio ad absurdum*. The extant sources, however, provide no evidence on this score.

Whatever advances in the Peripatetic program may have been made by Theophrastus and his followers in the Hellenistic period, in the subsequent centuries there seems to have been a general loss of interest in the foundational debates over the relative priority of categorical and hypothetical logic. Instead, these two kinds of logic came to be viewed as

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[25] Alexander notes that the wholly hypothetical syllogism discussed by Aristotle can be turned into a categorical syllogism if its two premises are replaced by the categorical propositions *Animal belongs to all man* and *Substance belongs to all animal* (*In Pr. An.* 348.16–19). Alexander's remark is, of course, correct in that the latter two categorical propositions do constitute the premises of a syllogism in Barbara. Crucially, however, Alexander does not say that these categorical propositions are equivalent to the original conditional premises stated by Aristotle at 47a28–30. In fact, Barnes (2012: 443 and 448 n. 3) argues that Alexander did not take them to be equivalent. Thus, Alexander's remark should not be read as a proposal for how to reduce wholly hypothetical to categorical syllogisms.

[26] See Barnes 1983: 311–16.

[27] Barnes writes: 'The old Peripatetic logicians ... held that all formally valid inferences could be construed within an extension of Aristotelian syllogistic. The task of establishing that thesis constitutes what I shall call the Peripatetic programme' (Barnes 1983: 282).

[28] See Bobzien 2000: 89–91.

resting on independent foundations, with the question of how one might be reduced to the other becoming an increasingly marginal issue.[29] This growing attitude of indifference is clearly evinced by Galen, who writes:

> As far as disputes of that sort are concerned, it is no great matter whether you discover the truth or remain in ignorance. For you need to know both kinds of syllogism [categorical and hypothetical]—that is what is useful. You may call one prior, or teach it to be prior, as you please—but you must not be ignorant of the other. (Galen, *Institutio logica* vii.3)

Galen's lack of interest in disputes over the relative priority of categorical and hypothetical logic seems to have been shared by most logicians from late antiquity through the Middle Ages.[30] During this period, only sporadic traces of the Peripatetic program can be found.[31] The 16th and 17th centuries, however, saw a renewed interest in the Peripatetic program, with many textbooks in circulation at the time containing proposals for how various kinds of hypothetical syllogism can be reduced to categorical form.[32] One of the most ambitious of these proposals appears in the logical writings of Gottfried Wilhelm Leibniz.[33]

Leibniz's commitment to the Peripatetic program is manifest in his view that:

> absolute [i.e., categorical] and hypothetical truths have one and the same laws and are contained in the same general theorems, so that all syllogisms become categorical. (Leibniz, *Generales inquisitiones* §137)

Leibniz attached great value to the reduction of hypothetical to categorical propositions, viewing it as an important source of unification and simplicity in his logical theory:

> If, as I hope, I can conceive all propositions as terms, and all hypothetical propositions as categorical, and if I can give a universal treatment of them all, this promises a wonderful ease in my symbolism and analysis

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[29] See, e.g., Speca 2001: 132.

[30] For example, Boethius wrote two separate treatises devoted to categorical and hypothetical syllogisms, respectively, and he did not undertake to reduce one kind of syllogism to the other. As Barnes points out, 'Boethius shows no interest in reducing hypothetical to categorical syllogisms; nor does he advert to the 'analogies' between the two types of syllogistic' (Barnes 1983: 303 n. 3). During the medieval period, most logicians in the Latin tradition either did not discuss hypothetical syllogisms or closely followed Boethius' treatment of them (see Ierodiakonou 1996: 111–14).

[31] One notable exception is Avicenna, who did inquire into the possibility of reducing certain hypothetical propositions to categorical ones (see *al-Shifā': al-Qiyās* 5.3 256.11–258.12 and 5.4 264.1–20; translated in Shehaby 1973: 55–7 and 62–3). For a discussion of Avicenna's proposed reduction, see Shehaby 1973: 241–2, Street 2004: 545–6 and 566.

[32] See, e.g., Crellius 1595: 218–21, Dietericus 1618: 355–6, Jungius 1638: 167 and 229–35, Wallis 1687: 235–55, Aldrich 1691: 27–9 (cf. Ashworth 1968: 188). Aldrich's textbook had an influence on the 19th-century logician Whately (1827: 118–21), who argued that all hypothetical syllogisms can be reduced to categorical ones. Following Whately, Boole (1952: 163) observed that 'a hypothetical judgment is reducible to a categorical form' (see Boole 1952: 146 and 162–4; cf. Prior 1949: 176, Nambiar 2000: 221–8).

[33] For Leibniz's commitment to the Peripatetic program in categorical logic, see Sommers 1982: 153–5, 1993: 169–78, Barnes 1983: 281. Barnes regards Leibniz as 'the modern hero of term logic', and observes that 'we may fairly name the theory championed by Leibniz Aristotelian or Peripatetic logic' (Barnes 1983: 281).

> of concepts, and will be a discovery of the greatest moment. (Leibniz, *Generales inquisitiones* §75)

In line with Theophrastus' analogy, Leibniz analyzed conditionals as categorical propositions by construing the antecedent of a conditional as the subject term of an a-proposition and the consequent as the predicate term.[34] For the purposes of this analysis, Leibniz introduced a novel kind of abstract term to serve as the reductive analogue of the propositions that occur as the antecedents and consequents of conditionals:

> I reduce hypothetical propositions to categorical propositions in which these abstract terms occur as components. For example, the hypothetical proposition *If Peter is wise, then Peter is just*, I reduce to the following categorical proposition: *Peter's being wise is Peter's being just*. In this way, the same rules hold for hypothetical propositions that hold for categorical propositions. (Leibniz, *De abstracto et concreto*, A VI.4 992)

In this passage, Leibniz proposes to reduce the conditional *If Peter is wise, then Peter is just* to the categorical proposition *Peter's being wise is Peter's being just*. He intends this latter proposition to be a universal affirmative proposition, the phrase '*A is B*' being one of his standard ways of expressing the a-proposition BaA. The subject and predicate terms of this proposition are abstract terms which Leibniz designates by the phrases '*Peter's being wise*' and '*Peter's being just*'. In Leibniz's implementation of Theophrastus' analogy, these terms serve as the reductive analogues, [*Peter is wise*] and [*Peter is just*], of the antecedent and consequent of the conditional. In what follows, we refer to such abstract terms as 'propositional terms'.[35]

In developing his theory of propositional terms, Leibniz does not provide an account of what these terms are meant to denote. In this respect, he differs from other logicians working in the Peripatetic tradition, who put forward various proposals for how to specify the denotation of the term analogues of propositions in Theophrastus' analogy. For example, Avicenna suggested that the term analogue of a proposition denotes the times at which that proposition is true.[36] The 17th-century logician John Wallis, on the other hand, took the term analogue of a proposition to denote the 'cases' in which that proposition is true.[37] By contrast, Leibniz does not subscribe to either a temporal or a case-based account of propositional terms. Instead, he adopts a more proof-theoretic approach, characterizing the nature of propositional terms, not by specifying their denotation, but rather by positing logical principles that govern their operation in a deductive system. This proof-theoretic approach comports well with the fact that Leibniz regards propositional terms as abstract terms, with

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[34] Leibniz writes: 'A hypothetical proposition is nothing other than a categorical proposition when the antecedent is turned into the subject and the consequent into the predicate' (A VI.4 811 n. 6). See also A VI.4 862–3.

[35] The label 'propositional term' is borrowed from Sommers 1982: 156, 1993: 172–3, Barnes 1983: 315, and Swoyer 1995: 110–11.

[36] Avicenna *al-Shifā': al-Qiyās* 5.4 262.1–275.14 and 6.1 295.5–304.5; translated in Shehaby 1973: 61–72 and 91–9. Such a temporal interpretation is also adopted, e.g., by Boole (1854: 159–84 and 241) and Schröder (1891: 1–24 and 59–63).

[37] Wallis 1687: 242–4. Such a case-based interpretation is also adopted, e.g., by Whately (1827: 118–21), De Morgan (1847: 23), and Boole (1847: 48–50). Similarly, Sommers takes a propositional term [φ] to denote the states of affairs in which φ (Sommers 1970: 3 and 16–17, 1982: 153–6), or the worlds characterized by φ (Sommers 1993: 176–8, Sommers & Englebretsen 2000: 206–7).

the important qualification that they are merely 'logical, or, notional' abstracts.[38] He thus adopts a nominalistic conception of propositional terms, according to which they are to be regarded 'not as corresponding to things, but as shorthands for discourse'.[39]

Given such nominalistic scruples, it is perhaps understandable why Leibniz did not undertake to provide an account of the denotation of propositional terms. Nonetheless, it is clear that Leibniz would agree with both Avicenna and Wallis that the term [*A is*] that serves as the reductive analogue of the proposition *A is* in Theophrastus' analogy cannot simply be identified with the term *A*.[40] Instead, Leibniz insists that the term [*A is*], which he expresses by the phrase '*A's being*', is a 'new term' distinct from *A*:

> Let *A* be a term, and *A is* or *A is true* be a proposition. Then, *A true*, or *A's being true*, or *A's being* will be a new term, from which a new proposition can in turn arise. (Leibniz, *Generales inquisitiones* §198.7)

It is clear from this passage that Leibniz would take the propositional term *man's being* to be distinct from the term *man*. Consequently, he would reject the simplistic reduction of Aristotle's conditional *If man is, then animal is* to the categorical proposition *Animal belongs to all man*. Instead, he would reduce this conditional to the a-proposition in which the propositional term *animal's being* is predicated of the propositional term *man's being*.[41]

As we shall see, Leibniz's theory of propositional terms opens a promising route to the fulfillment of the Peripatetic program. Not only does this theory allow Leibniz to achieve Theophrastus' aim of reducing all wholly hypothetical syllogisms to categorical ones, it also allows him to overcome the challenges to the Peripatetic program raised by Aristotle's reliance upon the method of *reductio ad absurdum*.

§2. The rule of *reductio* in categorical logic. Throughout the history of logic, one of the main objections that has been raised against the Peripatetic program concerns Aristotle's use of *reductio ad absurdum* in his assertoric syllogistic. As we have seen, Aristotle's

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[38] A VI.4 740; see also A VI.4 987, 992.

[39] A VI.4 996. In accordance with Leibniz's nominalism, an abstract term can be used without ascribing to it any object that serves as its denotation. For example, regarding the abstract term *heat*, Leibniz writes: 'when I speak of heat (*calor*), there is no need to make mention of some vague subject or to say that something is hot—and to that extent I am a nominalist, at least provisionally' (A VI.4 996). Likewise, Leibniz takes himself to be able to employ abstract propositional terms in his calculus without ascribing to them any objects that they denote. Thus, contrary to Sommers's contention, Leibniz's use of propositional terms does not commit him to 'an ontology of states of affairs' (Sommers 1982: 159). Instead, as Leibniz puts it, 'it suffices to posit only substances as things and to assert truths about these' (A VI.4 996).

[40] On Avicenna's temporal interpretation, the term [*Man is*] denotes the set of times at which the proposition *Man is* is true. On Wallis's case-based interpretation, [*Man is*] denotes the set of cases in which this proposition is true. On neither interpretation is the term [*Man is*] coextensive with the term *man*, which denotes the set of all humans.

[41] In keeping with scholastic terminology, Leibniz refers to propositions of the form *A is* as '*secundi adjecti*', and to those of the form *A is B* as '*tertii adjecti*' (*GI* §§144–51; cf. A VI.4 1160 and Alsted 1628: 313, 1630: 453 and 455). This terminology derives from Aristotle's distinction between existential statements such as *Man is*, on the one hand, and predicative statements such as *Man is just*, on the other, in which, as Aristotle puts it, ''is' is predicated additionally as a third item' (*De int.* 10 19b14–22; see Prior 1955: 164–5, Parkinson 1966: xlv n. 1, Nuchelmans 1992: 7–35, Weidemann 2014: 341–2). Thus, the class of propositions *secundi adjecti* discussed by Leibniz includes the existential statements *Man is*, *Animal is*, and *Substance is* that serve as antecedents and consequents in Aristotle's example of a wholly hypothetical syllogism (see n. 14).

use of this method opened him to the charge of presupposing principles of hypothetical syllogistic in his theory of the categorical syllogism. Thus, Neoplatonic authors such as Philoponus and Pseudo-Ammonius claimed that any deduction that employs the method of *reductio ad absurdum* is not a categorical syllogism but a hypothetical one.[42] In fact, Aristotle himself acknowledges that arguments by *reductio* are not 'ostensive' syllogisms but only 'syllogisms from a hypothesis'.[43] As such, Aristotle concedes, they cannot be fully justified within his theory of the three figures of categorical syllogisms:

> And similarly for arguments that are brought to a conclusion through an impossibility; for these too cannot be analyzed [into the three figures of categorical syllogisms[44]]. Instead, the part that leads to the impossibility can be so analyzed since it is proved by a syllogism, but the other part cannot since the conclusion is reached from a hypothesis. (Aristotle, *Prior Analytics* 1.44 50a29–32)

In this passage, Aristotle states that every argument by *reductio* contains a part that can be analyzed into the three syllogistic figures. This part of the argument consists of a direct deduction occurring within the subordinate proof initiated by the assumption for *reductio*.[45] Aristotle maintains, however, that no such syllogistic analysis is available for the final step of the argument, in which the assumption for *reductio* is discharged and the desired conclusion inferred. Thus, Aristotle admits that the rule of *reductio* that licenses this final step cannot be justified within his theory of the three figures of categorical syllogisms, but must instead be posited as a separate rule of inference.

For this reason, those who take the rule of *reductio* to fall within the purview of propositional logic contend that Aristotle's theory of the categorical syllogism presupposes modes of propositional reasoning. Among modern commentators, this view has been endorsed, for example, by Łukasiewicz:

> No one can fully understand Aristotle's proofs who does not know that there exists besides the Aristotelian system another system of logic more fundamental than the theory of the syllogism. It is the logic of propositions. . . . The first system of propositional logic was invented about half a century after Aristotle: it was the logic of the Stoics. . . . It seems that Aristotle did not suspect the existence of another system of logic besides his theory of the syllogism. Yet he uses intuitively the laws of propositional logic in his proofs of imperfect syllogisms [such as Baroco and Bocardo]. (Łukasiewicz 1957: 47–9)

In order to capture its supposed reliance on propositional logic, Łukasiewicz regards Aristotle's syllogistic as a theory formulated within a propositional object language employing propositional operators such as negation, conjunction, and the material conditional. He is thus able to formulate *reductio* as a rule of inference between compound propositions in this language and justify the rule by appeal to the laws of classical propositional logic.

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[42] Philoponus *In Pr. An.* 246.19–21, 247.17–32, and 302.1–6, Pseudo-Ammonius *In Pr. An.* 65.31–2, 66.13–14, 67.6–12, 69.15–16, and 69.24–5; see also Pseudo-Themistius *In Pr. An.* 94.35–7.

[43] *Pr. An.* 1.23 40b23–9.

[44] See *Pr. An.* 1.44 50b2–3 in conjunction with 1.44 50a16 and 1.32 46b40–47a1.

[45] See also *Pr. An.* 1.23 41a32–4; cf. Alexander *In Pr. An.* 259.9–11, 261.24–8, and 262.3–4, Pseudo-Ammonius *In Pr. An.* 66.39–67.4, Lear 1980: 34, Crivelli 2011: 125.

This way of modeling Aristotle's syllogistic has been rightly rejected by commentators such as Corcoran and Smiley. As Smiley points out, 'the deductive machinery of propositional logic ... is conspicuous by its absence from Aristotle's writings'.[46] Thus, Łukasiewicz is wrong to suggest that Aristotle's syllogisms ought to be understood as conditionals formulated in a propositional object language. Instead, as Corcoran and Smiley maintain, the syllogisms discussed by Aristotle should be regarded as arguments consisting of only simple categorical propositions.[47]

This raises the question as to how the rule of *reductio* is to be formulated with respect to an object language that admits only categorical propositions and does not include an operator of propositional negation. Corcoran and Smiley provide such a formulation of the rule by appealing to the contradictory relations that obtain between categorical propositions, as specified in the traditional square of opposition.[48] These relations are captured by the following operation, $\textsc{con}(\cdots)$, which maps each categorical proposition to its contradictory:

φ	$\textsc{con}(\varphi)$
AaB	AoB
AeB	AiB
AiB	AeB
AoB	AaB

By means of this operation, the rule of *reductio* employed by Aristotle in his indirect proofs of Baroco and Bocardo can be formulated as follows:[49]

$$\frac{\varphi, \psi \vdash \chi}{\varphi, \textsc{con}(\chi) \vdash \textsc{con}(\psi)}$$

The apparent similarity between this formulation of the rule of *reductio* and the first Stoic *thema* helps to explain why ancient logicians were led to claim that Aristotle's proofs by *reductio* rely on this Stoic principle. There is, however, a crucial difference between the first Stoic thema and the rule of *reductio* as it appears in systems of categorical logic such as those put forward by Corcoran and Smiley. The difference is that, for the Stoics, as well as for Łukasiewicz, the operation $\textsc{con}(\cdots)$ is defined in terms of a primitive object-language operator of propositional negation. By contrast, no such operator is available in purely categorical systems of logic. This is because these systems do not countenance any primitive propositional operators for constructing compound propositions, but only admit simple, categorical propositions which result from applying the a-, e-, i-, or o-copula to a

[46] Smiley 1973: 137.

[47] Corcoran 1972: 696–700, 1974: 94–112, Smiley 1973: 136–42, Smith 1989: xvi–xxii.

[48] Corcoran 1972: 696–8, 1974: 101–12, Smiley 1973: 141–2, Smith 1989: xix–xxi, Moss 2011: 182–3, van Rooij 2012: 87.

[49] It should be noted that this rule does not underwrite Aristotle's proof by *reductio* of Darapti (*Pr. An.* 1.7 29a36–9; see also 1.6 28a22–3). In this proof, Aristotle employs a variant of the rule in which the contradictory of χ, $\textsc{con}(\chi)$, is replaced by the contrary of χ, where the contrary of an a-proposition is the corresponding e-proposition, and vice versa; see Thom 1981: 39–41, Crivelli 2011: 156–8. The contrariety of a- and e-propositions is also assumed by Aristotle in his proof by *reductio* of the rule of a-conversion (*Pr. An.* 1.2 25a17–19). This gives rise to the well-known problem of existential import, which we discuss in §6 below.

pair of terms.[50] Consequently, as Smiley points out, the operation $\mathrm{CON}(\cdots)$ which figures in the above formulation of the rule of *reductio* must be treated as 'part of the metatheory and not a connective belonging to the system itself'.[51]

For a categorical logician, this metatheoretical account of *reductio* has the advantage of avoiding the need to posit any principles in the object language that exceed the syntactic bounds of categorical logic. At the same time, there is a potential cost that comes with relegating the rule of *reductio* to the metatheory. In particular, such a metatheoretical formulation of the rule fails to be schematic in the sense that its object-language instances cannot be obtained simply by replacing the schematic letters 'φ', 'ψ', and 'χ' with propositions of the object language.[52] If, for example, the schematic letter 'χ' appearing in the expression '$\mathrm{CON}(\chi)$' is replaced by a categorical proposition, the resulting expression still includes the metatheoretical operator '$\mathrm{CON}(\cdots)$' and hence is not a proposition of the object language. Consequently, in order to determine the object-language instances of the rule of *reductio*, one must know how to apply the metatheoretical operation $\mathrm{CON}(\cdots)$ codified by the traditional square of opposition. Thus, while the systems of categorical logic put forward by Corcoran and Smiley are able to implement the rule of *reductio* in a purely categorical object language, they do so only at the expense of adopting a nonschematic formulation of this rule which presupposes substantive metatheoretical knowledge of the traditional square of opposition.[53]

Now, a categorical logician could avoid the need to make such a compromise by giving separate schematic formulations for each of the specific categorical rules of inference that are licensed by the general rule of *reductio*. For example, the rule of *reductio* employed in Aristotle's indirect proofs of Baroco and Bocardo could be formulated as follows:

$$\frac{\varphi, AaB \vdash CaD}{\varphi, CoD \vdash AoB}$$

[50] See Smith 1989: xvii, Patterson 1995: 17–18, Whitaker 1996: 80–1, Charles 2000: 380, Malink 2013: 25.

[51] Smiley 1973: 141; similarly, van Rooij 2012: 87.

[52] According to this criterion of schematicity, the instances of a schema are all and only those expressions which result from replacing the schematic letters appearing in the schema with object-language expressions of the appropriate syntactic type. This substitutional criterion of schematicity is presupposed, e.g., by Vaught (1967: 473) and Quine (1986: 50–51). By contrast, Corcoran (2006: 219–23) discusses a broader notion of schematicity which allows for a schema to include additional 'side conditions' that impose substantive constraints on which expressions are to count as its object-language instances (see n. 53 below).

[53] For comparison, consider the following formulation of Tarski's T-schema: '*x is a true sentence if and only if p*' (Tarski 1956: 155–6). Tarski adds to this formulation of the T-schema a side condition to the effect that its instances are to be obtained by substituting 'in place of the symbol "*p*" in this schema any sentence, and in the place of "*x*" any individual name of this sentence'. The application of this formulation of the T-schema presupposes substantive metatheoretical knowledge as to which expressions of the object language count as names of which sentences. Hence, this formulation fails to be schematic in the sense just described (despite being schematic in the broader sense discussed by Corcoran (2006); see n. 52 above). By contrast, the following formulation of the T-schema is fully schematic: '*"p" is a true sentence if and only if p*' (Tarski 1956: 159). Given that the quotation marks surrounding the first occurrence of '*p*' are part of the object language, the application of this schema no longer presupposes any substantive metatheoretical knowledge, since its object-language instances are obtained simply by substituting sentences for the schematic letter '*p*'.

Unlike the general metatheoretical formulation of *reductio*, this specific rule is schematic in the sense that its object-language instances can be obtained simply by replacing the schematic letters 'φ', 'A', 'B', 'C' and 'D' with expressions of the object language that are of the appropriate syntactic type. Additional schematic rules of this sort involving different combinations of categorical propositions would then be needed to account for Aristotle's proofs by *reductio* of other syllogistic moods, such as Darii, Ferio, Camestres, and Festino.[54] For instance, the specific rule of *reductio* underwriting Aristotle's indirect proof of Darii from Celarent would be formulated as follows:

$$\frac{\varphi, AeB \vdash CeD}{\varphi, CiD \vdash AiB}$$

Taking into account all possible combinations of a-, e-, i-, and o-propositions, there are sixteen schematic rules of this sort that would be required to reproduce the full effect of the general rule of *reductio* in a categorical language which countenances only these four kinds of proposition. Thus, one way to license the method of *reductio* by means of principles that are schematic with respect to a categorical object language is to posit each of these sixteen rules as a separate principle.

This case-by-case approach has the obvious disadvantage of obscuring the fact that the sixteen specific rules of *reductio* are not unrelated, but are all underwritten by one and the same general principle. A categorical logician following Aristotle's lead would presumably be dissatisfied with this lack of generality in the formulation of *reductio*. For, Aristotle himself formulates the rule of *reductio* in general terms using the abstract notion of the contradictory of a proposition. Thus, he characterizes the assumption for *reductio* as the 'contradictory (ἀντίφασις) of the conclusion', and makes it clear that this characterization applies regardless of the specific kind of categorical proposition the conclusion happens to be.[55] This is in keeping with Aristotle's broader methodological maxim that a satisfactory formulation of a law should do more than merely assert each of its instances. Rather, the law should be formulated in terms of the most general properties in virtue of which it applies in any given case. Take, for example, the geometrical law that, for any given triangle, the sum of its interior angles is equal to the sum of two right angles. According to Aristotle, a satisfactory formulation of this law should not consist of separate statements for equilateral, isosceles, and scalene triangles. Although every triangle is of one of these three kinds, someone who has established three separate theorems to the effect that equilateral, isosceles, and scalene triangles each have the desired property does not yet possess a proper understanding of the law. Instead, Aristotle insists, to arrive at such an understanding, one must establish a fully general formulation of the law that applies to all triangles, simply insofar as they are triangles.[56] Likewise, a satisfactory formulation of the rule of *reductio* should not consist of multiple versions of the rule that apply separately to a-, e-, i-, and

[54] Aristotle uses the method of *reductio* to streamline his deductive system by deriving the perfect moods Darii and Ferio from Celarent (*Pr. An.* 1.7 29b1–19; see Alexander *In Pr. An.* 113.5–114.30, Weidemann 2004: 73–5, Barnes 2007: 364–6, Ebert & Nortmann 2007: 354–6). Moreover, Aristotle mentions proofs by *reductio* of Camestres (1.5 27a14–15) and Festino (1.7 29b5–6).

[55] *Pr. An.* 1.23 41a23–37, 2.11 61a19–21, 2.14 62b29–38. Aristotle indicates that this characterization applies regardless of whether the conclusion to be established by the *reductio* argument is an affirmative or negative categorical proposition (62b37–8).

[56] *Post. An.* 1.5 74a25–32, see also 74a17–25; cf. Philoponus *In Post. An.* 76.28–78.9, Barnes 1993: 123–4.

o-propositions—even if these are all the categorical propositions there are. Instead, a proper formulation of this rule should apply to all propositions in general, simply insofar as they are propositions.

Thus, all told, a categorical logician faces a challenge in providing a fully general formulation of *reductio* that is also schematic with respect to the object language of categorical logic. By contrast, a general and schematic formulation of *reductio* is readily available in the language of propositional logic, which includes an operator of propositional negation. This fact may help to account for the view, reiterated by a number of logicians over the centuries, that in employing the method of *reductio* Aristotle tacitly relies on principles of propositional logic. Of course, a categorical logician might respond to this charge by simply rejecting the requirement that the rule of *reductio* be given a general and schematic formulation.[57] Depending on which of these two desiderata is rejected, the method of *reductio* could then be implemented either by a multiplicity of schematic rules, each capturing a specific categorical application of *reductio*, or by a single general rule based on the metatheoretical relation of contradictoriness codified by the traditional square of opposition.

In principle, either of these approaches constitutes an acceptable way for a categorical logician to counter the charge of tacitly relying on propositional logic. Nonetheless, the question remains whether one can offer a more satisfactory response to the charge by developing a purely categorical system of logic in which the rule of *reductio* has a general and schematic formulation. At first blush, it is not obvious how, if at all, this can be achieved. For, it would require that the operation $\textsc{con}(\cdots)$ mapping each proposition to its contradictory be represented in the object language of categorical logic without introducing into the syntax any primitive propositional operators or other noncategorical elements. In what follows, we show how this aim can be achieved by utilizing the theory of propositional terms developed in the logical writings of Leibniz. The key elements of this theory are expounded in an essay titled *Specimina calculi rationalis*, in which Leibniz undertakes to justify the rule of *reductio* within a purely categorical system of logic.

§3. Leibniz's categorical proof of *reductio*. Throughout his logical writings, Leibniz makes frequent use of the method of *reductio ad absurdum*. For the most part, he employs this method freely without any further explanation. On some occasions, however, he is unwilling to take the validity of this method for granted, but instead seeks to derive the rule of *reductio* from more basic principles of categorical logic. For example, in his 1686 treatise *Generales inquisitiones de analysi notionum et veritatum*, Leibniz presents a series of proofs by *reductio* accompanied by the following remark:

> This mode of reasoning, i.e., *reductio ad absurdum*, has, I believe, already been established in what precedes. (Leibniz, *Generales inquisitiones* §91, A VI.4 766 n. 36)

In this passage, Leibniz regards the rule of *reductio* as something that needs to be 'established', that is, as something to be derived rather than simply posited as a principle. Contrary to what he suggests, no such derivation of the rule of *reductio* appears in the *Generales inquisitiones*. Leibniz does, however, supply the outline of such a derivation

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[57] For example, van Rooij rejects the desideratum of schematicity in his implementation of the Peripatetic program, arguing that the positing of a nonschematic metatheoretical rule of *reductio* does not imply that 'syllogistic depends on propositional logic' (van Rooij 2012: 87 n. 3).

in a related essay titled *Specimina calculi rationalis*, written sometime between 1686 and 1690.[58] Since Leibniz's treatment of the rule of *reductio* in this essay is somewhat intricate, it calls for a detailed point-by-point exegesis.

In §15 of the *Specimina calculi rationalis*, Leibniz describes how a theorem of the form $\varphi \vdash \psi$ can be established by means of the method of *reductio*:

> Let us posit that that [i.e., the proposition φ] is true, and that, nevertheless, this [i.e., the proposition ψ] is false, so that if something absurd will follow therefrom, then this [i.e., ψ] undoubtedly will be derivable from that [i.e., φ] (by the Lemma of the following paragraph). (Leibniz, *Specimina calculi rationalis* §15)

At the end of this passage, Leibniz indicates that the method of *reductio* is to be justified 'by the Lemma of the following paragraph'. In fact, the lemma Leibniz is here referring to is stated, not in §16, but in §17 of the *Specimina*, where it is formulated as follows:[59]

> In §15 we assumed that if, when A is posited, B follows, then *non-A* follows from *non-B*. Or, more generally, according to our way of subsuming hypothetical propositions under categorical ones, we assumed this consequence: *A is B*, therefore *non-B is non-A*.[60] (Leibniz, *Specimina calculi rationalis* §17)

In the first sentence of this passage, Leibniz supplies the lemma he promised in §15 as part of his argument justifying the method of *reductio*. While it is not entirely clear how Leibniz's formulation of this lemma ought to be construed, some guidance on this point can be gleaned from his subsequent remark in the passage that the lemma is a special case of the following, more general law of contraposition:

$$A\ is\ B \ \vdash\ non\text{-}B\ is\ non\text{-}A.$$

By the phrase '*A is B*' Leibniz means to express the a-proposition BaA which occurs, for example, in categorical syllogisms of the form Barbara.[61] Moreover, the expressions '*non-A*' and '*non-B*' are used by Leibniz to designate privative terms such as *non-rational*

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[58] The title '*Specimina calculi rationalis*' is not used by Leibniz but is adopted in the *Akademie* edition of Leibniz's works. For the dating of this essay between 1686 and 1690, see A VI.4 807, Lenzen 1986: 19, Schupp 2000: x–xi and xxxii.

[59] The phrase 'by the Lemma of the following paragraph' translates Leibniz's 'per Lemma prop. sequentis' (A VI.4 809). Schupp (2000: 184 n. 23) suggests that this phrase refers, not to §17, but to the subsequent sentences of §15. This reading, however, is implausible since these sentences do not contain a lemma that would in any way help to justify the rule of *reductio*. Moreover, Leibniz often uses the abbreviation 'prop.' to refer to numbered paragraphs of an essay rather than to particular propositions within such a paragraph (see, e.g., A VI.4 150, 815, 831–44). Finally, since the passage from §17 deals with what Leibniz explicitly identifies as an assumption made in §15, it is natural to identify this assumption with the lemma referred to in §15.

[60] This is the original version of the text, which Leibniz subsequently revised by replacing the clause '*A is B*, therefore *non-B is non-A*' with '*A is B*, therefore *it is false that non-B is A*' (see the textual notes at A VI.4 809). The most likely reason for this revision is that Leibniz was unable to provide a satisfactory proof of the law of contraposition expressed by the original clause (see Lenzen 1986: 32, Schupp 2000: 185–6 n. 29). We will discuss Leibniz's treatment of contraposition below. For now, we focus on Leibniz's justification of *reductio* based on contraposition, as intended in the original version of the text.

[61] See, e.g., A VI.4 206, 280, *GI* §28, §124, §129.

and *non-animal*. Such privative terms are discussed by Aristotle in the *De interpretatione*, and by late antiquity they had come to be regarded as an integral part of categorical logic. These terms figure prominently in Leibniz's categorical logic, and in many of his essays he takes it for granted that, for any term A, there is a corresponding privative term, *non-A*. For the sake of brevity, we will write '$\overline{A}$' to designate the privative term *non-A*. Using this notation, the above law of contraposition reads as follows:

$$BaA \vdash \overline{A}a\overline{B}.$$

In the passage from §17 of the *Specimina* just quoted, Leibniz claims that this law of contraposition comprehends the lemma needed for the proof of *reductio*. Leibniz formulates this lemma as follows: 'if, when A is posited, B follows, then *non-A* follows from *non-B*'. The obvious problem with viewing this lemma as an instance of the law of contraposition is that, in the latter, the letters 'A' and 'B' designate terms that serve as the subject or predicate of a categorical proposition. In the lemma, on the other hand, these letters seem to designate propositions. This, at least, is strongly suggested by the fact that B is said to 'follow' from A. Moreover, insofar as the lemma is to play a role in justifying the rule of *reductio*, the letters 'A' and 'B' must somehow represent the propositions φ and ψ which appear in Leibniz's characterization of *reductio* in §15 of the *Specimina*.

Fortunately, Leibniz supplies the background needed to resolve these apparent ambiguities in §16 of the *Specimina*. There, he explains how to interpret standard expressions of categorical logic such as '*non-A*' and 'AaB' when 'A' and 'B' are taken to designate propositions rather than ordinary terms such as *rational* and *animal*. He writes:

> If A is a proposition or statement, by *non-A* I mean that the proposition A is false. And when I say *A is B*, and A and B are propositions, I mean that B follows from A. . . . If *A is B* is called C, then C will be the same as *A's being B*. Thus, when we say *E is F* follows from *A is B*, this is the same as if we were to say *A's being B is E's being F*. (Leibniz, *Specimina calculi rationalis* §16)

In the second sentence of this passage, Leibniz states that, when 'A' and 'B' designate propositions, the categorical proposition BaA means that B follows from A. While this might be taken to imply that a proposition can serve as the subject or predicate of a categorical proposition, the last two sentences of the passage suggest that this is not quite what Leibniz intends. As he explains, it is not the proposition *E is F* itself that serves as the subject or predicate, but rather a certain syntactic transform of this proposition, which he signifies by the phrase '*E's being F*'.[62] As Leibniz makes clear in the *Generales inquisitiones*, this syntactic transformation has the effect of converting a proposition into a term:[63]

[62] The phrase '*E's being F*' translates '*E esse F*'. This latter phrase differs from that which signifies the original proposition, '*E est F*', in using the infinitive form of the verb, '*esse*', instead of the finite form, '*est*'. Sometimes Leibniz also adds the Greek definite article τό, writing '*τὸ E esse F*' instead of '*E esse F*' (e.g., *GI* §§138–42 and A VI.4 740).

[63] See Schupp 2000: 184 n. 25. This syntactic transformation reflects Leibniz's view that 'every proposition can be conceived of as a term' (*GI* §75, §109, and §197). When a proposition is conceived of as a term, Leibniz describes the proposition as giving rise to a 'new term' (*terminus novus*, §197 and §198.7; *contra* Parkinson 1966: 86 n. 2).

> If the proposition *A is B* is considered as a term, as we have explained that it can be, there arises an abstract term, namely *A's being B*. And if from the proposition *A is B* the proposition *C is D* follows, then from this there comes about a new proposition of the following kind: *A's being B is* (or *contains*) *C's being D*; or, in other words, *the B-ness of A contains the D-ness of C*, or *the B-ness of A is the D-ness of C*. (Leibniz, *Generales inquisitiones* §138)

In this passage, Leibniz stipulates that the phrases '*A's being B*' and '*the B-ness of A*' signify an abstract propositional term generated from the proposition *A is B*.[64] The phrases used by Leibniz to signify propositional terms are chosen so as to ensure that the categorical propositions into which they enter are grammatically well-formed. For present purposes, what is important is that the expressions for propositional terms can be derived from those for the corresponding propositions in a systematic manner, so that the latter can be put in one-to-one correspondence with the former. We will use square brackets to indicate the operation mapping any given proposition to the corresponding propositional term. Thus, for example, $[BaA]$ is the propositional term corresponding to the proposition BaA. In this notation, the complex categorical proposition *A's being B is C's being D* is written as follows:

$$[DaC]a[BaA].$$

In the passage just quoted, Leibniz takes this categorical proposition to express the claim that the proposition DaC follows from the proposition BaA. Thus, he endorses the following principle concerning a-predications between propositional terms:

$$BaA \vdash DaC \qquad \text{if and only if} \qquad \vdash [DaC]a[BaA].$$

In §16 of the *Specimina*, Leibniz gives a more general formulation of this principle, asserting that 'when I say *A is B*, and *A* and *B* are propositions, I mean that *B* follows from *A*'.[65] In this formulation, the letters '*A*' and '*B*' stand for any categorical propositions φ and ψ. Accordingly, the expression '*A is B*' designates the a-proposition in which the propositional term corresponding to ψ is predicated of the propositional term corresponding to φ, i.e., $[\psi]a[\varphi]$. On the other hand, when Leibniz writes that '*B* follows from *A*', he means that the proposition ψ follows from the proposition φ, i.e., $\varphi \vdash \psi$. Thus, the principle stated by Leibniz in §16 of the *Specimina* can be formulated as follows:

$$\varphi \vdash \psi \qquad \text{if and only if} \qquad \vdash [\psi]a[\varphi].$$

We will refer to this biconditional as the principle of 'propositional a-predication'.[66]

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[64] Leibniz refers to such terms not as 'propositional terms' but as 'complex terms' (see *GI* §61, §65, §75; cf. A VI.4 528–9). He describes these terms as 'abstract' at *GI* §§138–42, A VI.4 740, 992.

[65] Similarly, Leibniz writes: 'For a proposition to follow from a proposition is nothing other than for the consequent to be contained in the antecedent as a term in a term. By this method we reduce consequences to propositions, and propositions to terms' (*GI* §198.8). In §138 of the *Generales inquisitiones*, Leibniz asserts this principle only for the case in which the antecedent and consequent are a-propositions. Nonetheless, it is clear that, even in the *Generales inquisitiones*, Leibniz intends this principle to apply not only to a-propositions but to any categorical propositions; see *GI* §§139–142. For example, in §140b he applies the principle to i-propositions.

[66] The principle of propositional a-predication guarantees that the categorical relations that obtain between propositional terms exactly mirror the inferential relations that obtain between the

In addition to the principle of propositional a-predication, in §16 of the *Specimina* Leibniz introduces a second principle pertaining to propositional terms. This principle, which explicates the meaning of the privative of a propositional term, asserts that, 'if A is a proposition or statement, by *non-A* I mean that the proposition A is false'. Similarly, Leibniz writes in the *Generales inquisitiones*:

> If B is a proposition, *non-B* is the same as *B is false*, or, *B's being false*. (Leibniz, *Generales inquisitiones* §32, A VI.4 753 n. 18)

Let φ be the proposition designated by the letter 'B' in this passage. Then the expression '*non-B*' designates the privative of the propositional term corresponding to φ, i.e., $\overline{[\varphi]}$. In the passage just quoted, Leibniz states that this privative term is the same as the term *φ's being false*, which is the propositional term generated from the proposition *φ is false*. Leibniz uses phrases of the form '*φ is false*' to designate the contradictory of φ in proofs by *reductio*.[67] In our notation, the contradictory of φ is expressed by '$\textsc{con}(\varphi)$'. Thus, in the above passage, Leibniz states that the privative term $\overline{[\varphi]}$ is the same as the propositional term $[\textsc{con}(\varphi)]$.[68] Now, when Leibniz asserts that a term A is 'the same as' a term B, this is tantamount to asserting the pair of a-propositions AaB and BaA.[69] All told, then, Leibniz endorses the following two laws concerning the privatives of propositional terms:

$$\vdash [\textsc{con}(\varphi)]\, a\, \overline{[\varphi]} \qquad\qquad \vdash \overline{[\varphi]}\, a\, [\textsc{con}(\varphi)].$$

We will refer to these two laws, collectively, as the principle of 'propositional privation'.

The device of propositional terms, along with the principles of propositional a-predication and privation, provide the background needed to understand Leibniz's justification of the method of *reductio* in the *Specimina calculi rationalis*. In particular, these principles allow us to make sense of Leibniz's claim in §17 that *reductio* relies on the lemma that 'if, when A is posited, B follows, then *non-A* follows from *non-B*'. As it stands, this formulation of the lemma exhibits an incongruity. On the one hand, the expressions '*non-A*' and '*non-B*' seem to designate privatives of propositional terms. Like any other terms, these privative terms can stand in the categorical relation of a-predication to one

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corresponding propositions. This marks an important difference between Leibniz's theory of propositional terms and Boole's categorical analysis of hypothetical propositions. Unlike Leibniz, Boole maintains that the logic of hypothetical propositions of the form *If A is B, then C is D* 'does not depend upon any considerations which have reference to the terms A, B, C, D. . . . We may, in fact, represent the propositions *A is B*, *C is D*, by the arbitrary symbols X and Y, respectively' (Boole 1847: 48; cf. 1854: 164–7). Thus, in Boole's system, the hypothetical proposition *If A is B, then C is D* is represented by a formula such as '*X is Y*', which makes no reference to the constituent propositions *A is B* and *C is D*. Consequently, Boole's system cannot express the way in which the logic of hypothetical propositions supervenes on the deductive relations that obtain between their constituent propositions. Peirce (1873: 371) regards this as an 'obvious imperfection' of Boole's system. Likewise, Frege (1979: 14–15) argues that, for this reason, Boole's system lacks organic unity. Leibniz's categorical analysis of hypothetical propositions, on the other hand, is not open to such criticisms since the propositional terms appearing in the complex proposition $[DaC]a[BaA]$ preserve reference to the constituent propositions BaA and DaC.

[67] See, e.g., the proofs by *reductio* given by Leibniz in *GI* §§93–4 and in §15 of the *Specimina*. See also Leibniz's formulation of the rule of *reductio* at A VI.4 499.

[68] This is in accordance with Leibniz's statement that, when 'A' signifies a proposition, '*non-A*' signifies the contradictory of that proposition (*contradictoriam propositionis*, *GI* §13).

[69] A VI.4 285; cf. A VI.4 813, *GI* §30.

another. A term, however, cannot ‘follow from’ another term. Instead, what can stand in this relation of following are propositions and their contradictories. Thus, there are two natural ways of resolving the incongruity in Leibniz’s formulation of the lemma: either replace the privatives of propositional terms by the contradictories of the corresponding propositions, or replace the relation of following by that of a-predication. The former option yields the following version of the rule of *reductio*:

$$\frac{\varphi \vdash \psi}{\textsc{con}(\psi) \vdash \textsc{con}(\varphi)}$$

The latter option yields the following law of contraposition for propositional terms:

$$[\psi]\, a\, [\varphi] \vdash \overline{[\varphi]}\, a\, \overline{[\psi]}.$$

In opting for his incongruous formulation of the lemma, Leibniz presumably means to assert both the former rule of *reductio* and the latter law of contraposition at the same time. Moreover, he indicates that the former is to be derived from the latter. This is suggested by his subsequent remark in §17 to the effect that, ‘according to our way of subsuming hypothetical propositions under categorical ones’, the lemma can be subsumed under the general law of contraposition for terms, $BaA \vdash \overline{A}a\overline{B}$.

As we have just seen, Leibniz’s method of ‘subsuming hypothetical propositions under categorical ones’ is explicated in §16 and is captured by the principles of propositional a-predication and privation. Given these two principles, it is easy to see how Leibniz intended to derive the rule of *reductio* from the general law of contraposition. The proof proceeds as follows. Suppose that $\varphi \vdash \psi$. By propositional a-predication, this implies $\vdash [\psi]a[\varphi]$. We then have the following derivation:

1	$[\psi]a[\varphi]$	premise
2	$\overline{[\varphi]}\, a\, \overline{[\psi]}$	contraposition: 1
3	$[\textsc{con}(\varphi)]\, a\, \overline{[\varphi]}$	propositional privation
4	$\overline{[\psi]}\, a\, [\textsc{con}(\psi)]$	propositional privation
5	$[\textsc{con}(\varphi)]\, a\, [\textsc{con}(\psi)]$	Barbara: 2, 3, 4

Given $\vdash [\psi]a[\varphi]$, this derivation establishes $\vdash [\textsc{con}(\varphi)]\, a\, [\textsc{con}(\psi)]$. Hence, by propositional a-predication, $\textsc{con}(\psi) \vdash \textsc{con}(\varphi)$.

Thus, by means of the law of contraposition, Leibniz is able to establish a rule of *reductio* to the effect that $\varphi \vdash \psi$ entails $\textsc{con}(\psi) \vdash \textsc{con}(\varphi)$. In fact, however, Leibniz’s proofs by *reductio* in the *Specimina* and elsewhere require a slightly stronger version of this rule which is applicable to inferences with multiple premises, such as $\varphi, \psi \vdash \chi$. Consider, for example, the proof by *reductio* given by Leibniz in §17 of the *Specimina*:

> (1) *A is B* by hypothesis, I claim that it is false that (2) *non-B is A*. For, since *B* can be substituted for *A* (by 1), let it be substituted in 2, so that we will have *non-B is B*, which is false by §6. (Leibniz, *Specimina calculi rationalis* §17)

In this proof by *reductio*, the conclusion that *non-B is A* is false is derived from the premise *A is B* and the additional premise that *non-B is B* is false.[70] In our notation, the conclusion of this proof is formulated as '$\text{CON}(Aa\overline{B})$' and the second premise as '$\text{CON}(Ba\overline{B})$'. Thus, the claim established by Leibniz in the passage is:

$$BaA, \text{CON}(Ba\overline{B}) \vdash \text{CON}(Aa\overline{B}).$$

Leibniz proves this claim as follows:[71]

$$\begin{array}{r|l|l}
1 & BaA & \text{premise} \\
2 & \text{CON}(Ba\overline{B}) & \text{premise} \\
\hline
3 & \quad\mid Aa\overline{B} & \text{assumption for } \textit{reductio} \\
4 & \quad\mid BaA & \text{reiteration: 1} \\
5 & \quad\mid Ba\overline{B} & \text{Barbara: 3, 4} \\
6 & \text{CON}(Aa\overline{B}) & \textit{reductio}\text{: 2, 3–5}
\end{array}$$

In this proof, the conclusion of the *reductio* subproof stated in line 5 is derived not only from the assumption for *reductio* in line 3, but also from the original premise in line 1 (which is reiterated in line 4). The proof thus relies on the following rule of inference:

$$\frac{BaA, Aa\overline{B} \vdash Ba\overline{B}}{BaA, \text{CON}(Ba\overline{B}) \vdash \text{CON}(Aa\overline{B})}$$

Since the antecedent of this rule is an inference from two premises, BaA and $Aa\overline{B}$, the rule is not an instance of the single-premise version of *reductio* discussed above. Instead, it is an instance of the following two-premise version of *reductio*:

$$\frac{\varphi, \psi \vdash \chi}{\varphi, \text{CON}(\chi) \vdash \text{CON}(\psi)}$$

While Leibniz does not explicitly state this two-premise version of the rule in the *Specimina*, he does endorse this version of *reductio* elsewhere, in an essay titled *De formis syllogismorum mathematice definiendis*:

> In the method of *reductio*, we use the following principle: if the conclusion is false (i.e., if its contradictory is true), and one of the premises is true, then the other premise must necessarily be false, or, its contradictory must necessarily be true. The method of *reductio*, therefore, assumes

[70] Leibniz takes this latter premise to be a consequence of the assumption that $\overline{B}$ is a 'possible' term or, as Leibniz puts it, 'a being' (*ens*); see §§1–2 and §6 of the *Specimina*. Leibniz often reasons under the assumption that the terms appearing in a given proposition are possible. In Leibniz's proof by *reductio*, this assumption is in place for the term $\overline{B}$.

[71] When, in this proof, Leibniz writes that 'B can be substituted for A', he has in mind an application of the syllogistic mood Barbara, whereby the premise BaA licenses the substitution of B for A when A occurs as the predicate of an a-proposition. Accordingly, in §17 of the *Specimina*, Leibniz describes the application of Barbara as 'one-sided substitution' (*substitutio unilateralis*); see also A VI.4 143–5, 154, 275, 672.

> the principle of contradiction, but a contradiction obtains between a universal affirmative and particular negative, or, if the A is false then the O is true, and vice versa; likewise between the universal negative and the particular affirmative, or, if the E is false then the I will be true, and vice versa. (Leibniz, *De formis syllogismorum mathematice definiendis*, A VI.4 499)

It is clear from this passage that Leibniz intends the rule of *reductio* to apply not only to arguments with a single premise, but to two-premise arguments as well.[72] The two-premise version of *reductio*, however, cannot be derived from the categorical principles stated above. The most obvious way to establish this version of *reductio* is to posit the following two-premise version of the principle of propositional a-predication:

$$\varphi, \psi \vdash \chi \qquad \text{if and only if} \qquad \varphi \vdash [\chi]\, a\, [\psi]$$

Given this version of the principle, the proof of the two-premise rule of *reductio* proceeds just as before. Suppose that $\varphi, \psi \vdash \chi$. By the two-premise version of propositional a-predication, this implies $\varphi \vdash [\chi]\, a\, [\psi]$. We then have the following derivation:

1	φ	
2	$[\chi]\, a\, [\psi]$	$\varphi \vdash [\chi]\, a\, [\psi]$: 1
3	$\overline{[\psi]}\, a\, \overline{[\chi]}$	contraposition: 2
4	$[\text{CON}(\psi)]\, a\, \overline{[\psi]}$	propositional privation
5	$\overline{[\chi]}\, a\, [\text{CON}(\chi)]$	propositional privation
6	$[\text{CON}(\psi)]\, a\, [\text{CON}(\chi)]$	Barbara: 3, 4, 5

This derivation shows that $\varphi \vdash [\text{CON}(\psi)]\, a\, [\text{CON}(\chi)]$. Hence, by the two-premise version of propositional a-predication, we have $\varphi, \text{CON}(\chi) \vdash \text{CON}(\psi)$.

In his various statements of propositional a-predication, Leibniz does not distinguish between the one-premise version of the principle and its two-premise counterpart. Nevertheless, given the role that this principle plays in Leibniz's justification of the two-premise version of *reductio*, it seems clear that he endorses the two-premise version of propositional a-predication. More generally, once the principle is allowed to apply not only to arguments with a single premise but also to arguments with two premises, there is no reason to object to the application of the principle to arguments with any finite number of premises.[73] Thus, Leibniz's commitment to the one-premise and two-premise versions

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[72] After his statement of the two-premise version of *reductio* in this passage, Leibniz proceeds to utilize this version of the rule to derive the syllogistic moods Baroco and Bocardo from Barbara (A VI.4 500). For similar applications of the two-premise version of *reductio*, see A VI.4 144, 167, 815, C 208, 304–9.

[73] The shift from the one-premise to the two-premise version of propositional a-predication carries significant consequences for the interpretation of a-propositions of the form $[\psi]\, a\, [\varphi]$ (see n. 106 below). In the notation of Fitch-style proofs, the two-premise version of propositional a-predication licenses a rule of reiteration whereby a single proposition appearing in a superordinate proof can be reiterated within a subproof of ψ from φ used to establish $[\psi]\, a\, [\varphi]$. By contrast, the shift from the two-premise to the multi-premise version of propositional a-predication is much

of propositional a-predication can naturally be captured by the following more general version of the principle, in which Γ is any finite set of propositions:

$$\Gamma, \varphi \vdash \psi \qquad \text{if and only if} \qquad \Gamma \vdash [\psi]\, a\, [\varphi].$$

All told, then, the rule of *reductio* employed by Leibniz in the *Specimina* can be derived from the following principles:

Barbara:	$AaB,\ BaC \vdash AaC$
contraposition:	$AaB \dashv\vdash \overline{B}a\overline{A}$
propositional privation:	$\vdash [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]}$ and $\vdash \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)]$
propositional a-predication:	$\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash [\psi]\, a\, [\varphi]$

Apart from the syllogistic mood Barbara, which Leibniz takes for granted in the *Specimina*, his derivation of *reductio* relies on the principles of contraposition, propositional privation, and propositional a-predication. These are exactly the three principles that Leibniz highlights in §§16–17 of the *Specimina*.

As for the law of contraposition, Leibniz does not regard it as a primitive of his system but undertakes to derive it from more fundamental principles. In the *Specimina*, he makes a few attempts to provide such a derivation, none of which is ultimately successful.[74] Leibniz makes more significant progress toward a proof of contraposition in the *Generales inquisitiones*. In §99 of this treatise, he offers the following proof of contraposition:[75]

> *A is B* is the same as *A is non-non-B* (by §96), and this is the same (by §87) as *No A is non-B*, that is, *No non-B is A* (by the conversion of the universal negative), which (by §87) is the same as *Every non-B is non-A* = *A is B*. Q.E.D. (Leibniz, *Generales inquisitiones* §99)

In the first sentence of this passage, Leibniz infers the a-proposition $\overline{\overline{B}}aA$ from BaA. This inference relies on the principle of double privation, which Leibniz states in §96 and which can be represented by the following two laws:[76]

$$\vdash \overline{\overline{A}}aA \qquad\qquad \vdash Aa\overline{\overline{A}}.$$

less significant, since it does not pertain to the question of whether a proposition can be reiterated into a subproof, but only to that of how many propositions can be reiterated.

[74] Leibniz abandons his first attempt to prove contraposition in §17 of the first part of the *Specimina* (A VI.4 809); see Lenzen 1986: 32, Schupp 2000: 185–6 n. 29, and n. 60 above. He makes a new attempt at such a proof in §22 of the second part of the *Specimina* (A VI.4 813). This latter attempt fails because it relies on the assumption that *A is not B* entails *A is non-B*. As Leibniz points out elsewhere, this assumption is not valid (*GI* §92); cf. Schupp 2000: xxxviii and 191–2. For Leibniz's various attempts to establish contraposition in these and other passages, see Lenzen 1986: 13–14 and 27–32, 1988: 63–4.

[75] In §§93–4, Leibniz attempts to prove contraposition by means of the method of *reductio ad absurdum*. In §95, he expresses dissatisfaction with this way of establishing contraposition and seeks out an alternative proof (which he supplies in §99). Leibniz's dissatisfaction with his earlier proof by *reductio* is perhaps not surprising given the essential role that contraposition plays in his justification of *reductio* in the *Specimina*.

[76] In §96, Leibniz asserts that the terms $\overline{\overline{A}}$ and A coincide (see also *GI* §2, §96, §171.4, §189.2, §198.3, A VI.4 807, 811). This is equivalent to affirming the pair of a-propositions $\overline{\overline{A}}aA$ and $Aa\overline{\overline{A}}$ (see n. 69 above).

Leibniz proceeds to identify the proposition $\overline{\overline{B}}aA$ with the e-proposition $\overline{B}eA$, which he expresses by the phrase ‘*No A is non-B*’. This is justified by a traditional principle of categorical logic which is known as obversion and is stated by Leibniz in §87 as follows:[77]

> *No A is B* is the same as *A is non-B* .... We thus have a transition between infinite affirmatives and negatives.[78] (Leibniz, *Generales inquisitiones* §87)

In our notation, the principle of obversion is expressed by the following bidirectional rule of inference:

$$BeA \dashv\vdash \overline{B}aA.$$

A version of this rule is endorsed by Aristotle in the *De interpretatione*, when he states that the proposition *No non-man is just* is equivalent to *Every non-man is non-just*.[79] More detailed treatments of obversion were given by Neoplatonic authors in late antiquity, who referred to the rule as ‘the canon of Proclus’.[80] Obversion figured prominently in scholastic systems of categorical logic and is often invoked by Leibniz in his logical writings.[81]

In addition to the principles of double privation and obversion, Leibniz’s proof of contraposition in §99 of the *Generales inquisitiones* also relies on the rule of e-conversion, which he refers to as ‘the conversion of the universal negative’. Thus, Leibniz’s proof of contraposition proceeds as follows:

1	BaA	premise
2	$\overline{\overline{B}}aB$	double privation
3	$\overline{\overline{B}}aA$	Barbara: 1, 2
4	$\overline{B}eA$	obversion: 3
5	$Ae\overline{B}$	e-conversion: 4
6	$\overline{A}a\overline{B}$	obversion: 5

[77] The principle of obversion is also stated by Leibniz at *GI* §97–8, §190, A VI.4 126, C 230, 301.

[78] By ‘infinite affirmatives’ Leibniz means a-propositions with a privative predicate term (see *GI* §80). This terminology derives from Aristotle’s characterization of privative terms as ‘indefinite’ (ἀόριστον); *De int.* 2 16a29–32, 3 16b12–15, 10 19b5–12, 20a31–6.

[79] *De int.* 10 20a39–40; see Soreth 1972: 413–6, Whitaker 1996: 148, Weidemann 2014: 366–7.

[80] The canon of Proclus is described by the Neoplatonic commentator Stephanus as follows: ‘Aristotle demonstrated with examples which propositions follow from which, but Proclus gives a general method for finding these propositions. The method is as follows: if a proposition is given, keep the same subject term and the same quantity, but change the quality and the predicate term. The quality you change as follows: if the given proposition is affirmative, make it negative, and conversely if it is negative, make it affirmative. The predicate term you change as follows: if it is definite, make it indefinite, and conversely if it is indefinite, make it definite’ (Stephanus *In De int.* 46.24–32). Similarly, Ammonius *In De int.* 181.30–182.25; cf. Soreth 1972: 410–11 and 422, Weidemann 2014: 366–7, Helmig 2017: 189.

[81] For a discussion of the role that obversion plays in Leibniz’s logic, see Lenzen 1986: 9–14. For the role played by obversion in scholastic logic, see, e.g., Parsons 2014: 75–6. Some medieval logicians rejected the e-a-direction of obversion (i.e., $BeA \vdash \overline{B}aA$) for reasons relating to the problem of existential import; see, e.g., Buridan, *Treatise on Consequences* Book 1, Chapter 8, 17th Conclusion (cf. n. 130 below).

Given this proof of contraposition from e-conversion, obversion, and double privation, Leibniz's derivation of the rule of *reductio* in the *Specimina calculi rationalis* can be seen to rely on the following six principles:

Barbara:	$AaB,\, BaC \vdash AaC$
e-conversion:	$AeB \vdash BeA$
obversion:	$AeB \dashv\vdash \overline{A}aB$
double privation:	$\vdash \overline{\overline{A}}aA \quad$ and $\quad \vdash Aa\overline{\overline{A}}$
propositional privation:	$\vdash [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]} \quad$ and $\quad \vdash \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)]$
propositional a-predication:	$\Gamma, \varphi \vdash \psi \quad$ iff $\quad \Gamma \vdash [\psi]\, a\, [\varphi]$

These six principles together entail the rule of *reductio*. At the same time, it should be noted that the operation $\mathrm{CON}(\cdots)$ appearing in the principle of propositional privation has so far been left undefined. Consequently, it has not yet been shown whether and, if so, how the rule of *reductio* can be given a general and schematic formulation in the language of categorical logic. To achieve this further end, we must find a way of defining the operation $\mathrm{CON}(\cdots)$ within the object language of categorical logic. As we shall see, Leibniz's theory of propositional terms provides an elegant solution to this problem.

§4. The syntax of Leibniz's categorical calculus. Leibniz's derivation of the rule of *reductio* in the *Specimina calculi rationalis* does not rely on any specific account of the operation $\mathrm{CON}(\cdots)$, which maps each categorical proposition to its contradictory. Nonetheless, as it turns out, the principles employed by Leibniz in this derivation suffice to determine an explicit definition of the operation $\mathrm{CON}(\cdots)$ within the object language of his categorical logic. Specifically, these principles entail that, for any proposition φ, its contradictory $\mathrm{CON}(\varphi)$ is provably equivalent to the a-proposition $\overline{[\varphi]}\, a\, [\varphi]$. To see this, we first observe that, given the principles listed above, $\overline{[\varphi]}\, a\, [\varphi]$ can be derived from $\mathrm{CON}(\varphi)$ as follows:

1	$\mathrm{CON}(\varphi)$	premise
2	$\quad \varphi$	assumption for propositional a-predication
3	$\quad \mathrm{CON}(\varphi)$	reiteration: 1
4	$[\mathrm{CON}(\varphi)]\, a\, [\varphi]$	propositional a-predication: 2–3
5	$\overline{[\varphi]}\, a\, [\varphi]$	Barbara: 4, propositional privation

Conversely, $\mathrm{CON}(\varphi)$ can be derived from $\overline{[\varphi]}\, a\, [\varphi]$ as follows:[82]

[82] This derivation employs a variant of the rule of *reductio* according to which $\varphi, \psi \vdash \mathrm{CON}(\chi)$ entails $\varphi, \chi \vdash \mathrm{CON}(\psi)$. This variant of *reductio* can be established from Leibniz's categorical principles in the same way as the rule of *reductio* stated above, according to which $\varphi, \psi \vdash \chi$ entails $\varphi, \mathrm{CON}(\chi) \vdash \mathrm{CON}(\psi)$.

1	$\overline{[\varphi]}\, a\, [\varphi]$	premise
2	$[\text{CON}(\varphi)]\, a\, [\varphi]$	Barbara: 1, propositional privation
3	$\quad \varphi$	assumption for *reductio*
4	$\quad\quad [\text{CON}(\varphi)]\, a\, [\varphi]$	assumption for *reductio*
5	$\quad\quad \varphi$	reiteration: 3
6	$\quad\quad \text{CON}(\varphi)$	propositional a-predication: 4, 5
7	$\quad \text{CON}([\text{CON}(\varphi)]\, a\, [\varphi])$	*reductio*: 3, 4–6
8	$\text{CON}(\varphi)$	*reductio*: 2, 3–7

Thus, the principles employed by Leibniz in his derivation of *reductio* entail that, for any categorical proposition φ, $\text{CON}(\varphi)$ is provably equivalent to the a-proposition $\overline{[\varphi]}\, a\, [\varphi]$.

The Fitch-style notation employed in the above two proofs encodes a number of implicit structural assumptions concerning the nature of the derivability relation $\vdash$. For example, the representation of proofs in sequential form encodes a commitment to the rule of cut, which licenses the construction of complex derivations through the consecutive application of primitive rules of inference. However, not all of the structural rules implicit in the Fitch-style notation are needed to derive the equivalence of $\text{CON}(\varphi)$ and $\overline{[\varphi]}\, a\, [\varphi]$. In fact, the only structural rule needed to establish this equivalence is the following weak rule of cut:[83]

$$\text{If } \varphi \vdash \psi \text{ and } \vdash \varphi, \text{ then } \vdash \psi.$$

When this weak rule of cut is added to Leibniz's categorical principles stated above, the following is a theorem of the resulting calculus:[84]

$$\text{CON}(\varphi) \dashv\vdash \overline{[\varphi]}\, a\, [\varphi].$$

This theorem provides us with an explicit definition of the operation $\text{CON}(\cdots)$ in the object language of Leibniz's categorical calculus. With this definition in hand, the rule of *reductio* can be formulated as follows:

$$\frac{\varphi, \psi \vdash \chi}{\varphi, \overline{[\chi]}\, a\, [\chi] \vdash \overline{[\psi]}\, a\, [\psi]}$$

This formulation of *reductio* is fully general in that it comprehends all possible applications of *reductio* in the language of Leibniz's categorical logic. At the same time, the formulation is schematic in that its object-language instances are obtained simply by replacing the schematic letters 'φ', 'ψ', and 'χ' with any arbitrary categorical propositions. Hence, the above formulation of *reductio* is both general and schematic with respect to an object language which includes only categorical propositions and countenances no primitive propositional operators.

[83] This weak rule of cut is a special case of the general rule of cut: if $\Gamma, \varphi \vdash \psi$ and $\Delta \vdash \varphi$, then $\Gamma \cup \Delta \vdash \psi$.

[84] For a proof of this claim, see Theorems 7.25 and 7.26 and the discussion following Theorem 7.26 in the appendix below.

Of course, this formulation of *reductio* involves propositional terms, and in this way goes beyond the syntax of Aristotle's assertoric syllogistic.[85] Nevertheless, the extended syntax employed by Leibniz in his derivation of *reductio* remains strictly categorical in that every proposition is of the form AaB, AeB, AiB, or AoB, where A and B are any terms, propositional or otherwise. Thus, while the principles of Leibniz's calculus license modes of reasoning that are not available in Aristotle's assertoric syllogistic, the increased expressive and deductive power of Leibniz's system is not due to the positing of a new, noncategorical kind of proposition. Instead, it is due to the positing of a new kind of term along with the logical principles governing its operation. As we have seen, these principles are strong enough not only to justify the rule of *reductio*, but also to provide a general and schematic formulation of this rule within categorical logic. In this way, Leibniz's theory of propositional terms provides the basis for a compelling response to the charge, raised at various points throughout the centuries, that the method of *reductio ad absurdum* presupposes laws of propositional logic.

In addition, Leibniz's theory of propositional terms allows for a significant simplification of the standard syntax of categorical logic. Recall that Aristotle's assertoric syllogistic countenances four kinds of primitive proposition, namely, a-, e-, i-, and o-propositions. Leibniz's theory allows us to eliminate e-, i-, and o-propositions from the list of syntactic primitives by reducing them all to a-propositions. For example, using the categorical definition of $\textsc{con}(\cdots)$, an o-proposition AoB can be defined as $\textsc{con}(AaB)$, i.e.:

$$\overline{[AaB]}\, a\, [AaB].$$

In this way, o-propositions can be eliminated from the list of syntactic primitives by defining them as a-propositions relating propositional terms and their privatives. Furthermore, as Leibniz points out, the law of obversion can be used to reduce e- to a-propositions by defining AeB as $\overline{A}aB$.[86] Thus, e-propositions can be eliminated as well, and the law of obversion can be omitted from the list of Leibniz's categorical principles. Finally, AiB can be defined as $\textsc{con}(AeB)$, i.e.:

$$\overline{[\,\overline{A}aB]}\, a\, [\,\overline{A}aB].$$

In this way, by utilizing Leibniz's device of propositional terms, e-, i-, and o-propositions can all be defined as a-propositions relating complex terms.

Although this specific way of reducing i- and o- to a-propositions is not explicitly discussed by Leibniz, it was clearly a theoretical ambition of his to reduce all categorical propositions to a-propositions. This ambition is part of Leibniz's broader commitment to his conceptual containment theory of truth, according to which the truth-conditions of any proposition consist in one term being conceptually contained in another. As Leibniz puts

[85] While the assertoric syllogistic developed by Aristotle in *Prior Analytics* 1.1–7 does not countenance propositional terms, it is clear from the rest of the *Analytics* that Aristotle intended his syllogistic to be applicable not only to terms such as *animal* and *rational* but also to a wider range of terms of varying complexity. Indeed, it is arguable that Aristotle would even have been open to the introduction of propositional terms in his broader theory of the categorical syllogism. Thus, Mendell argues that, in the application of his syllogistic, 'Aristotle sometimes allows propositions as terms .... The ease with which one can nominalize any sentence in Greek suggests that a proposition can and does function as a term' (Mendell 1998: 176).

[86] C 301, A VI.4 126, 223, 787 n. 52, *GI* §190; see Adams 1994: 60.

it in the *Calculus consequentiarum*, 'in every proposition the predicate is said to inhere in the subject'.[87] Similarly, he writes in §132 of the *Generales inquisitiones*:

> Every true proposition can be proved. For since, as Aristotle says, the predicate inheres in the subject, or, the concept of the predicate is involved in the concept of the subject when this concept is perfectly understood, then surely it must be possible for the truth to be shown by the analysis of terms into their values, or, those terms which they contain. (Leibniz, *Generales inquisitiones* §132)

As this passage makes clear, Leibniz took the conceptual containment theory of truth to apply to all true propositions. Since, for Leibniz, conceptual containment between terms is expressed by a-propositions, this implies that every proposition must be analyzable into an a-proposition.[88] This is unproblematic for propositions of the form AeB since, as we have just seen, e-propositions can be reduced to a-propositions by means of the law of obversion.[89] It is less clear how to reduce i-propositions to a-propositions, yet Leibniz leaves no doubt that the conceptual containment theory of truth is meant to apply to i-propositions as well:

> The predicate or consequent always inheres in the subject or antecedent, and the nature of truth in general or the connection between the terms of a statement, consists in this very thing, as Aristotle also observed. ... This is true for every affirmative truth, universal or particular, necessary or contingent, and in both an intrinsic and extrinsic denomination. And here lies hidden a wonderful secret. (Leibniz, *Principia logico-metaphysica*, A VI.4 1644)

In this passage, Leibniz does not provide any specific account of how the conceptual containment theory of truth can be extended to cover i-propositions. He does put forward one proposal for how this can be done in the *Calculus consequentiarum*, suggesting that the proposition AiB be reduced to the claim that there exists some species of B which contains A.[90] This proposal, however, presents some difficulties. For one thing, it is unclear how to guarantee the validity of the rule of i-conversion on this interpretation. More importantly, the proposal involves an existential quantification over the species of B, and existential quantifications are not expressible in the language of Leibniz's categorical calculus in any straightforward way.[91] By contrast, the above definitions of i- and o-propositions utilizing

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[87] A VI.4 223.

[88] See Adams 1994: 57–63; similarly, Parkinson 1965: 6 and 33.

[89] Leibniz appeals to obversion in order to apply the conceptual containment theory of truth to e-propositions at A VI.4 223.

[90] A VI.4 223; see also *GI* §190.

[91] In the *Generales inquisitiones*, Leibniz explores the possibility of expressing existential quantifications in the language of his calculus by means of what he calls 'indefinite' letters (*GI* §§16–31; see Lenzen 1984a: 7–13, 2004: 47–50, Hailperin 2004: 329). However, the use of indefinite letters in the calculus leads to complications analogous to those which arise in connection with the elimination of existential quantifiers in modern quantificational logic (see, e.g., *GI* §§21–31). Leibniz was aware of some of these complications and was never entirely satisfied with the use of indefinite letters, indicating that it would be preferable to omit them from the language of his calculus (see *GI* §162 in conjunction with §128 and A VI.4 766 n. 35; cf. Schupp 1993: 153, 168, and 182–3). In the absence of indefinite letters, the language of Leibniz's calculus lacks the resources to express existential quantification.

the device of propositional terms avoid these difficulties. In particular, these definitions allow for the reduction of i- and o- to a-propositions to be carried out in a purely categorical language without existential quantification.[92]

Leibniz's categorical principles thus suffice to underwrite a systematic reduction of e-, i-, and o-propositions to a-propositions. In this way, these principles help to vindicate Leibniz's conceptual containment theory of truth by ensuring that this theory is applicable to all categorical propositions countenanced by the standard syntax of Aristotle's assertoric syllogistic. In fact, as we shall see, these principles are strong enough to underwrite an even more comprehensive implementation of the conceptual containment theory of truth, one which covers not only categorical propositions but hypothetical propositions as well.

§5. Propositional reasoning in Leibniz's categorical calculus. The principles posited by Leibniz in his categorical proof of *reductio* imply that all e-, i-, and o-propositions can be reduced to a-propositions. Hence, the categorical calculus employed by Leibniz in this proof can be formulated, without any loss of expressive power, in a language in which the only primitive propositions are those of the form AaB. The language of Leibniz's calculus can thus be defined as follows:

DEFINITION 5.1. *Given a countably infinite set of simple terms, the* terms *and* propositions *of the language of Leibniz's categorical calculus are defined as follows*:[93]

1. *Every simple term is a term.*
2. *If A is a term, then $\overline{A}$ is a term.*
3. *If A and B are terms, then AaB is a proposition.*
4. *If AaB is a proposition, then $[AaB]$ is a term.*

In this language, the operation $\mathrm{CON}(\cdots)$ can be defined as the function mapping any proposition φ to the a-proposition $\overline{[\varphi]}\,a\,[\varphi]$. By means of this operation, the other three kinds of categorical proposition can be defined as follows:

DEFINITION 5.2. *For any terms A and B*:

1. $\mathrm{CON}(AaB)$ *is the proposition* $\overline{[AaB]}\,a\,[AaB]$.
2. AeB *is the proposition* $\overline{A}aB$.
3. AiB *is the proposition* $\mathrm{CON}(AeB)$.
4. AoB *is the proposition* $\mathrm{CON}(AaB)$.

Based on these definitions, we are now in a position to give a precise formulation of Leibniz's categorical calculus. In doing so, we take a 'calculus' to be a binary relation

[92] Leibniz's categorical principles thus provide a solution to the recalcitrant problem of how to define i- and o-propositions in a purely affirmative categorical language. By contrast, Boole's (1854: 226–31) attempt to define i- and o-propositions in an affirmative categorical language using 'indefinite symbols' gives rise to various difficulties and was subject to acute criticism by Peirce (1873: 371–4, 1880: 22), Venn (1881: 161–70 and 306–7), and Schröder (1891: 91–5); cf. Brady 1997: 174–6, Sánchez-Valencia 2004: 432–42 and 470–2.

[93] The simple terms are to be chosen in such a way that no simple term is identical with any other term or proposition. Following Leibniz, we assume that there are infinitely many simple terms (A VI.4 271).

between sets of propositions and single propositions. To indicate that this relation obtains between a set of propositions Γ and a proposition φ, we write $\Gamma \vdash \varphi$. As usual, $\vdash \varphi$ is shorthand for $\emptyset \vdash \varphi$. Moreover, we use $\psi_1, \ldots, \psi_n \vdash \varphi$ as shorthand for $\{\psi_1, \ldots, \psi_n\} \vdash \varphi$, and $\Gamma, \psi_1, \ldots, \psi_n \vdash \varphi$ as shorthand for $\Gamma \cup \{\psi_1, \ldots, \psi_n\} \vdash \varphi$.

The six principles of Leibniz's categorical calculus identified above are: Barbara, e-conversion, obversion, double privation, propositional privation, and propositional a-predication. Given the above definition of e-propositions, the principle of obversion is superfluous and can be omitted from this list. In addition to the five remaining principles, we must posit structural rules which regulate how theorems are to be derived from these principles in the calculus. A basic structural rule that any calculus must satisfy in order to allow for sequential proofs is the weak rule of cut stated above. While Leibniz does not explicitly assert this rule, it is clear that he takes it for granted throughout his logical writings. In addition, Leibniz endorses the following structural rule of weakening:[94]

$$\varphi, \psi \vdash \varphi.$$

In endorsing this rule, Leibniz parts ways with Aristotle, who rejects weakening on the grounds that the conclusion of a syllogism must not appear among its premises.[95] We will explore the consequences of siding with Aristotle in rejecting this rule below, but for now we follow Leibniz and posit weakening as a structural rule. All told, then, Leibniz's categorical calculus consists of the following principles:

DEFINITION 5.3. *Leibniz's* categorical calculus, $\vdash$, *is the smallest relation between sets of propositions and propositions which satisfies the following principles*:

Barbara:	$AaB, BaC \vdash AaC$
e-conversion:	$AeB \vdash BeA$
double privation:	$\vdash \overline{\overline{A}}aA$ *and* $\vdash Aa\overline{\overline{A}}$
propositional privation:	$\vdash [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]}$ *and* $\vdash \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)]$
propositional a-predication:	$\Gamma, \varphi \vdash \psi$ *iff* $\Gamma \vdash [\psi]\, a\, [\varphi]$
weakening:	$\varphi, \psi \vdash \varphi$
weak cut:	*If* $\varphi \vdash \psi$ *and* $\vdash \varphi$, *then* $\vdash \psi$.

Here, A, B, C are any terms; φ, ψ any propositions; and Γ any finite set of propositions.

In referring to the calculus constituted by these principles as 'Leibniz's categorical calculus', we do not mean to imply that these are the only categorical principles posited by Leibniz in his logical writings. For example, in many of his essays, Leibniz posits additional categorical principles that pertain to the logic of composite terms such as *rational animal*. The introduction of composite terms along with the principles governing their operation significantly increases the expressive and deductive power of Leibniz's

[94] Leibniz states this rule at A VI.4 149.

[95] See *Pr. An.* 1.1 24b19, *Top.* 1.1 100a25–6, *Soph. Ref.* 5 167a25–6, 6 168b25–6; cf. Alexander *In Pr. An.* 18.12–20.29, Ammonius *In Pr. An.* 27.34–28.20, Frede 1974: 20–1, Bolton 1994: 110–12, Barnes 2007: 487–90, Striker 2009: 80.

calculus.[96] Since, however, these terms play no role in Leibniz's derivation of *reductio* in the *Specimina calculi rationalis*, for the purposes of the present discussion we set them aside. Our subsequent remarks concerning the logic of propositional terms apply equally well to the extended version of Leibniz's calculus which includes composite terms.

By utilizing the device of propositional terms, Leibniz's categorical calculus is capable of simulating operations of propositional logic. For example, for any proposition φ, its propositional negation, $\neg\varphi$, can be defined as the categorical proposition $\mathrm{CON}(\varphi)$, i.e., $\overline{[\varphi]}\,a\,[\varphi]$. Moreover, following Theophrastus' analogy, the conditional $\varphi \rightarrow \psi$ can be defined as the categorical proposition $[\psi]\,a\,[\varphi]$. By means of these definitions, any complex propositional formula involving negation and implication is expressible in the language of Leibniz's categorical calculus. For example, the propositional formula $\neg(\varphi \rightarrow \neg\psi) \rightarrow \psi$ just is the categorical proposition:

$$[\psi]\,a\,[\mathrm{CON}([\mathrm{CON}(\psi)]\,a\,[\varphi])].$$

Thus, the language of Leibniz's categorical calculus has all the expressive power of the implication-negation fragment of the language of propositional logic. This naturally gives rise to the question as to what kind of propositional logic is determined by the principles of Leibniz's categorical calculus. We have already seen, for example, that the rule of *reductio* is derivable in this calculus. But apart from this rule, what other laws of propositional logic can be derived in the calculus? The answer is: all the laws of classical propositional logic. This can be seen from the following theorem, which states that all the principles of Łukasiewicz's axiomatization of classical propositional logic are derivable in Leibniz's categorical calculus:

THEOREM 5.4. *For any propositions φ and ψ, let $\neg\varphi$ be the proposition $\mathrm{CON}(\varphi)$, and $\varphi \rightarrow \psi$ the proposition $[\psi]\,a\,[\varphi]$. Then the following claims hold in Leibniz's categorical calculus*:

Łukasiewicz 1:	$\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$
Łukasiewicz 2:	$\vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
Łukasiewicz 3:	$\vdash (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$
modus ponens:	$\varphi \rightarrow \psi\,,\ \varphi \vdash \psi$
reflexivity:	$\varphi \vdash \varphi$
monotonicity:	*If* $\Gamma \vdash \varphi$, *then* $\Gamma \cup \Delta \vdash \varphi$.
cut:	*If* $\Gamma, \varphi \vdash \psi$ *and* $\Delta \vdash \varphi$, *then* $\Gamma \cup \Delta \vdash \psi$.

Here, φ, ψ, χ are any propositions, and Γ, Δ any finite sets of propositions.

It is not difficult to verify that each of these principles is derivable in Leibniz's categorical calculus. First, consider the structural rules of reflexivity, monotonicity, and cut. Reflexivity is an immediate consequence of the rule of weakening.[97] Moreover, the full

[96] For example, these additional principles are strong enough to ensure that the terms of Leibniz's calculus constitute a Boolean algebra; see Lenzen 1984b: 200–2, Malink & Vasudevan 2016: 707–9.

[97] Since we defined a calculus to be a relation between sets (as opposed to sequences) of propositions and propositions, reflexivity can be expressed as $\{\varphi, \varphi\} \vdash \varphi$, which is an instance of weakening. Leibniz asserts reflexivity at A VI.4 149.

rule of cut stated in the theorem can be derived from the rule of weak cut in Leibniz's categorical calculus.[98] This is due to the fact that, given propositional a-predication, the syllogistic mood Barbara in effect functions as a method for cutting propositional terms.[99] Finally, given the full rule of cut, monotonicity follows from weakening.[100]

Next, consider the logical principles of Łukasiewicz's axiomatization of classical propositional logic. Modus ponens is derivable in Leibniz's categorical calculus from reflexivity and propositional a-predication. Łukasiewicz 1 follows from weakening by two applications of propositional a-predication. Łukasiewicz 2 is likewise derivable by repeated applications of propositional a-predication, cut, and reflexivity.[101] Finally, for Łukasiewicz 3, it suffices to establish that:

$$\neg\varphi \to \neg\psi \vdash \psi \to \varphi.$$

The proof of this theorem is as follows:

1	$[\mathrm{CON}(\psi)]\, a\, [\mathrm{CON}(\varphi)]$	premise
2	$\overline{[\psi]}\, a\, \overline{[\varphi]}$	Barbara: 1, propositional privation
3	$\overline{\overline{[\varphi]}}\, a\, [\psi]$	e-conversion: 2
4	$[\varphi]\, a\, [\psi]$	Barbara: 3, double privation

This last proof highlights the role that the syllogistic principles of Barbara and e-conversion play in establishing the laws of classical propositional logic.

Thus, Leibniz's categorical calculus validates all the laws of classical propositional logic. Conversely, these are the only laws of propositional logic that are validated by the calculus.[102] In this sense, Leibniz's categorical calculus is both sound and complete with

[98] See Theorem 7.23 below.

[99] Thus, for example, Barbara can be used to establish the following special case of cut: if $\varphi \vdash \psi$ and $\chi \vdash \varphi$, then $\chi \vdash \psi$. To see this, suppose $\varphi \vdash \psi$ and $\chi \vdash \varphi$. Given propositional a-predication, we have $\vdash [\psi]\, a\, [\varphi]$ and $\vdash [\varphi]\, a\, [\chi]$. By Barbara, propositional a-predication, and weak cut, it follows that $\vdash [\psi]\, a\, [\chi]$ (see Theorem 7.6 below). Hence, by propositional a-predication, $\chi \vdash \psi$.

[100] First, we establish monotonicity for the case in which Γ is empty and Δ is the singleton set $\{\psi\}$: if $\vdash \varphi$, then $\psi \vdash \varphi$. By weakening, we have $\varphi, \psi \vdash \varphi$. Hence, given $\vdash \varphi$, it follows by cut that $\psi \vdash \varphi$. Next, we establish monotonicity for the case in which Γ is nonempty and Δ is the singleton set $\{\psi\}$: if $\Gamma \vdash \varphi$ and $\Gamma \neq \emptyset$, then $\Gamma \cup \{\psi\} \vdash \varphi$. Let χ be any proposition in Γ. By weakening, we have $\chi, \psi \vdash \chi$. Hence, given $\Gamma \vdash \varphi$, it follows by cut that $\Gamma \cup \{\psi\} \vdash \varphi$. The rule of monotonicity stated above, in which Γ and Δ are any finite sets of propositions, follows immediately from these two cases.

[101] By reflexivity and propositional a-predication, we have $\varphi \to \psi, \varphi \vdash \psi$ and $\psi \to \chi, \psi \vdash \chi$. Hence, by cut: $\psi \to \chi, \varphi \to \psi, \varphi \vdash \chi$. Moreover, by reflexivity and propositional a-predication, we have $\varphi \to (\psi \to \chi), \varphi \vdash \psi \to \chi$. Hence, by cut: $\varphi \to (\psi \to \chi), \varphi \to \psi, \varphi \vdash \chi$. Łukasiewicz 2 follows by three applications of propositional a-predication.

[102] This is because classical propositional logic is Post-complete in the sense that it has no consistent schematic extension. Thus, if the categorical calculus were to validate any non-classical law of propositional logic, it would be inconsistent in the sense that every proposition would be derivable. To see that the categorical calculus is not inconsistent in this sense, it suffices to construct a semantics with respect to which the calculus is sound (and in which some propositions are not satisfied). One such semantics is given in n. 103 below.

respect to classical propositional logic.[103] In other words, the propositional logic generated by Leibniz's categorical calculus is exactly classical propositional logic.

This result entails that all the standard connectives of classical propositional logic are definable in the language of Leibniz's categorical calculus. For example, if the conjunction of two propositions, $\varphi \wedge \psi$, is defined as $\neg(\varphi \rightarrow \neg\psi)$, i.e.:

$$\text{CON}([\text{CON}(\psi)]\, a\, [\varphi]),$$

this definition validates the classical laws of conjunction, such as $\varphi \wedge \psi \vdash \varphi$ and $\varphi, \psi \vdash \varphi \wedge \psi$. Similar categorical definitions can be given for the other connectives of classical propositional logic. Since every proposition in the language of Leibniz's categorical calculus is of the form AaB, this means that all hypothetical propositions can be reduced to a-propositions. As a result, Leibniz's conceptual containment theory of truth applies not only to all categorical propositions but to all hypothetical propositions as well. In this way, Leibniz manages to achieve the aim of the Peripatetic program of reducing hypothetical to categorical logic, vindicating his claim that:

> absolute [i.e., categorical] and hypothetical truths have one and the same laws and are contained in the same general theorems, so that all syllogisms become categorical. (Leibniz, *Generales inquisitiones* §137)

The fact that Leibniz's categorical calculus validates all the laws of classical propositional logic bears on a dispute in the literature concerning the nature of the conditional expressed by propositions of the form $[\varphi]\, a\, [\psi]$ in Leibniz's logic. The two main positions in this dispute have their clearest articulation in the work of Héctor-Neri Castañeda and Wolfgang Lenzen. Castañeda argues that the propositional terms of Leibniz's calculus are subject to all the laws of classical propositional logic.[104] Consequently, he maintains that the relation of a-predication when applied to propositional terms obeys the laws of

[103] More generally, Leibniz's categorical calculus is both sound and complete with respect to the class of partially ordered sets endowed with (i) a unary operation of negation obeying the laws of double negation and contraposition, and (ii) a designated element which is less than its own negation. Specifically, let an interpretation of the language of Leibniz's categorical calculus consist of an algebraic structure $\langle \mathbb{A}, \leq, ', \bot \rangle$ and a function μ mapping each term of the language to an element of $\mathbb{A}$, such that:

1. $\langle \mathbb{A}, \leq \rangle$ is a partially ordered set.
2. $'$ is a unary operation on $\mathbb{A}$ such that, for any $x, y \in \mathbb{A}$: (i) $x'' = x$, and (ii) if $x \leq y$, then $y' \leq x'$.
3. $\bot \in \mathbb{A}$ such that $\bot \leq \top$, where $\top$ is the element $\bot'$.
4. For any term A of the language: $\mu(\overline{A}) = \mu(A)'$.
5. For any terms A, B of the language:

$$\mu([AaB]) = \begin{cases} \top & \text{if } \mu(B) \leq \mu(A), \\ \bot & \text{otherwise.} \end{cases}$$

We say that a proposition AaB is satisfied in an interpretation $(\langle \mathbb{A}, \leq, ', \bot \rangle, \mu)$ iff $\mu(B) \leq \mu(A)$. We write $\Gamma \models \varphi$ to indicate that there is no interpretation that satisfies every proposition in Γ but does not satisfy φ. It can then be shown that Leibniz's categorical calculus is sound and complete with respect this semantics, i.e.: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$. For reasons of space, we omit the proof of this claim.

[104] Castañeda 1990: 24–5. A similar view is held by Doull 1991: 21–2.

material implication. Lenzen, on the other hand, contends that this relation is not material implication but is instead some weaker relation of strict implication.[105]

While the position advanced in the present article is in agreement with Castañeda's view, it should be acknowledged that there are certain *prima facie* considerations that speak in favor of the view adopted by Lenzen. For example, given that Leibniz interprets the relation of a-predication as conceptual containment, one might doubt whether he would be willing to accept that, for any two propositional terms, one is conceptually contained in the other—as is required by the laws of material implication. Leibniz himself does not discuss the question of whether such laws hold for conceptual containment between propositional terms. Indeed, he does not offer any account of what it means for one propositional term to be conceptually contained in another. Nevertheless, there are a number of claims that Leibniz makes in his logical writings that can help to adjudicate between the differing accounts put forward by Castañeda and Lenzen.

As we have seen, Leibniz's derivation of *reductio* in the *Specimina calculi rationalis* relies on the following two-premise version of the principle of propositional a-predication:

$$\varphi, \psi \vdash \chi \qquad \text{if and only if} \qquad \varphi \vdash [\chi]\, a\, [\psi].$$

Using the notation introduced above, this reads:

$$\varphi, \psi \vdash \chi \qquad \text{if and only if} \qquad \varphi \vdash \psi \to \chi.$$

While this principle is valid when '$\to$' is interpreted as material implication, it is invalid when this operator is interpreted as strict implication.[106] Thus, the fact that Leibniz is committed to this principle speaks against interpreting a-predication between propositional terms as strict implication.

A further consideration that speaks against this interpretation derives from the various claims that Leibniz makes about negation, falsehood, and contradiction in his logical writings. On the one hand, Leibniz maintains that 'a negative statement is nothing other than that statement which says that the affirmative statement is false'.[107] So, for example, '*Not: A is B*, or *A is not B*, is the same as: the statement *A is B* is false'.[108] Thus, for Leibniz, to assert that a proposition φ is false just is to assert the negation of that proposition, $\neg\varphi$. At the same time, Leibniz defines falsehood for propositions as follows: 'that proposition is false which contains opposite propositions, as in $\odot$ and *non*-$\odot$'.[109] In other words, 'a falsehood is that which implies a contradiction'.[110] Taken together, these claims entail that, for Leibniz, to assert the negation of a proposition is to assert that that proposition contains a contradiction. Thus, Leibniz maintains that, for any proposition φ, if $\neg\varphi$ is true, then there

[105] Lenzen 1987 and 2004: 35–9. A similar view is held by Juniewicz 1987: 51. Kauppi (1960: 259) goes even further than Lenzen, arguing that the relation of a-predication between propositional terms fails to obey not only the laws of material implication but those of strict implication as well. In particular, she argues that the law of explosion, $(\varphi \wedge \neg\varphi) \to \psi$, fails to hold in Leibniz's calculus. Kauppi's argument for this view, however, is not convincing; see Malink & Vasudevan 2017: 153 n. 12.

[106] Strict implication does obey the one-premise version of this law: $\varphi \vdash \psi$ if and only if $\vdash \varphi \to \psi$. However, this one-premise version of the law does not suffice to derive the two-premise version of *reductio* which Leibniz clearly intends to establish in §§15–17 of the *Specimina*.

[107] A VI.4 811 n. 6.

[108] A VI.4 811; see also *GI* §5, §32a, §84, A VI.4 807 n. 1, 808.

[109] *GI* §196; similarly, §130b.

[110] A VI.4 912.

is a proposition ψ such that the conditionals $\varphi \to \psi$ and $\varphi \to \neg\psi$ are both true. While this claim holds for any interpretation in which '$\to$' is taken to be material implication, it does not hold in general when this operator is taken to be strict implication.

Now, one piece of evidence adduced by Lenzen that would seem to speak in favor of his interpretation is Leibniz's claim that '*A contains B* is a true proposition if *A non-B* implies a contradiction'.[111] Lenzen argues that, when A and B are taken to be propositions, this passage asserts the standard definition of strict implication in modal propositional logic, according to which, for any propositions φ and ψ:[112]

$$\varphi \to \psi \qquad \text{if and only if} \qquad \neg\Diamond(\varphi \wedge \neg\psi).$$

Lenzen's interpretation is based on the assumption that, when Leibniz says that a proposition φ 'implies a contradiction', he intends to say not only that it is false but that it is impossible, i.e., $\neg\Diamond\varphi$. This, however, is not Leibniz's intended meaning. Recall that, for Leibniz, 'a falsehood is that which implies a contradiction', so that to say that a proposition implies a contradiction just is to say that that proposition is false.[113] In order for a false proposition to be impossible, on Leibniz's view, not only must it imply a contradiction, but it must have the further property that one of the contradictions it implies can be disclosed through a finite as opposed to an infinite process of analysis. Thus, Leibniz writes:

> That term or proposition is false which contains opposites, however they are proved. That term or proposition is impossible which contains opposites that are proved by reduction to finitely many terms. So, $A = AB$, if a proof has been produced through a finite analysis, must be distinguished from $A = AB$, if a proof has been produced through an analysis *ad infinitum*. From this there already results what has been said about the necessary, the possible, the impossible, and the contingent. (Leibniz, *Generales inquisitiones* §130b)

As this passage shows, there are propositions which imply a contradiction, and hence are false, but are not impossible since the contradictions they imply cannot be revealed through any finite process of analysis. Thus, contrary to what Lenzen suggests, Leibniz's claim that '*A contains B* is a true proposition if *A non-B* implies a contradiction' does not assert that $\varphi \to \psi$ is true just in case $\varphi \wedge \neg\psi$ is impossible. Instead, it asserts that $\varphi \to \psi$ is true just in case $\varphi \wedge \neg\psi$ is false, which is the usual definition of material implication.[114]

Independently of these considerations, we have already seen that the laws of material implication are a direct consequence of the principles posited by Leibniz in his derivation of *reductio* in the *Specimina calculi rationalis*. Consequently, in order to deny that the relation of a-predication between propositional terms obeys the laws of material implication, at least one of these principles must be rejected. Lenzen, for example, does not include any version of the principle of propositional privation in his reconstruction of Leibniz's calculus.[115] This omission is what makes it possible for Lenzen to interpret a-predication between propositional terms as non-classical strict implication. By contrast, Castañeda

[111] A VI.4 862.
[112] Lenzen 2004: 36.
[113] See nn. 109 and 110 above.
[114] This reading fits well with the formulation of this claim given by Leibniz in §199 of the *Generales inquisitiones*: '*A contains B* is the same as *A non-B is not true*'.
[115] See Lenzen 1986, 2004: 35–9.

endorses all of the principles posited by Leibniz in the *Specimina*, including propositional privation, and hence acknowledges that a-predication between propositional terms obeys the laws of material implication.[116]

In the subsequent development of the Peripatetic program since Leibniz, a number of logicians have taken the relation of a-predication between propositional terms to obey the laws of material implication.[117] Like Leibniz, many of these logicians endorse the principle of propositional privation, or some close variant of it, in their treatment of propositional terms.[118] Admittedly, this does not sit well with the temporal and case-based semantics for conditionals adopted by some categorical logicians at various points in the history of the Peripatetic program.[119] On these interpretations, a conditional $\varphi \to \psi$ is taken to be true just in case every time, or case, at which φ is true is also a time, or case, at which ψ is true. Assuming that there are propositions which are 'variable' in the sense that they are true at some times, or cases, but not at others, this semantics fails to validate the laws of material implication.[120] Thus, for example, if a proposition φ is variable in this sense, it follows that neither $\varphi \to \neg\varphi$ nor $\neg\varphi \to \varphi$ is true.[121]

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[116] See Castañeda 1990: 23–5. In fact, Castañeda (1990: 23, Axiom 6.2) posits a stronger version of propositional privation, which applies not only to propositional terms but to non-propositional terms as well. Consequently, he attributes to Leibniz the view that all terms, propositional and non-propositional alike, obey the laws of classical propositional logic. This view, however, conflicts with several claims made by Leibniz concerning the logic of non-propositional terms (see Malink & Vasudevan 2016: 716–18).

[117] For example, Schröder 1891, 1905, Couturat 1914, and van Rooij 2012. In the systems developed by Schröder and Couturat, $[\varphi]\, a\, [\psi]$ is provably equivalent to the Boolean disjunction of $[\varphi]$ and the complement of $[\psi]$ (see Schröder 1891: 68, Couturat 1914: 85; cf. Peckhaus 1998: 20–1 and 29–30). This guarantees that $[\varphi]\, a\, [\psi]$ obeys the laws of material implication. Likewise, van Rooij (2012: 96–8) establishes that the propositional terms in his system obey the laws of classical propositional logic.

[118] Schröder (1891: 52 and 65, 1905: 484) and Couturat (1914: 84) take propositional terms to be subject to the law $[[\varphi] = 1] = [\varphi]$. Moreover, they take this law to apply to all Boolean compounds of propositional terms (see, e.g., Schröder 1891: 68, Couturat 1914: 85). In particular, when applied to the privatives of propositional terms, this law states: $[\overline{[\varphi]} = 1] = \overline{[\varphi]}$. Since Schröder and Couturat take $\overline{[\varphi]} = 1$ to be equivalent to $\overline{[\varphi]}\, a\, [\varphi]$, this is equivalent to $[\overline{[\varphi]}\, a\, [\varphi]] = \overline{[\varphi]}$. Given that Schröder (1890: 184–5) and Couturat (1914: 6) use '=' to indicate mutual a-predication, this is exactly the principle of propositional privation. Similarly, van Rooij (2012: 96–7, Axiom 11) posits a variant of the law $[[\varphi] = 1] = [\varphi]$, in which $[\varphi] = 1$ is replaced by a syllogistic i-proposition which ensures that the extension of the term $[\varphi]$ is both nonempty and a superset of the extension of every propositional term.

[119] For categorical logicians who adopt such a temporal or case-based semantics, see nn. 36 and 37 above.

[120] For example, this is the case for the temporal semantics of conditionals given by Boole (1854: 163–4). Following Aristotle, Boole admits variable propositions that are true at some times and false at others. For Aristotle's acknowledgement of temporally variable propositions, see *Categories* 5 4a23–6 and 4a34–4b1; cf. Crivelli 2004: 183–9.

[121] By contrast, the laws of material implication can be validated in a temporal or case-based semantics provided that variable propositions are excluded from consideration. For example, Schröder (1891: 1–24 and 59–63) adopts a temporal semantics for conditionals in which every proposition is either always true or never true (see Schröder 1891: 51–2, 63–5, 312; 1905: 484). An alternative strategy is adopted by Sommers, who takes a propositional term $[\varphi]$ to denote the worlds that are characterized by the proposition φ (Sommers 1993: 176–8, Sommers & Englebretsen 2000: 206–7). Sommers stipulates that the domain of worlds includes only a single world, namely, the actual one (1993: 179–81, Sommers & Englebretsen 2000: 206–7). In both Schröder's temporal semantics and Sommers's world-based semantics, propositional terms

Of course, the fact that the principles of Leibniz's categorical calculus are incompatible with certain temporal or case-based semantics for conditionals does not pose any problem for Leibniz. For, nowhere does he suggest that the truth-conditions of a-propositions relating propositional terms should be taken to consist in the subsumption of one set of times, or cases, under another. Indeed, Leibniz has good reason for not wanting to commit himself to such a semantics. Leibniz makes it clear that he intends his calculus to be an abstract deductive system which admits of multiple interpretations.[122] Thus, for example, he asserts that the terms and propositions of his calculus admit of both extensional and intensional interpretations.[123] On an extensional interpretation, the semantic value of a term is taken to be the set of individuals falling under that term, and a proposition of the form AaB is true just in case every individual falling under B also falls under A. On an intensional interpretation, by contrast, the semantic value of a term is the set of concepts that are, in some sense, conceptual parts of that term, and AaB is true just in case every conceptual part of A is also a conceptual part of B.[124] In general, Leibniz tends to prefer intensional over extensional interpretations of his calculus, in part because the former do 'not depend on the existence of individuals'.[125] Now, both the temporal and the case-based interpretation outlined above are extensional interpretations, in which the individuals falling under a propositional term are taken to be either times or cases. Hence, given Leibniz's general preference for intensional interpretations of his calculus, one should not expect his theory of propositional terms to be based on semantic intuitions deriving from extensional interpretations of this sort.[126] More generally, given his desire

can take only one of two values, and hence obey all the laws of classical propositional logic. In fact, Schröder's semantics was criticized by Ladd Franklin (1892: 128–30) on the grounds that it gives rise to the paradoxes of material implication. In particular, she regarded Schröder's assumption of omnitemporal bivalence for propositional terms as 'a most unfortunate restriction. Why exclude from an Algebra which is intended to cover all possible instances of ... reasoning such propositions as "sometimes when it rains I am pleased and sometimes when it rains I am indifferent"?' (Ladd Franklin 1892: 129). A similar objection was raised by Peirce in a letter to Schröder from 1897 (see Houser 1990: 228–9). Elsewhere, however, Peirce himself adopts an assumption of bivalence for propositional terms, similar to that adopted by Sommers (Peirce 1885: 184 and 187).

122 See, e.g., A VI.4 834–5; cf. Lenzen 2000: 79–82, Mugnai 2017: 175–6.

123 See, e.g., A VI.4 199–200, 247–8, 838–9; cf. Swoyer 1995: 104–6, Bassler 1998: 132–6. In addition, Leibniz considers geometrical and arithmetical interpretations of his calculus, in which the semantic value of a term is taken to be a line segment or a number, respectively (see *GI* §§113–27).

124 For such intensional interpretations of Leibniz's calculus, see Lenzen 1983: 145, van Rooij 2014: 188–92.

125 A VI.4 200; see also A VI.4 247–8.

126 This is not to say that the principles of Leibniz's categorical calculus cannot be validated on an extensional interpretation of propositional terms. For example, these principles are valid on the temporal and world-based interpretations given by Schröder and Sommers (see n. 121 above). Another extensional interpretation that validates all the principles of Leibniz's categorical calculus is given by Barnes (1983). In this semantics, '[φ] is the term *being such that* φ produced from the sentence 'φ'. Plainly, [φ] is true of any object just in case that object is such that φ; and, trivially, an object is such that φ just in case φ. For example, [*Verdi is Italian*] is true of Berlioz just in case Berlioz is such that Verdi is Italian, and Berlioz is such that Verdi is Italian just in case Verdi is Italian' (Barnes 1983: 315; with notational adjustments). On this interpretation, the extension of a propositional term [φ] is the set of all individuals in the domain of discourse if the proposition φ is true, and the empty set if φ is false. Yet another extensional interpretation that validates all the principles of Leibniz's categorical calculus is given by van Rooij (2012: 95–9).

to construct a calculus which admits of multiple interpretations, he would presumably want to avoid an appeal to any specific semantic intuitions. This may help to explain why, in developing his theory of propositional terms, Leibniz is guided not so much by semantic intuitions but rather by proof-theoretic considerations as to which laws ought to be derivable in the calculus. A clear example of this is Leibniz's use of propositional terms in the *Specimina calculi rationalis*. In this essay, the principles of propositional a-predication and propositional privation are posited by Leibniz with the specific aim of deriving the rule of *reductio* in his categorical calculus. These principles entail that, whatever semantic interpretation of propositional terms one might adopt, this interpretation must validate all the laws of classical propositional logic.

§6. The nonmonotonic categorical calculus. The fact that Leibniz's categorical calculus validates the laws of classical propositional logic allows us to address a problem in Aristotle's assertoric syllogistic which we have not yet discussed, namely, the problem of existential import. This problem arises from Aristotle's rule of a-conversion, $AaB \vdash BiA$. As it happens, this rule is not derivable in Leibniz's categorical calculus. Moreover, the rule cannot be added to the calculus on pain of inconsistency. To see this, consider the result of applying the rule of a-conversion to propositional terms:

$$[\psi]\, a\, [\varphi] \vdash [\varphi]\, i\, [\psi].$$

In Leibniz's categorical calculus, the i-proposition $[\varphi]\, i\, [\psi]$ is equivalent to $\psi \wedge \varphi$, i.e., to the categorical proposition $\textsc{con}([\textsc{con}(\psi)]\, a\, [\varphi])$.[127] Thus, the rule of a-conversion implies that, for any propositions φ and ψ:

$$\varphi \to \psi \vdash \psi \wedge \varphi.$$

This schema is not classically valid, and so adding it to the theory of classical propositional logic leads to inconsistency. Hence, since Leibniz's categorical calculus contains classical propositional logic, the calculus is rendered inconsistent by adding to it the rule of a-conversion.[128]

Nevertheless, the availability of classical conjunction in the language of Leibniz's categorical calculus allows us to formulate an alternative kind of a-proposition for which the rule of a-conversion does hold. Specifically, this rule will hold for the conjunctive a-proposition $A\tilde{a}B$ which asserts both that AaB and that B is nonempty in the sense that BiB. In order to preserve Aristotle's proof by *reductio* of moods such as Baroco, we must also introduce a corresponding disjunctive variant of the o-proposition, $A\tilde{o}B$, which is just

In this semantics, propositional terms are taken to be 0-ary predicates of individuals. Accordingly, the extension of a propositional term $[\varphi]$ is the empty set if the proposition φ is false, and the singleton set whose sole member is the empty sequence of individuals if φ is true.

127 Given Barbara and propositional privation, we have $\overline{[\varphi]}a[\psi] \dashv\vdash [\textsc{con}(\varphi)]a[\psi]$. Thus, by the rule of *reductio*, $\textsc{con}(\overline{[\varphi]}a[\psi]) \dashv\vdash \textsc{con}([\textsc{con}(\varphi)]a[\psi])$. By definition, the proposition on the right-hand side is $\psi \wedge \varphi$, and the proposition on the left-hand side is $[\varphi]\, i\, [\psi]$ (see Definition 5.2). See Barnes 1983: 317 n. 2.

128 For this reason, Barnes (1983: 317) concludes that, 'if the old Peripatetic programme is unsuccessful, the underlying cause of its failure is to be found in the fact that AaB in Aristotle's system entails *AiB*'. To avoid this difficulty, Barnes (1983: 317–8) proposes to implement the Peripatetic program in a system of categorical logic which does not include the rule of a-conversion. Alternatively, van Rooij (2012: 93–6) admits the rule of a-conversion but restricts it to non-propositional terms in his implementation of the Peripatetic program.

the contradictory of $A\widetilde{a}B$. Leaving the e- and i-propositions unchanged, we thus arrive at the following alternative definitions of syllogistic propositions in the language of Leibniz's categorical calculus:

$A\widetilde{a}B$	is the proposition	$AaB \wedge BiB$
$A\widetilde{e}B$	is the proposition	AeB
$A\widetilde{i}B$	is the proposition	AiB
$A\widetilde{o}B$	is the proposition	$\neg(AaB \wedge BiB)$.

This strategy of rendering a- and o-propositions as propositional compounds has been employed since medieval times to address the problem of existential import.[129] The above definitions of $A\widetilde{a}B$ and $A\widetilde{o}B$, however, are distinctive in that they are formulated in a purely categorical language that does not include any primitive propositional connectives. With respect to these alternative definitions, all the rules of Aristotle's assertoric syllogistic, including a-conversion and *reductio*, are derivable in Leibniz's categorical calculus.[130] For example, the rule of a-conversion, $A\widetilde{a}B \vdash B\widetilde{i}A$, can be established as follows:

1	$AaB \wedge BiB$	premise
2	AaB	conjunction elimination: 1
3	BiB	conjunction elimination: 1
4	AiB	Darii: 2, 3
5	BiA	i-conversion: 4

This proof of a-conversion makes essential use of the classical laws of conjunction elimination. The validity of these laws in Leibniz's categorical calculus relies, in turn, on the structural rule of weakening. Consider, for example, the following proof in the calculus of $\varphi \wedge \psi \vdash \psi$:

1.	$\neg\psi, \varphi \vdash \neg\psi$	weakening
2.	$\neg\psi \vdash \varphi \rightarrow \neg\psi$	from 1, by propositional a-predication
3.	$\neg(\varphi \rightarrow \neg\psi) \vdash \neg\neg\psi$	from 2, by *reductio*
4.	$\neg(\varphi \rightarrow \neg\psi) \vdash \psi$	from 3, by double negation

Without the structural rule of weakening, conjunction elimination would not be valid in Leibniz's categorical calculus, and hence the above proof of a-conversion just described would not be available. As noted above, however, the rule of weakening is rejected by Aristotle.[131] Furthermore, Aristotle rejects the principle of monotonicity on the grounds that a syllogism must not contain any superfluous premises not needed to derive the

[129] See, e.g., Moody 1953: 52, Thompson 1953: 256–7, Prior 1955: 169, Wedin 1990: 135, Ebert & Nortmann 2007: 333, Read 2015: 538–43.

[130] This does not include the traditional law of obversion, which is not posited by Aristotle as a principle of his assertoric syllogistic. With respect to the alternative definitions given above, the e-a-direction of obversion no longer holds, i.e., $B\widetilde{e}A \nvdash \overline{B}\widetilde{a}A$ (cf. n. 81).

[131] See n. 95.

conclusion.[132] For a Peripatetic logician inclined to follow Aristotle on this point, this raises the question as to what modes of propositional reasoning can be validated in a nonmonotonic system of categorical logic which does not include weakening among its structural rules. To address this question, we will consider the following nonmonotonic calculus obtained from Leibniz's categorical calculus by omitting the principle of weakening:

DEFINITION 6.1. *The* nonmonotonic categorical calculus, $\mid\!\sim$, *is the smallest relation between sets of propositions and propositions which satisfies the following principles*:

Barbara:	$AaB, BaC \mid\!\sim AaC$
e-conversion:	$AeB \mid\!\sim BeA$
double privation:	$\mid\!\sim \overline{\overline{A}}aA$ *and* $\mid\!\sim Aa\overline{\overline{A}}$
propositional privation:	$\mid\!\sim [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]}$ *and* $\mid\!\sim \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)]$
propositional a-predication:	$\Gamma, \varphi \mid\!\sim \psi$ *iff* $\Gamma \mid\!\sim [\psi]\, a\, [\varphi]$
weak cut:	*If* $\varphi \mid\!\sim \psi$ *and* $\mid\!\sim \varphi$, *then* $\mid\!\sim \psi$.

Here, A, B, C *are any terms;* φ, ψ *any propositions; and* Γ *any finite set of propositions.*

While some laws of classical propositional logic remain derivable in the nonmonotonic categorical calculus, others do not. For example, while the rule of *reductio* is still derivable, Łukasiewicz's first axiom, $\varphi \to (\psi \to \varphi)$, is not. Accordingly, related laws such as that of explosion no longer hold, i.e.: $\varphi, \neg\varphi \not\mid\!\sim \psi$.[133]

So, what exactly are the laws of propositional logic that are derivable in the nonmonotonic categorical calculus? It turns out that this calculus gives rise to a natural system of relevance logic known as $\mathrm{RMI}_{\overset{\neg}{\to}}$. The axioms of this system are the negation-implication axioms of the relevance logic R, together with the mingle axiom, $\varphi \to (\varphi \to \varphi)$.[134] The following theorem shows that all the axioms and rules of $\mathrm{RMI}_{\overset{\neg}{\to}}$ are derivable in the nonmonotonic categorical calculus:[135]

THEOREM 6.2. *For any propositions* φ *and* ψ*, let* $\neg\varphi$ *be the proposition* $\mathrm{CON}(\varphi)$*, and* $\varphi \to \psi$ *the proposition* $[\psi]\, a\, [\varphi]$*. Then the following claims hold in the nonmonotonic categorical calculus*:

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[132] See *Top.* 8.11 161b28–30, in conjunction with *Pr. An.* 1.1 24b18–20; cf. Alexander *In Top.* 13.28–14.2 and 568.18–23, Frede 1974: 22, Barnes 1980: 168–9, Smith 1997: 143, Ebert & Nortmann 2007: 227–8, Striker 2009: 81–2, Castagnoli 2016: 14–23.

[133] This law of explosion is likewise rejected by Aristotle (*Pr. An.* 2.15 64a20–2; see Priest 2006: 12 n. 16, 2007: 132, Malink 2013: 61). Moreover, Aristotle appeals to the nonmonotonicity of his syllogistic theory to refute an argument for a variant of the law of explosion (*Soph. Ref.* 5 167b21–36, 6 168b22–5; see Castagnoli 2016: 9–20).

[134] For an exposition of $\mathrm{RMI}_{\overset{\neg}{\to}}$, see Avron 1984: 334–5, 2016: 19–21.

[135] The principles RMI 1–6 listed in this theorem are exactly the axioms posited by Avron (1984: 335, 2016: 21) in his Hilbert-style axiomatization of $\mathrm{RMI}_{\overset{\neg}{\to}}$. In Avron 2016: 21, this Hilbert-style system is referred to as $\mathrm{HRMI}_{\overset{\neg}{\to}}$. Avron (2016: 16–21) uses the label '$\mathrm{RMI}_{\overset{\neg}{\to}}$' to refer to the monotonic closure of the system described in the theorem below. Since the principle of monotonicity runs counter to the spirit of relevance logic, we opt to use the label '$\mathrm{RMI}_{\overset{\neg}{\to}}$' for the system which results from Avron's $\mathrm{RMI}_{\overset{\neg}{\to}}$ by omitting the structural rule of monotonicity.

RMI 1:	$\mid\!\sim \varphi \rightarrow (\varphi \rightarrow \varphi)$
RMI 2:	$\mid\!\sim (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
RMI 3:	$\mid\!\sim (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
RMI 4:	$\mid\!\sim (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$
RMI 5:	$\mid\!\sim (\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$
RMI 6:	$\mid\!\sim \neg\neg\varphi \rightarrow \varphi$
modus ponens:	$\varphi \rightarrow \psi, \varphi \mid\!\sim \psi$
reflexivity:	$\varphi \mid\!\sim \varphi$
cut:	*If* $\Gamma, \varphi \mid\!\sim \psi$ *and* $\Delta \mid\!\sim \varphi$, *then* $\Gamma \cup \Delta \mid\!\sim \psi$.

Here, φ, ψ, χ *are any propositions, and* Γ, Δ *any finite sets of propositions.*

To establish this theorem, we first note that the structural rule of reflexivity is still valid in the nonmonotonic categorical calculus.[136] Given this, it is easy to verify that the principles RMI 1–6 are also valid in the calculus. Take, for example, the mingle axiom RMI 1. Since we defined a calculus to be a relation between sets (as opposed to sequences) of propositions and propositions, we have by reflexivity: $\varphi, \varphi \mid\!\sim \varphi$. Thus, by two applications of propositional a-predication: $\mid\!\sim \varphi \rightarrow (\varphi \rightarrow \varphi)$. In similar fashion, RMI 2 can be established as follows:

1.	$\varphi \rightarrow (\varphi \rightarrow \psi) \mid\!\sim \varphi \rightarrow (\varphi \rightarrow \psi)$	reflexivity
2.	$\varphi \rightarrow (\varphi \rightarrow \psi), \varphi \mid\!\sim \varphi \rightarrow \psi$	from 1, by propositional a-predication
3.	$\varphi \rightarrow (\varphi \rightarrow \psi), \varphi \mid\!\sim \psi$	from 2, by propositional a-predication
4.	$\varphi \rightarrow (\varphi \rightarrow \psi) \mid\!\sim \varphi \rightarrow \psi$	from 3, by propositional a-predication
5.	$\mid\!\sim (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$	from 4, by propositional a-predication

RMI 3–6 can be established in a similarly straightforward fashion, and the same is true for the rules of modus ponens and cut.[137]

Thus, the nonmonotonic categorical calculus validates all the laws of $\mathrm{RMI}_{\overrightarrow{\neg}}$. Conversely, it can be shown that these are the only laws of propositional logic that are validated by the calculus.[138] In this sense, the nonmonotonic categorical calculus is both sound and complete with respect to the relevance logic $\mathrm{RMI}_{\overrightarrow{\neg}}$. In other words, the propositional logic generated by this calculus is exactly $\mathrm{RMI}_{\overrightarrow{\neg}}$.

Since the rule of *reductio* is valid in $\mathrm{RMI}_{\overrightarrow{\neg}}$, syllogisms such as Baroco and Bocardo are still derivable in the nonmonotonic categorical calculus. The rule of a-conversion, however, remains a challenge for a Peripatetic logician who wishes to accommodate the entirety of Aristotle's assertoric syllogistic in this framework. Since $\mathrm{RMI}_{\overrightarrow{\neg}}$ is significantly weaker than

[136] See Theorem 7.11 below.

[137] See Theorems 7.14–7.18 and Theorem 7.23 below.

[138] The proof of this claim is somewhat involved, and relies on an algebraic semantics for $\mathrm{RMI}_{\overrightarrow{\neg}}$ introduced by Avron (1984). We therefore defer this proof to the appendix; see Theorem 7.48 below.

classical propositional logic, one might reconsider the option of positing a-conversion as an additional principle:

$$AaB \mathrel{|\!\sim} BiA.$$

This option, however, is still not viable because the rule of a-conversion leads to inconsistency even in the weaker setting of the nonmonotonic categorical calculus. To see this, note that the rule of a-conversion, when applied to propositional terms, implies the following law:[139]

$$\mathrel{|\!\sim} (\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi).$$

This law is known as Boethius' Thesis. In $\mathrm{RMI}_{\overrightarrow{\neg}}$, it is equivalent to the following schema, which is known as Aristotle's Thesis:[140]

$$\mathrel{|\!\sim} \neg(\varphi \rightarrow \neg\varphi).$$

Systems of propositional logic that satisfy these two theses are commonly referred to as 'connexive' logics. The addition of these connexive theses to even quite weak systems of propositional logic leads to inconsistency.[141] This is true, in particular, for the system $\mathrm{RMI}_{\overrightarrow{\neg}}$.[142] Thus, since the nonmonotonic categorical calculus contains $\mathrm{RMI}_{\overrightarrow{\neg}}$, adding to this calculus the rule of a-conversion renders it inconsistent.

It remains an open question, then, how to accommodate Aristotle's rule of a-conversion in the framework of Leibniz's categorical calculus based on his theory of propositional terms. Nevertheless, the above considerations serve to illustrate the complex interplay between categorical and propositional logic in this framework. We have seen how different choices of categorical principles license different modes of propositional reasoning, and how different categorical calculi generate different systems of propositional logic. For example, the presence or absence of weakening in the categorical calculus corresponds to the difference between classical propositional logic and the relevance logic $\mathrm{RMI}_{\overrightarrow{\neg}}$. Similarly, the presence or absence of a-conversion corresponds to the difference between connexive and nonconnexive logics.[143] Thus, in the framework of Leibniz's categorical calculus, the problem of existential import manifests itself as the problem of how to devise a consistent connexive logic that validates Aristotle's and Boethius' Theses.

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[139] Since the argument given in n. 127 still holds for the nonmonotonic categorical calculus, the rule of a-conversion implies $\varphi \rightarrow \psi \mathrel{|\!\sim} \neg(\psi \rightarrow \neg\varphi)$. But, by RMI 5, propositional a-predication, and *reductio*, we have $\neg(\psi \rightarrow \neg\varphi) \mathrel{|\!\sim} \neg(\varphi \rightarrow \neg\psi)$. Hence, $\varphi \rightarrow \psi \mathrel{|\!\sim} \neg(\varphi \rightarrow \neg\psi)$. By propositional a-predication, this implies $\mathrel{|\!\sim} (\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)$.

[140] Both Boethius' Thesis and Aristotle's Thesis are endorsed by Aristotle (*Pr. An.* 2.4 57b3–14; see McCall 2012: 415–16).

[141] See McCall 1966: 416–17.

[142] The schema $\neg\varphi \rightarrow (\varphi \rightarrow \neg\varphi)$ is valid in $\mathrm{RMI}_{\overrightarrow{\neg}}$ (see Avron 1984: 335–6, 2016: 21). Hence, by contraposition: $\neg(\varphi \rightarrow \neg\varphi) \rightarrow \varphi$. Since the antecedent of this conditional just is Aristotle's Thesis, adding this thesis to $\mathrm{RMI}_{\overrightarrow{\neg}}$ leads to inconsistency in the sense that every proposition is provable.

[143] The link between a-conversion and connexivity has been noted by McCall and Angell, who employ a connexive conditional to define a- and i-propositions in such a way that a-conversion is valid (McCall 1967: 349–56, 2012: 433, Angell 1986: 216–23). By contrast, in the framework of Leibniz's categorical calculus, we adopt the opposite approach, deriving connexivity from a-conversion. This is more in keeping with the Peripatetic program of deriving laws of propositional logic from categorical principles rather than the other way around.

As we have seen in this article, Leibniz's implementation of the Peripatetic program based on the principles of propositional privation and a-predication accomplishes far more than merely licensing the method of *reductio ad absurdum*. These principles allow for entire systems of propositional logic to be built up from purely categorical foundations. Leibniz's theory of propositional terms thus provides a fruitful framework for exploring the deep interconnections that exist between categorical and hypothetical logic, or, between the logic of terms and the logic of propositions.

§7. Appendix. In this appendix, we establish that the propositional logic generated by the nonmonotonic categorical calculus is exactly the relevance logic $\mathrm{RMI}_{\overset{\neg}{\to}}$. We begin by supplying some preliminary definitions needed to give a precise statement of this result (§7.1). We then establish the completeness of the nonmonotonic categorical calculus with respect to the relevance logic $\mathrm{RMI}_{\overset{\neg}{\to}}$ (§7.2). Next, we provide a Hilbert-style axiomatization of the nonmonotonic categorical calculus (§7.3). Based on this axiomatization, we establish the soundness of the nonmonotonic categorical calculus with respect to $\mathrm{RMI}_{\overset{\neg}{\to}}$ (§7.4).

7.1. Preliminaries. In what follows, we take for granted the terminology introduced in Definitions 5.1–5.3 and Definition 6.1 above. We first define a language for propositional logic in which to formulate the calculus $\mathrm{RMI}_{\overset{\neg}{\to}}$.

DEFINITION 7.1. *Given a countably infinite set of primitive expressions referred to as* atomic formulae, *the* formulae *are defined as follows*:[144]

1. *Every atomic formula is a formula.*
2. *If φ is a formula, then $\neg\varphi$ is a formula.*
3. *If φ and ψ are formulae, then $\varphi \to \psi$ is a formula.*

DEFINITION 7.2. *The calculus* $\mathrm{RMI}_{\overset{\neg}{\to}}$, $\vdash_{\mathrm{RMI}}$, *is the smallest relation between sets of formulae and formulae which satisfies the following principles*:

RMI 1:	$\vdash_{\mathrm{RMI}} \varphi \to (\varphi \to \varphi)$
RMI 2:	$\vdash_{\mathrm{RMI}} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
RMI 3:	$\vdash_{\mathrm{RMI}} (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
RMI 4:	$\vdash_{\mathrm{RMI}} (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
RMI 5:	$\vdash_{\mathrm{RMI}} (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$
RMI 6:	$\vdash_{\mathrm{RMI}} \neg\neg\varphi \to \varphi$
modus ponens:	$\varphi \to \psi, \varphi \vdash_{\mathrm{RMI}} \psi$
reflexivity:	$\varphi \vdash_{\mathrm{RMI}} \varphi$
cut:	*If* $\Gamma, \varphi \vdash_{\mathrm{RMI}} \psi$ *and* $\Delta \vdash_{\mathrm{RMI}} \varphi$, *then* $\Gamma \cup \Delta \vdash_{\mathrm{RMI}} \psi$.

Here, φ, ψ, χ *are any formulae, and* Γ, Δ *any finite sets of formulae.*

[144] The atomic formulae are to be chosen in such a way that no atomic formula is identical with any other formula.

As usual, $\vdash_{\mathrm{RMI}} \varphi$ is shorthand for $\emptyset \vdash_{\mathrm{RMI}} \varphi$. Moreover, we use $\psi_1, \ldots, \psi_n \vdash_{\mathrm{RMI}} \varphi$ as shorthand for $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathrm{RMI}} \varphi$, and $\Gamma, \psi_1, \ldots, \psi_n \vdash_{\mathrm{RMI}} \varphi$ as shorthand for $\Gamma \cup \{\psi_1, \ldots, \psi_n\} \vdash_{\mathrm{RMI}} \varphi$.

The formulae of the propositional language specified in Definition 7.1 can be viewed as schematic expressions to be instantiated by the categorical propositions specified in Definition 5.1. The following definition indicates how such a categorical instantiation is to be carried out:

DEFINITION 7.3. *A* categorical instantiation *is a function σ mapping each formula* (*as specified in Definition 7.1*) *to a proposition* (*as specified in Definition 5.1*) *such that*:

1. $\sigma(\neg\varphi) = \mathrm{CON}(\sigma(\varphi))$
2. $\sigma(\varphi \to \psi) = [\sigma(\psi)]\, a\, [\sigma(\varphi)]$.

If Γ is a finite set of formulae and φ is a formula, we write $\Gamma \mid\!\sim \varphi$ to express that, for every categorical instantiation σ:

$$\{\sigma(\psi) : \psi \in \Gamma\} \mid\!\sim \sigma(\varphi).$$

Given Definitions 7.1–7.3, the soundness and completeness of the nonmonotonic categorical calculus with respect to the relevance logic $\mathrm{RMI}_{\overset{\neg}{\to}}$ can now be expressed as follows. For any finite set of formulae Γ and any formula φ:

$$\Gamma \mid\!\sim \varphi \qquad \text{if and only if} \qquad \Gamma \vdash_{\mathrm{RMI}} \varphi.$$

The aim of this appendix is to establish this claim (see Theorems 7.28 and 7.48).

7.2. Completeness. In this section, we establish the completeness of the nonmonotonic categorical calculus with respect to $\mathrm{RMI}_{\overset{\neg}{\to}}$ (Theorem 7.28). To render the proof of this completeness result more perspicuous, we introduce the following shorthand:

DEFINITION 7.4. *For any propositions φ and ψ* (*as specified in Definition 5.1*):

1. *$\neg\varphi$ is the proposition* $\mathrm{CON}(\varphi)$.
2. *$\varphi \to \psi$ is the proposition* $[\psi]\, a\, [\varphi]$.
3. *$\varphi \wedge \psi$ is the proposition* $\neg(\varphi \to \neg\psi)$.

This definition gives rise to a systematic, but harmless, ambiguity in our use of the symbols '$\neg$' and '$\to$'. When applied to formulae, these symbols are primitive connectives in the language of propositional logic introduced in Definition 7.1. When applied to propositions, on the other hand, these symbols function as notational shorthand for propositions of the categorical language introduced in Definition 5.1. Which of the two readings is intended will be clear from the context as determined by whether the symbols '$\neg$' and '$\to$' are applied to formulae or propositions. Unless otherwise specified, we will use the schematic letters 'φ', 'ψ', 'χ', 'ξ' to stand for arbitrary propositions as opposed to formulae; and 'Γ', 'Δ' to stand for arbitrary finite sets of propositions as opposed to sets of formulae. Thus, for example, the following theorem is to be understood as applying to any propositions φ and ψ:

THEOREM 7.5. *If $\mid\!\sim \varphi$ and $\mid\!\sim \varphi \to \psi$, then $\mid\!\sim \psi$.*

Proof. This follows from propositional a-predication and weak cut. □

In what follows, we write 'p.a.' as shorthand for 'propositional a-predication'.

THEOREM 7.6. *If $\mid\!\sim AaB$ and $\mid\!\sim BaC$, then $\mid\!\sim AaC$.*

Proof. By Barbara and two applications of p.a., we have $\mid\!\sim AaB \to (BaC \to AaC)$. Hence, by two applications of T7.5, we have $\mid\!\sim AaC$. □

THEOREM 7.7. *If $\varphi \mid\!\sim AaB$ and $\mid\!\sim CaA$, then $\varphi \mid\!\sim CaB$.*

Proof. By Barbara and p.a., we have $CaA \mid\!\sim [CaB]\,a\,[AaB]$. Given $\mid\!\sim CaA$, we have $\mid\!\sim [CaB]\,a\,[AaB]$ by weak cut. Now, given $\varphi \mid\!\sim AaB$, we have $\mid\!\sim [AaB]\,a\,[\varphi]$ by p.a. So, by T7.6, $\mid\!\sim [CaB]\,a\,[\varphi]$. The desired result follows by p.a. □

THEOREM 7.8. *If $AaB \mid\!\sim \varphi$ and $\mid\!\sim AaC$, then $CaB \mid\!\sim \varphi$.*

Proof. By Barbara and p.a., we have $AaC \mid\!\sim [AaB]\,a\,[CaB]$. Given $\mid\!\sim AaC$, we have $\mid\!\sim [AaB]\,a\,[CaB]$ by weak cut. Now, given $AaB \mid\!\sim \varphi$, we have $\mid\!\sim [\varphi]\,a\,[AaB]$ by p.a. So, by T7.6, we have $\mid\!\sim [\varphi]\,a\,[CaB]$. The desired result follows by p.a. □

THEOREM 7.9. *If $\mid\!\sim AaB$, then $\mid\!\sim \overline{B}a\overline{A}$.*

Proof. Given $\mid\!\sim AaB$, we have $\mid\!\sim \overline{\overline{A}}aB$ by double privation and T7.6. The desired result follows from e-conversion and weak cut. □

THEOREM 7.10. *If $\varphi \mid\!\sim AaB$, then $\varphi \mid\!\sim \overline{B}a\overline{A}$.*

Proof. Given $\varphi \mid\!\sim AaB$, we have $\varphi \mid\!\sim \overline{\overline{A}}aB$ by double privation and T7.7. Hence, by p.a., we have $\mid\!\sim [\overline{\overline{A}}aB]\,a\,[\varphi]$. But, by e-conversion and p.a., we have $\mid\!\sim [\overline{B}a\overline{A}]\,a\,[\overline{\overline{A}}aB]$. Hence, by T7.6, we have $\mid\!\sim [\overline{B}a\overline{A}]\,a\,[\varphi]$. The desired result follows by p.a. □

THEOREM 7.11 (Reflexivity). $\varphi \mid\!\sim \varphi$.

Proof. By propositional privation and T7.9, we have both $\mid\!\sim \overline{\overline{[\varphi]}}\,a\,\overline{[\mathrm{CON}(\varphi)]}$ and $\mid\!\sim \overline{[\mathrm{CON}(\varphi)]}\,a\,\overline{\overline{[\varphi]}}$. Hence, by double privation and T7.6, $\mid\!\sim [\varphi]\,a\,[\varphi]$. The desired result follows by p.a. □

THEOREM 7.12 (RMI 1). $\mid\!\sim \varphi \to (\varphi \to \varphi)$.

Proof. By reflexivity, we have $\varphi, \varphi \mid\!\sim \varphi$. The desired result follows by two applications of p.a. □

THEOREM 7.13 (RMI 2). $\mid\!\sim (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$.

Proof. By reflexivity, we have $\varphi \to (\varphi \to \psi) \mid\!\sim \varphi \to (\varphi \to \psi)$. Hence, by two applications of p.a., $\varphi \to (\varphi \to \psi), \varphi \mid\!\sim \psi$. The desired result follows by two applications of p.a. □

THEOREM 7.14 (RMI 3). $\mid\!\sim (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$.

Proof. This follows from Barbara by two applications of p.a. □

THEOREM 7.15 (RMI 4). $\mid\!\sim (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$.

Proof. By reflexivity, we have $\varphi \to (\psi \to \chi) \mid\!\sim \varphi \to (\psi \to \chi)$. Hence, by two applications of p.a., $\varphi \to (\psi \to \chi), \psi, \varphi \mid\!\sim \chi$. The desired result follows by three applications of p.a. □

THEOREM 7.16 (RMI 5). $\mathrel{|\!\sim} (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$.

Proof. By e-conversion, we have $\overline{[\psi]}\, a\, [\varphi] \mathrel{|\!\sim} \overline{[\varphi]}\, a\, [\psi]$. Hence, by propositional privation and T7.7, $\overline{[\psi]}\, a\, [\varphi] \mathrel{|\!\sim} \psi \to \neg\varphi$. By propositional privation and T7.8, this implies $\varphi \to \neg\psi \mathrel{|\!\sim} \psi \to \neg\varphi$. The desired result follows by p.a. $\square$

THEOREM 7.17 (RMI 6). $\mathrel{|\!\sim} \neg\neg\varphi \to \varphi$.

Proof. By propositional privation, we have $\mathrel{|\!\sim} \overline{[\mathrm{CON}(\varphi)]}\, a\, [\mathrm{CON}(\mathrm{CON}(\varphi))]$. But, by propositional privation and T7.9, we have $\mathrel{|\!\sim} \overline{\overline{[\varphi]}}\, a\, \overline{[\mathrm{CON}(\varphi)]}$. The desired result follows by double privation and T7.6. $\square$

THEOREM 7.18 (Modus ponens). $\varphi \to \psi, \varphi \mathrel{|\!\sim} \psi$.

Proof. This follows from reflexivity by p.a. $\square$

THEOREM 7.19. $\mathrel{|\!\sim} \varphi \to \neg\neg\varphi$.

Proof. By RMI 5 and p.a., we have $\neg\varphi \to \neg\varphi \mathrel{|\!\sim} \varphi \to \neg\neg\varphi$. Hence, the desired result follows from reflexivity by p.a. and weak cut. $\square$

THEOREM 7.20. $\mathrel{|\!\sim} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.

Proof. By reflexivity, we have $\varphi \to \psi \mathrel{|\!\sim} \varphi \to \psi$. Hence, by T7.19 and T7.7, we have $\varphi \to \psi \mathrel{|\!\sim} \varphi \to \neg\neg\psi$. By p.a., this implies $\mathrel{|\!\sim} (\varphi \to \psi) \to (\varphi \to \neg\neg\psi)$. But, by RMI 5, $\mathrel{|\!\sim} (\varphi \to \neg\neg\psi) \to (\neg\psi \to \neg\varphi)$. The desired result follows by T7.6. $\square$

THEOREM 7.21. $\mathrel{|\!\sim} (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

Proof. By RMI 5 and p.a., we have $\neg\varphi \to \neg\psi \mathrel{|\!\sim} \psi \to \neg\neg\varphi$. Hence, by RMI 6 and T7.7, we have $\neg\varphi \to \neg\psi \mathrel{|\!\sim} \psi \to \varphi$. The desired result follows by p.a. $\square$

THEOREM 7.22. $\mathrel{|\!\sim} \varphi \to (\psi \to \chi)$ *iff* $\mathrel{|\!\sim} (\varphi \wedge \psi) \to \chi$.

Proof. For the left-to-right direction, suppose $\mathrel{|\!\sim} \varphi \to (\psi \to \chi)$. By T7.20 and T7.6, we have $\mathrel{|\!\sim} \varphi \to (\neg\chi \to \neg\psi)$. By RMI 4 and T7.5, this implies $\mathrel{|\!\sim} \neg\chi \to (\varphi \to \neg\psi)$. By T7.20 and T7.5, we have $\mathrel{|\!\sim} \neg(\varphi \to \neg\psi) \to \neg\neg\chi$. Hence, by T7.17 and T7.6, we have $\mathrel{|\!\sim} (\varphi \wedge \psi) \to \chi$.

For the right-to-left direction, suppose $\mathrel{|\!\sim} (\varphi \wedge \psi) \to \chi$. By T7.19 and T7.6, we have $\mathrel{|\!\sim} \neg(\varphi \to \neg\psi) \to \neg\neg\chi$. By T7.21 and T7.5, this implies $\mathrel{|\!\sim} \neg\chi \to (\varphi \to \neg\psi)$. By RMI 4 and T7.5, we have $\mathrel{|\!\sim} \varphi \to (\neg\chi \to \neg\psi)$. Hence, by T7.21 and T7.6, we have $\mathrel{|\!\sim} \varphi \to (\psi \to \chi)$. $\square$

THEOREM 7.23 (Cut). *If* $\Gamma, \varphi \mathrel{|\!\sim} \psi$ *and* $\Delta \mathrel{|\!\sim} \varphi$, *then* $\Gamma \cup \Delta \mathrel{|\!\sim} \psi$.

Proof. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and $\Delta = \{\delta_1, \ldots, \delta_m\}$. Given that $\Gamma, \varphi \mathrel{|\!\sim} \psi$, we have by repeated applications of p.a.:

$$\mathrel{|\!\sim} \varphi \to (\gamma_1 \to (\gamma_2 \to \cdots (\gamma_n \to \psi) \cdots)). \tag{1}$$

Since $\Delta \mathrel{|\!\sim} \varphi$, we have by repeated applications of p.a.:

$$\mathrel{|\!\sim} \delta_1 \to (\delta_2 \to \cdots (\delta_m \to \varphi) \cdots).$$

By repeated applications of T7.22, we have:

$$\mid\!\sim ((\cdots(\delta_1 \wedge \delta_2)\cdots) \wedge \delta_m) \rightarrow \varphi. \tag{2}$$

Given (1) and (2), we have by T7.6:

$$\mid\!\sim ((\cdots(\delta_1 \wedge \delta_2)\cdots) \wedge \delta_m) \rightarrow (\gamma_1 \rightarrow (\gamma_2 \rightarrow \cdots(\gamma_n \rightarrow \psi)\cdots)).$$

By repeated applications of T7.22, we have:

$$\mid\!\sim \delta_1 \rightarrow (\delta_2 \rightarrow (\cdots(\delta_m \rightarrow (\gamma_1 \rightarrow (\gamma_2 \rightarrow \cdots(\gamma_n \rightarrow \psi)\cdots))))).$$

Hence, by repeated applications of p.a., we obtain $\Gamma \cup \Delta \mid\!\sim \psi$. □

THEOREM 7.24 (Reductio). *If* $\Gamma, \varphi \mid\!\sim \psi$, *then* $\Gamma, \neg\psi \mid\!\sim \neg\varphi$.

Proof. Suppose $\Gamma, \varphi \mid\!\sim \psi$. Then, by p.a., we have $\Gamma \mid\!\sim \varphi \rightarrow \psi$. By T7.20 and p.a., $\varphi \rightarrow \psi \mid\!\sim \neg\psi \rightarrow \neg\varphi$. Hence, by T7.23, $\Gamma \mid\!\sim \neg\psi \rightarrow \neg\varphi$. The desired result follows by p.a. □

It can easily be verified that none of the theorems established so far (Theorems 7.5–7.24) rely on the definition of the operator CON$(\cdots)$ given in Definition 5.2. Thus, all of these theorems would still be provable in the nonmonotonic categorical calculus if CON$(\cdots)$ were not introduced as a defined notion but rather treated as a primitive propositional operator. The same is true of the following two theorems, since their proofs likewise do not rely on the definition of CON$(\cdots)$:

THEOREM 7.25. $\neg\varphi \mid\!\sim \overline{[\varphi]}\, a\, [\varphi]$.

Proof. By reflexivity, we have $\varphi, \varphi \mid\!\sim \varphi$. Hence, by T7.24, $\varphi, \neg\varphi \mid\!\sim \neg\varphi$, and so, by p.a., $\neg\varphi \mid\!\sim \varphi \rightarrow \neg\varphi$. The desired result follows by propositional privation and T7.7. □

THEOREM 7.26. $\overline{[\varphi]}\, a\, [\varphi] \mid\!\sim \neg\varphi$.

Proof. By T7.18, $\varphi, \varphi \rightarrow \neg\varphi \mid\!\sim \neg\varphi$. Hence, by T7.24, $\varphi, \neg\neg\varphi \mid\!\sim \neg(\varphi \rightarrow \neg\varphi)$. By T7.19, p.a., and cut, we have $\varphi \mid\!\sim \neg(\varphi \rightarrow \neg\varphi)$. Hence, by p.a., $\mid\!\sim \varphi \rightarrow \neg(\varphi \rightarrow \neg\varphi)$. By T7.16 and T7.5, it follows that $\mid\!\sim (\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$. By p.a., $\varphi \rightarrow \neg\varphi \mid\!\sim \neg\varphi$. The desired result follows by propositional privation and T7.8. □

Since the proofs of Theorems 7.25 and 7.26 do not rely on the definition of CON(φ) given in Definition 5.2, it follows that this definition is conservative with respect to the nonmonotonic categorical calculus. In other words, the definition of CON(φ) does not allow us to establish any new theorems in the calculus that could not be established if CON$(\cdots)$ were treated as an undefined primitive of the language.

We now establish the completeness of the nonmonotonic categorical calculus with respect to $\text{RMI}_{\underset{\rightarrow}{\neg}}$. To this end, we first note that the rule of cut holds not only for propositions but for formulae:

THEOREM 7.27. *For any finite sets of formulae* Γ, Δ *and any formulae* φ, ψ*: if* $\Gamma, \varphi \mid\!\sim \psi$ *and* $\Delta \mid\!\sim \varphi$, *then* $\Gamma \cup \Delta \mid\!\sim \psi$.

Proof. Suppose $\Gamma, \varphi \mid\!\sim \psi$ and $\Delta \mid\!\sim \varphi$, where Γ, Δ are finite sets of formulae and φ, ψ are formulae. Let σ be an arbitrary categorical instantiation. Then, we have $\{\sigma(\chi) : \chi \in \Gamma\} \cup \{\sigma(\varphi)\} \mid\!\sim \sigma(\psi)$ and $\{\sigma(\chi) : \chi \in \Delta\} \mid\!\sim \sigma(\varphi)$. Hence, by T7.23, $\{\sigma(\chi) : \chi \in \Gamma \cup \Delta\} \mid\!\sim \sigma(\psi)$. Since σ was chosen arbitrarily, it follows that $\Gamma \cup \Delta \mid\!\sim \psi$. □

THEOREM 7.28 (Completeness). *For any finite set of formulae* Γ *and any formula* φ*: if* $\Gamma \vdash_{\mathrm{RMI}} \varphi$*, then* $\Gamma \mid\!\sim \varphi$.

Proof. Since T7.11–T7.18 hold for arbitrary propositions φ, ψ, χ, these theorems also hold when φ, ψ, χ are taken to be arbitrary formulae. Hence, given Definition 7.2, the desired result follows from T7.11–T7.18 and T7.27. □

7.3. A Hilbert-style axiomatization of $\mid\!\sim$**.** In this section, we present a Hilbert-style axiomatization of the nonmonotonic categorical calculus (Theorem 7.39). We begin by defining the notion of a 'proof' in this calculus.

DEFINITION 7.29. *An* indexed proposition *is a pair* (Γ, φ) *where* Γ *is a finite set of propositions and* φ *is a proposition.*

DEFINITION 7.30. *Let* φ *be a proposition and* Γ *a finite set of propositions. A* proof of φ from Γ *is a sequence of indexed propositions* $(\Gamma_1, \varphi_1), \ldots, (\Gamma_n, \varphi_n)$*, such that* $(\Gamma_n, \varphi_n) = (\Gamma, \varphi)$ *and, for each* $i = 1, \ldots, n$*, at least one of the following conditions holds*:

1. $\Gamma_i = \{\varphi_i\}$
2. $\Gamma_i = \emptyset$ *and* φ_i *is a proposition of one of the following forms*:

$$
\begin{array}{ll}
\textit{Ax1}: & AaB \to (BaC \to AaC) \\
\textit{Ax2}: & AeB \to BeA \\
\textit{Ax3}: & \overline{\overline{A}}aA \\
\textit{Ax4}: & Aa\overline{\overline{A}} \\
\textit{Ax5}: & \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)] \\
\textit{Ax6}: & [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]} \\
\textit{Ax7}: & \varphi \to (\varphi \to \varphi) \\
\textit{Ax8}: & (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) \\
\textit{Ax9}: & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)).
\end{array}
$$

3. *For some* $j, k < i$, $\Gamma_i = \Gamma_j \cup \Gamma_k$ *and* $\varphi_k = \varphi_j \to \varphi_i$.

We write $\Pr(\Gamma, \varphi)$ *to express that there exists a proof of* φ *from* Γ.

THEOREM 7.31. *If* $\Pr(\Gamma, \varphi)$ *and* $\Pr(\Delta, \varphi \to \psi)$*, then* $\Pr(\Gamma \cup \Delta, \psi)$.

Proof. This follows directly from Definition 7.30. □

THEOREM 7.32. *If* $\Pr(\Gamma, AaB)$ *and* $\Pr(\Delta, BaC)$*, then* $\Pr(\Gamma \cup \Delta, AaC)$.

Proof. This follows from Ax1 by two application of T7.31. □

Note that a special case of this theorem is that, for any propositions φ, ψ, χ: if $\Pr(\Gamma, \varphi \to \psi)$ and $\Pr(\Delta, \psi \to \chi)$, then $\Pr(\Gamma \cup \Delta, \varphi \to \chi)$.

THEOREM 7.33. *If* $\Pr(\emptyset, \varphi \to (\psi \to \chi))$ *and* $\Pr(\emptyset, \chi \to \xi)$*, then* $\Pr(\emptyset, \varphi \to (\psi \to \xi))$.

Proof. By Ax1, $\Pr(\emptyset, (\chi \to \xi) \to ((\psi \to \chi) \to (\psi \to \xi)))$. Given $\Pr(\emptyset, \chi \to \xi)$, we have $\Pr(\emptyset, (\psi \to \chi) \to (\psi \to \xi))$ by T7.31. Given $\Pr(\emptyset, \varphi \to (\psi \to \chi))$, the desired result follows by T7.32. □

THEOREM 7.34. $\Pr(\emptyset, (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$.

Proof. By Ax9, $\Pr(\emptyset, (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)))$. Hence, by Ax1 and T7.32, we have $\Pr(\emptyset, (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to (\varphi \to \chi))))$. The desired result follows from Ax8 and T7.33. □

THEOREM 7.35. $\Pr(\emptyset, \varphi \to \varphi)$.

Proof. By Ax3, Ax6, and T7.32, we have $\Pr(\emptyset, \overline{\overline{[\mathrm{CON}(\varphi)]}}\, a\, \overline{[\varphi]})$. Hence, by Ax2 and T7.31, we have $\Pr(\emptyset, \overline{\overline{[\varphi]}}\, a\, \overline{[\mathrm{CON}(\varphi)]})$. By Ax4 and T7.32, it follows that $\Pr(\emptyset, [\varphi]\, a\, \overline{[\mathrm{CON}(\varphi)]})$. Now, by Ax2, Ax5, and T7.31, we have $\Pr(\emptyset, \overline{[\mathrm{CON}(\varphi)]}\, a\, [\varphi])$. Hence, the desired result follows by T7.32. □

The following theorem is a variant of the Moh-Church deduction theorem for relevant sequent calculi:

THEOREM 7.36. *If* $\Pr(\Gamma \cup \{\varphi\}, \psi)$, *then* $\Pr(\Gamma - \{\varphi\}, \varphi \to \psi)$.

Proof. Let $(\varphi_1, \Gamma_1), \ldots, (\varphi_n, \Gamma_n)$ be a proof of ψ from $\Gamma \cup \{\varphi\}$. We will show, by induction on i, that one of the following two conditions holds, for each $i = 1, \ldots, n$:

(i) $\varphi \in \Gamma_i$ and $\Pr(\Gamma_i - \{\varphi\}, \varphi \to \varphi_i)$

(ii) $\varphi \notin \Gamma_i$ and $\Pr(\Gamma_i, \varphi_i)$.

This will suffice to establish the desired result. For, since $\varphi \in \Gamma_n = \Gamma \cup \{\varphi\}$, the disjunction of (i) and (ii) holds for (Γ_n, φ_n) iff $\Pr(\Gamma_n - \{\varphi\}, \varphi \to \varphi_n)$. But since $\Gamma_n - \{\varphi\} = (\Gamma \cup \{\varphi\}) - \{\varphi\} = \Gamma - \{\varphi\}$ and $\varphi \to \varphi_n = \varphi \to \psi$, this means $\Pr(\Gamma - \{\varphi\}, \varphi \to \psi)$.

To show that the disjunction of (i) and (ii) holds for every (Γ_i, φ_i), we will consider each of the three cases described in Definition 7.30. First, suppose $\Gamma_i = \{\varphi_i\}$. If $\varphi \neq \varphi_i$, the claim follows from the fact that $\Pr(\{\varphi_i\}, \varphi_i)$. If $\varphi = \varphi_i$, the claim follows from T7.35.

Second, suppose that $\Gamma_i = \emptyset$ and φ_i is an instance of one of Ax1–Ax9. Then the claim follows from the fact that $\Pr(\emptyset, \varphi_i)$.

Finally, suppose that, for some $j, k < i$, $\Gamma_i = \Gamma_j \cup \Gamma_k$ and $\varphi_k = \varphi_j \to \varphi_i$. There are four cases to consider:

1. $\varphi \notin \Gamma_j$ and $\varphi \notin \Gamma_k$: By the induction hypothesis, $\Pr(\Gamma_j, \varphi_j)$ and $\Pr(\Gamma_k, \varphi_j \to \varphi_i)$. Then, by T7.31, $\Pr(\Gamma_i, \varphi_i)$. But since $\varphi \notin \Gamma_i$, this establishes the claim.

2. $\varphi \in \Gamma_j$ and $\varphi \notin \Gamma_k$: By the induction hypothesis, $\Pr(\Gamma_j - \{\varphi\}, \varphi \to \varphi_j)$ and $\Pr(\Gamma_k, \varphi_j \to \varphi_i)$. We then have the following proof:

$$
\begin{array}{r|ll}
1 & (\Gamma_j - \{\varphi\}, \varphi \to \varphi_j) & \text{by hypothesis} \\
2 & (\Gamma_k, \varphi_j \to \varphi_i) & \text{by hypothesis} \\
3 & (\emptyset, (\varphi_j \to \varphi_i) \to ((\varphi \to \varphi_j) \to (\varphi \to \varphi_i))) & \text{Ax1} \\
4 & (\Gamma_k, (\varphi \to \varphi_j) \to (\varphi \to \varphi_i)) & \text{T7.31: 2, 3} \\
5 & ((\Gamma_j - \{\varphi\}) \cup \Gamma_k, \varphi \to \varphi_i) & \text{T7.31: 1, 4}
\end{array}
$$

Since $(\Gamma_j - \{\varphi\}) \cup \Gamma_k = (\Gamma_j \cup \Gamma_k) - \{\varphi\}$, we have $\Pr(\Gamma_i - \{\varphi\}, \varphi \rightarrow \varphi_i)$.

3. $\varphi \notin \Gamma_j$ and $\varphi \in \Gamma_k$: By the induction hypothesis, $\Pr(\Gamma_j, \varphi_j)$ and $\Pr(\Gamma_k - \{\varphi\}, \varphi \rightarrow (\varphi_j \rightarrow \varphi_i))$. We then have the following proof:

1	(Γ_j, φ_j)	by hypothesis
2	$(\Gamma_k - \{\varphi\}, \varphi \rightarrow (\varphi_j \rightarrow \varphi_i))$	by hypothesis
3	$(\emptyset, \varphi \rightarrow (\varphi_j \rightarrow \varphi_i)) \rightarrow (\varphi_j \rightarrow (\varphi \rightarrow \varphi_i)))$	Ax9
4	$(\Gamma_k - \{\varphi\}, \varphi_j \rightarrow (\varphi \rightarrow \varphi_i))$	T7.31: 2, 3
5	$(\Gamma_j \cup (\Gamma_k - \{\varphi\}), \varphi \rightarrow \varphi_i)$	T7.31: 1, 4

Since $\Gamma_j \cup (\Gamma_k - \{\varphi\}) = (\Gamma_j \cup \Gamma_k) - \{\varphi\}$, we have $\Pr(\Gamma_i - \{\varphi\}, \varphi \rightarrow \varphi_i)$.

4. $\varphi \in \Gamma_j$ and $\varphi \in \Gamma_k$: By the induction hypothesis, $\Pr(\Gamma_j - \{\varphi\}, \varphi \rightarrow \varphi_j)$ and $\Pr(\Gamma_k - \{\varphi\}, \varphi \rightarrow (\varphi_j \rightarrow \varphi_i))$. We then have the following proof:

1	$(\Gamma_j - \{\varphi\}, \varphi \rightarrow \varphi_j)$	by hypothesis
2	$(\Gamma_k - \{\varphi\}, \varphi \rightarrow (\varphi_j \rightarrow \varphi_i))$	by hypothesis
3	$(\emptyset, (\varphi \rightarrow (\varphi_j \rightarrow \varphi_i)) \rightarrow ((\varphi \rightarrow \varphi_j) \rightarrow (\varphi \rightarrow \varphi_i)))$	T7.34
4	$(\Gamma_k - \{\varphi\}, (\varphi \rightarrow \varphi_j) \rightarrow (\varphi \rightarrow \varphi_i))$	T7.31: 2, 3
5	$((\Gamma_j - \{\varphi\}) \cup (\Gamma_k - \{\varphi\}), \varphi \rightarrow \varphi_i)$	T7.31: 1, 4

Since $(\Gamma_j - \{\varphi\}) \cup (\Gamma_k - \{\varphi\}) = (\Gamma_j \cup \Gamma_k) - \{\varphi\}$, we have $\Pr(\Gamma_i - \{\varphi\}, \varphi \rightarrow \varphi_i)$.

This completes the induction. □

THEOREM 7.37. $\Pr(\Gamma, \varphi \rightarrow \psi)$ *iff* $\Pr(\Gamma \cup \{\varphi\}, \psi)$.

Proof. For the left-to-right direction, let $(\Gamma_1, \varphi_1), \ldots, (\Gamma_n, \varphi_n)$ be a proof of $\varphi \rightarrow \psi$ from Γ. Then $(\Gamma_n, \varphi_n) = (\Gamma, \varphi \rightarrow \psi)$. If we add to the end of this proof the two indexed formulae $(\{\varphi\}, \varphi)$ and $(\Gamma \cup \{\varphi\}, \psi)$, the result is a proof of ψ from $\Gamma \cup \{\varphi\}$.

For the right-to-left direction, suppose $\Pr(\Gamma \cup \{\varphi\}, \psi)$. By T7.36, we have $\Pr(\Gamma - \{\varphi\}, \varphi \rightarrow \psi)$. If $\varphi \notin \Gamma$, then $\Gamma - \{\varphi\} = \Gamma$ and so we have $\Pr(\Gamma, \varphi \rightarrow \psi)$. If, on the other hand, $\varphi \in \Gamma$, then we have the following proof:

1	$(\Gamma - \{\varphi\}, \varphi \rightarrow \psi)$	by hypothesis
2	$(\emptyset, (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \psi)))$	Ax1
3	$(\Gamma - \{\varphi\}, (\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \psi))$	T7.31: 1, 2
4	$(\emptyset, \varphi \rightarrow (\varphi \rightarrow \varphi))$	Ax7
5	$(\Gamma - \{\varphi\}, \varphi \rightarrow (\varphi \rightarrow \psi))$	T7.32: 3, 4

Hence, $\Pr(\Gamma - \{\varphi\}, \varphi \rightarrow (\varphi \rightarrow \psi))$. But since $(\Gamma - \{\varphi\}) \cup \{\varphi\} = \Gamma$, the desired result follows from the left-to-right direction of the theorem, which has already been established. □

THEOREM 7.38. $\Gamma \mathrel{|\!\sim} \varphi$ *iff* $\Pr(\Gamma, \varphi)$.

Proof. In the left-to-right direction, the proof proceeds by induction on the definition of $\mid\!\sim$ given in Definition 6.1. For the case of Barbara, we have the following proof:

1	$(\{AaB\}, AaB)$	Definition 7.30
2	$(\{BaC\}, BaC)$	Definition 7.30
3	$(\emptyset, AaB \to (BaC \to AaC))$	Ax1
4	$(\{AaB\}, BaC \to AaC)$	T7.31: 1, 3
5	$(\{AaB, BaC\}, AaC)$	T7.31: 2, 4

The cases of e-conversion, double privation, and propositional privation are equally straightforward. The case of propositional a-predication follows from Theorem 7.37. For the case of weak cut, suppose $\Pr(\{\varphi\}, \psi)$ and $\Pr(\emptyset, \varphi)$. By T7.37, we have $\Pr(\emptyset, \varphi \to \psi)$. Hence, by T7.31, we have $\Pr(\emptyset, \psi)$.

For the right-to-left direction, let $(\Gamma_1, \varphi_1), \ldots, (\Gamma_n, \varphi_n)$ be a proof of φ from Γ. We will show, by induction on i, that $\Gamma_i \mid\!\sim \varphi_i$ for $i = 1, \ldots, n$. If $\Gamma_i = \{\varphi_i\}$, then the result follows from T7.11. Next suppose $\Gamma_i = \emptyset$ and φ_i is of one of the forms Ax1–9. The case in which φ_i is an instance of Ax1 follows from Barbara and p.a. The case of Ax2 follows from e-conversion and p.a. The cases of Ax3–4 and Ax5–6 follow from double privation and propositional privation, respectively. The case of Ax7 follows from RMI 1 (T7.12). The case of Ax8 follows from RMI 2 (T7.13). The case of Ax9 follows from RMI 4 (T7.15). Finally, suppose that, for some $j, k < i$, $\Gamma_i = \Gamma_j \cup \Gamma_k$ and $\varphi_k = \varphi_j \to \varphi_i$. By the induction hypothesis, $\Gamma_j \mid\!\sim \varphi_j$ and $\Gamma_k \mid\!\sim \varphi_j \to \varphi_i$. By p.a., we have $\Gamma_k, \varphi_j \mid\!\sim \varphi_i$. Hence, by T7.23, we have $\Gamma_i \mid\!\sim \varphi_i$. This completes the induction. □

The following theorem provides a Hilbert-style axiomatization of the nonmonotonic categorical calculus.

THEOREM 7.39. *The set of propositions provable in the nonmonotonic categorical calculus,* $\{\varphi : \mid\!\sim \varphi\}$*, can be defined as follows*:

Ax1: $\mid\!\sim AaB \to (BaC \to AaC)$

Ax2: $\mid\!\sim AeB \to BeA$

Ax3: $\mid\!\sim \overline{\overline{A}}aA$

Ax4: $\mid\!\sim Aa\overline{\overline{A}}$

Ax5: $\mid\!\sim \overline{[\varphi]}\, a\, [\mathrm{CON}(\varphi)]$

Ax6: $\mid\!\sim [\mathrm{CON}(\varphi)]\, a\, \overline{[\varphi]}$

Ax7: $\mid\!\sim \varphi \to (\varphi \to \varphi)$

Ax8: $\mid\!\sim (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

Ax9: $\mid\!\sim (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

MP: *If* $\mid\!\sim \varphi$ *and* $\mid\!\sim \varphi \to \psi$*, then* $\mid\!\sim \psi$.

Proof. This is an immediate consequence of Definition 7.30 and Theorem 7.38. □

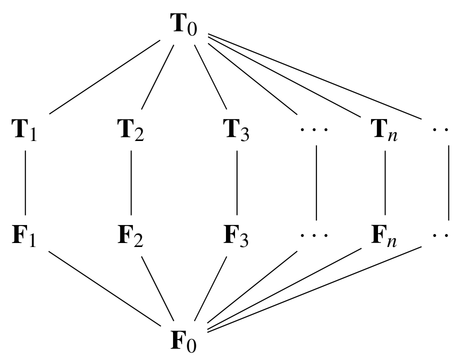
7.4. Soundness. In this section, we establish the soundness of the nonmonotonic categorical calculus with respect to $\mathrm{RMI}_{\overrightarrow{\neg}}$ (Theorem 7.48). The proof of this result will make essential use of an algebraic semantics for $\mathrm{RMI}_{\overrightarrow{\neg}}$ presented in Avron 1984.

DEFINITION 7.40. *Let $\mathfrak{A}$ be the lattice $\langle \mathbb{A}, \leq \rangle$ defined by*:

1. $\mathbb{A} = \{\mathbf{T}_n : n = 0, 1, 2, \dots\} \cup \{\mathbf{F}_n : n = 0, 1, 2, \dots\}$
2. $\leq$ *is the smallest partial order on $\mathbb{A}$ satisfying*:

   (i) $\mathbf{F}_0 \leq x \leq \mathbf{T}_0$, *for all $x \in \mathbb{A}$*
   (ii) $\mathbf{F}_n \leq \mathbf{T}_n$, *for all $n \geq 1$.*

The lattice $\mathfrak{A}$ is depicted by the following Hasse diagram:



For any $x, y \in \mathbb{A}$, we write $x < y$ to express that $x \leq y$ and $x \neq y$. Moreover, for any $x \in \mathbb{A}$, we write x' for the element of $\mathbb{A}$ defined by:

$$x' = \begin{cases} \mathbf{T}_n & \text{if } x = \mathbf{F}_n, \\ \mathbf{F}_n & \text{if } x = \mathbf{T}_n. \end{cases}$$

Note that, for any $x, y \in \mathbb{A}$, we have (i) $x'' = x$, and (ii) $x \leq y$ iff $y' \leq x'$.

DEFINITION 7.41. *An* Avron model *for the language of the nonmonotonic categorical calculus* (*as specified in Definition 5.1*) *is a function μ mapping every term of this language to an element of $\mathfrak{A}$, such that, for any terms A, B*:

1. $\mu(\overline{A}) = \mu(A)'$
2. $\mu([AaB]) = \begin{cases} \max\{\mu(B)', \mu(A)\} & \text{if } \mu(B) \leq \mu(A), \\ \min\{\mu(B)', \mu(A)\} & \text{if } \mu(A) < \mu(B), \\ \mathbf{F}_0 & \text{otherwise.} \end{cases}$

An Avron model μ satisfies *the proposition AaB (written $\mu \models AaB$) iff $\mu(B) \leq \mu(A)$. We write $\models AaB$ to express that, for every Avron model μ, $\mu \models AaB$. If φ is a formula* (*as specified in Definition 7.1*), *we write $\models \varphi$ to express that, for every categorical instantiation σ, $\models \sigma(\varphi)$.*

The following theorem is a straightforward consequence of Definition 7.41:

THEOREM 7.42. *For any Avron model μ, and any terms A, B*:

1. $\mu \models AaB$ *iff* $\mu([AaB]) \in \{\mathbf{T}_n : n \geq 0\}$
2. $\mu([Aa\overline{A}]) = \mu(A)$
3. $\mu([\overline{A}aA]) = \mu(\overline{A})$.

We next observe the weak soundness of the nonmonotonic categorical calculus with respect to the class of Avron models.

THEOREM 7.43. *For any proposition φ: if $\mid\!\sim \varphi$, then $\models \varphi$.*

Proof. Given the Hilbert-style axiomatization of the nonmonotonic categorical calculus presented in T7.39, the proof of this theorem is tedious but straightforward. Take, for example, the case of Ax1. Let μ be any Avron model. Then we must show that:

$$\mu \models AaB \rightarrow (BaC \rightarrow AaC).$$

Clearly, this claim holds if $\mu(A) = \mathbf{T}_0$ or $\mu(C) = \mathbf{F}_0$. Moreover, the claim holds if $\mu(B) = \mathbf{T}_0$ or $\mu(B) = \mathbf{F}_0$. Also, the claim holds if $\mu(A) = \mathbf{F}_0$ or $\mu(C) = \mathbf{T}_0$. Now, if neither $\mu(A) \leq \mu(B)$ nor $\mu(B) \leq \mu(A)$, then $\mu([AaB]) = \mathbf{F}_0$ and so the claim holds. Also, if neither $\mu(B) \leq \mu(C)$ nor $\mu(C) \leq \mu(B)$, then $\mu([BaC \rightarrow AaC]) = \mathbf{T}_0$ and so the claim holds. Thus, it only remains to consider the case in which there is some $n \geq 1$ such that $\mu(A), \mu(B), \mu(C) \in \{\mathbf{T}_n, \mathbf{F}_n\}$. In this case, $\mu([AaB]), \mu([BaC \rightarrow AaC]) \in \{\mathbf{T}_n, \mathbf{F}_n\}$. Hence, the claim fails to hold only if $\mu([AaB]) = \mathbf{T}_n$ and $\mu([BaC \rightarrow AaC]) = \mathbf{F}_n$. But, in this case, $\mu([BaC]) = \mathbf{T}_n$ and $\mu([AaC]) = \mathbf{F}_n$. This implies $\mu(A) = \mathbf{F}_n$ and $\mu(C) = \mu(B) = \mathbf{T}_n$, which is incompatible with $\mu([AaB]) = \mathbf{T}_n$. Hence, the claim holds.

We leave it to the reader to verify that the remaining axiom schemata Ax2–Ax9 and the rule of MP listed in T7.39 are sound with respect to the class of Avron models. □

Given Definition 7.3, the following is an immediate consequence of Theorem 7.43:

THEOREM 7.44. *For any formula φ: if $\mid\!\sim \varphi$, then $\models \varphi$.*

DEFINITION 7.45. *A* propositional Avron model *for the propositional language specified in Definition 7.1 is a function μ mapping every formula of this language to an element of $\mathfrak{A}$, such that, for any formulae φ, ψ:*

1. $\mu(\neg\varphi) = \mu(\varphi)'$
2. $\mu(\varphi \rightarrow \psi) = \begin{cases} \max\{\mu(\varphi)', \mu(\psi)\} & \textit{if } \mu(\varphi) \leq \mu(\psi), \\ \min\{\mu(\varphi)', \mu(\psi)\} & \textit{if } \mu(\psi) < \mu(\varphi), \\ \mathbf{F}_0 & \textit{otherwise}. \end{cases}$

A propositional Avron model μ satisfies *the formula φ (written $\mu \models_{\mathrm{RMI}} \varphi$) iff $\mu(\varphi) \in \{\mathbf{T}_n : n \geq 0\}$. We write $\models_{\mathrm{RMI}} \varphi$ to express that, for every propositional Avron model μ, $\mu \models_{\mathrm{RMI}} \varphi$.*

THEOREM 7.46. *For any formula φ: if $\models \varphi$, then $\models_{\mathrm{RMI}} \varphi$.*

Proof. Let $t_1, t_2, \ldots$ be the simple terms introduced in Definition 5.1, and let $p_1, p_2, \ldots$ be the atomic formulae introduced in Definition 7.1. Then, there is a unique categorical instantiation, σ^*, such that for all $n \geq 1$:

$$\sigma^*(p_n) = t_n a \overline{t_n}.$$

For any given Avron model μ, let μ^* be the unique propositional Avron model such that for all $n \geq 1$:

$$\mu^*(p_n) = \mu(t_n).$$

We first show that, for any given Avron model μ and any formula φ, $\mu([\sigma^*(\varphi)]) = \mu^*(\varphi)$. The proof proceeds by induction on the formulae. For any atomic formula p_n, we have by T7.42.2:

$$\mu([\sigma^*(p_n)]) = \mu([t_n a \overline{t_n}]) = \mu(t_n) = \mu^*(p_n).$$

Moreover, for any proposition φ, we have $\mu([\textsc{con}(\varphi)]) = \mu(\overline{[\varphi]})$ by T7.42.3. Now suppose, for induction, that the claim holds for the formula φ. Then, we have:

$$\begin{aligned} \mu([\sigma^*(\neg\varphi)]) &= \mu([\textsc{con}(\sigma^*(\varphi))]) \\ &= \mu(\overline{[\sigma^*(\varphi)]}) \\ &= \mu([\sigma^*(\varphi)])' \\ &= \mu^*(\varphi)' \\ &= \mu^*(\neg\varphi). \end{aligned}$$

Finally, suppose that the claim holds for the formulae φ, ψ. Then, we have:

$$\begin{aligned} \mu([\sigma^*(\varphi \to \psi)]) &= \mu([[\sigma^*(\psi)]\, a\, [\sigma^*(\varphi)]]) \\ &= \begin{cases} \max\{\mu([\sigma^*(\varphi)])', \mu([\sigma^*(\psi)])\} & \text{if } \mu([\sigma^*(\varphi)]) \leq \mu([\sigma^*(\psi)]), \\ \min\{\mu([\sigma^*(\varphi)])', \mu([\sigma^*(\psi)])\} & \text{if } \mu([\sigma^*(\psi)]) < \mu([\sigma^*(\varphi)]), \\ \mathbf{F}_0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \max\{\mu^*(\varphi)', \mu^*(\psi)\} & \text{if } \mu^*(\varphi) \leq \mu^*(\psi), \\ \min\{\mu^*(\varphi)', \mu^*(\psi)\} & \text{if } \mu^*(\psi) < \mu^*(\varphi), \\ \mathbf{F}_0 & \text{otherwise} \end{cases} \\ &= \mu^*(\varphi \to \psi). \end{aligned}$$

This completes the induction.

To prove the theorem, let φ be any formula such that $\models \varphi$. Then, in particular, $\models \sigma^*(\varphi)$. Hence, by T7.42.1, $\mu([\sigma^*(\varphi)]) \in \{\mathbf{T}_n : n \geq 0\}$ for any Avron model μ. It follows that $\mu^*(\varphi) \in \{\mathbf{T}_n : n \geq 0\}$ for any Avron model μ. In other words, for any Avron model μ, $\mu^* \models_{\mathrm{RMI}} \varphi$. But since every propositional Avron model is of the form μ^* for some Avron model μ, we have $\models_{\mathrm{RMI}} \varphi$. □

The following theorem asserts the weak completeness of $\mathrm{RMI}_{\overrightarrow{\neg}}$ with respect to the class of propositional Avron models. A proof of this theorem is given by Avron (1984: 337–40).

THEOREM 7.47 (Avron 1984). *For any formula φ: if $\models_{\mathrm{RMI}} \varphi$, then $\vdash_{\mathrm{RMI}} \varphi$.*

Given this completeness theorem, we are finally in a position to prove the soundness of the nonmonotonic categorical calculus with respect to $\mathrm{RMI}_{\overrightarrow{\neg}}$:

THEOREM 7.48 (Soundness). *For any finite set of formulae Γ and any formula φ: if $\Gamma \mid\!\sim \varphi$, then $\Gamma \vdash_{\mathrm{RMI}} \varphi$.*

Proof. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. Then, given $\Gamma \mid\!\sim \varphi$, we have by p.a.:

$$\mid\!\sim \psi_1 \to (\psi_2 \to (\cdots(\psi_n \to \varphi)\cdots)).$$

By T7.44, T7.46, and T7.47, we have:

$$\vdash_{\mathrm{RMI}} \psi_1 \to (\psi_2 \to (\cdots(\psi_n \to \varphi)\cdots)).$$

The desired result follows from repeated applications of T7.18 and cut. □

Theorems 7.28 and 7.48 together establish that the propositional logic generated by the nonmonotonic categorical calculus is exactly the relevance logic $\mathrm{RMI}_{\overset{\neg}{\to}}$.

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