LEIBNIZ ON THE LOGIC OF CONCEPTUAL CONTAINMENT AND COINCIDENCE

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Abstract. In a series of early essays written around 1679, Leibniz sets out to explore the logic of conceptual containment. In his more mature logical writings from the mid-1680s, however, his focus shifts away from the logic of containment to that of coincidence, or mutual containment. This shift in emphasis is indicative of the fact that Leibniz's logic has its roots in two distinct theoretical frameworks: (i) the traditional theory of the categorical syllogism based on rules of inference such as Barbara, and (ii) equational systems of arithmetic and geometry based on the rule of substitution of equals. While syllogistic reasoning is naturally modeled in a logic of containment, substitutional reasoning of the sort performed in arithmetic and geometry is more naturally modeled in a logic of coincidence. In this paper, we argue that Leibniz's logic of conceptual containment and his logic of coincidence can in fact be viewed as two alternative axiomatizations, in different but equally expressive languages, of one and the same logical theory. Thus, far from being incoherent, the varying syllogistic and equational themes that run throughout Leibniz's logical writings complement one another and fit together harmoniously.

1. Conceptual Containment and Coincidence

Among the notions that figure most prominently in Leibniz's logical and metaphysical writings is that of conceptual containment. It is well known that Leibniz subscribed to a containment theory of truth, according to which the truth of any proposition consists in one term's being conceptually contained in another. This theory applies most directly to simple predicative propositions of the form A *is* B. In Leibniz's view, whenever such a proposition is true, its truth consists in the predicate term, B, being conceptually contained in the subject term, A. Famously, this is so not only for necessary propositions such as *God is wise*, but also for contingent propositions such as *Caesar is just*. Thus, Leibniz writes:[1]

> The predicate or consequent always inheres in the subject or antecedent, and the nature of truth in general or the connection between the terms of a statement, consists in this very thing, as Aristotle also observed . . . This is true for every affirmative truth, universal or particular, necessary or contingent, and in both an intrinsic and extrinsic denomination. And here lies hidden a wonderful secret. (*Principia logico-metaphysica*, A VI.4 1644)

Leibniz's containment theory of truth implies that every proposition expresses the conceptual containment of one term in another.[2] Or, as Leibniz puts it, 'in every proposition the predicate is said to inhere in the subject'.[3] Given his commitment to this theory, it is perhaps not surprising that in several of his logical writings Leibniz undertakes to develop a calculus in which conceptual containment figures as the sole primitive relation between terms. For example, in an essay titled *De calculo analytico generali*, written around 1679, he develops a calculus in which every simple proposition is of the form *A is B*.[4] Similar containment calculi appear in a number of essays written by Leibniz in the late 1670s, including *De characteristica logica*, *Propositiones primitivae*, and *Specimen calculi universalis*.[5]

By contrast, in Leibniz's more mature logical writings from the mid-1680s there is a notable shift in focus away from the relation of conceptual containment to that of coincidence, or mutual containment, between terms. In these later writings, Leibniz develops calculi in which the primitive propositions are not of the form *A is B* but instead of the form *A coincides with B* (or, equivalently, *A is the same as B*). The canonical version of such a coincidence calculus is developed by Leibniz in his 1686 treatise *Generales inquisitiones de analysi notionum et veritatum*. In this treatise, Leibniz regards propositions of the form *A coincides with B* as 'the most simple' kind of proposition.[6] Propositions expressing containment are then defined

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[1] In what follows, we adopt the following abbreviations for editions of Leibniz's writings:

A	*Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, ed. by the Deutsche Akademie der Wissenschaften, Berlin: Akademie Verlag. Cited by series, volume, and page.
C	*Opuscules et fragments inédits de Leibniz*. Ed. by L. Couturat, Paris: Félix Alcan, 1903.
GM	*Leibnizens mathematische Schriften*. Ed. by C. I. Gerhardt, Halle: H. W. Schmidt, 1849–63. Cited by volume and page.
GP	*Die philosophischen Schriften von Gottfried Wilhelm Leibniz*. Ed. by C. I. Gerhardt. Berlin: Weidmann, 1875–90. Cited by volume and page.

In addition, we sometimes use the abbreviation '*GI*' for Leibniz's essay *Generales inquisitiones de analysi notionum et veritatum* (A VI.4 739–88). This essay consists of a series of numbered sections, which we designate by references such as '*GI* §16' and '*GI* §72'.

[2] See Adams 1994: 57–63; similarly, Parkinson 1965: 6 and 33.

[3] A VI.4 223; see also A VI.4 218 n. 1, 551, *GI* §132.

[4] See A VI.4 148–50.

[5] A VI.4 119-20, 140–5, and 280–8; see also A VI.4 274–9.

[6] *GI* §157 and §163. Accordingly, the axioms of the calculus developed by Leibniz in the *Generales inquisitiones* almost all take the form of coincidence propositions; see the preliminary axiomatizations of the calculus in *GI* §§1–15, §171, and §189, as well as the final axiomatization in §198.

in terms of coincidence. Specifically, Leibniz defines A *is* B as the proposition A *coincides with* AB, where AB is a composite term formed from the terms A and B.[7] By means of this definition, Leibniz is able to derive the laws of conceptual containment as theorems of his coincidence calculus. Thus, in the *Generales inquisitiones* coincidence is treated as the sole primitive relation between terms, with conceptual containment and all other relations being defined as coincidences between complex terms.[8] This same approach, aimed at developing a coincidence calculus, is adopted by Leibniz in a series of related essays written shortly after the *Generales inquisitiones*, including *Specimina calculi rationalis*, *Specimen calculi coincidentium*, and *Specimen calculi coincidentium et inexistentium*.[9]

Despite this shift in emphasis from conceptual containment to coincidence in his logical writings, Leibniz never abandons his commitment to the containment theory of truth. Indeed, he reaffirms this theory in numerous subsequent works, such as the *Discours de Metaphysique* (1686), *Principia logico-metaphysica* (around 1689), and the *Nouveaux Essais* (1704).[10] Even in the *Generales inquisitiones* (1686), Leibniz takes for granted the containment theory of truth. This can be seen, for example, from the following passage:

> Every true proposition can be proved. For since, as Aristotle says, the predicate inheres in the subject, or, the concept of the predicate is involved in the concept of the subject when this concept is perfectly understood, surely it must be possible for the truth to be shown by the analysis of terms into their values, or, those terms which they contain. (*Generales inquisitiones* §132)

Thus, Leibniz appears to persist in his commitment to the primacy of conceptual containment even in those writings in which his central aim is to develop a logic of coincidence. Accordingly, in the *Generales inquisitiones*, Leibniz undertakes to analyze the meaning of coincidence in terms of containment:

> A proposition is that which states what term is or is not contained in another. Hence, . . . a proposition is also that which says whether or not a term coincides with another; for those terms which coincide are mutually contained in one another. (*Generales inquisitiones* §195)

In spite of this characterization of coincidence as mutual containment, Leibniz opts to formulate the calculus developed in the *Generales inquisitiones* in a language in which coincidence, and not containment, figures as the sole primitive relation

[7] *GI* §83, §113, A VI.4 751 n. 13, 808. For other definitions of containment in terms of coincidence, see *GI* §16, §17, §189.4, and §198.9.

[8] As Castañeda points out, in the *Generales inquisitiones* Leibniz develops 'a strict equational calculus in which all propositions are about the coincidence of terms' (Castañeda 1976: 483). See also Schupp 1993: 156.

[9] A VI.4 807–14, 816–22, 830–45; see also A VI.4 845–55 and C 421–3. Prior to the *Generales inquisitiones*, no systematic attempt is made by Leibniz to develop a calculus of coincidence. As far as we can see, the earliest indication of Leibniz's desire to develop such a calculus appears in an essay written around 1685 titled *Ad Vossii Aristarchum* (see A VI.4 622–4).

[10] See A VI.4 1539–41, 1644, and VI.6 396-8.

between terms. Indeed, it is striking that the passage just quoted appears immediately before Leibniz's final axiomatization of his coincidence calculus in §198, in which containment is defined in terms of coincidence rather than the other way around.[11]

Thus, we are left with an exegetical puzzle. Given Leibniz's unwavering commitment to the containment theory of truth, why does he choose to treat, not containment, but coincidence as the sole primitive relation between terms in his mature logical writings? If, as Leibniz insists, every proposition expresses a containment between terms, why does he nonetheless maintain that 'it will be best to reduce propositions from predication and from being-in-something to coincidence'?[12]

If containment and coincidence were straightforwardly interdefinable in the object language of Leibniz's calculus, the choice to treat the latter relation as primitive would be a mere notational preference of only marginal interest. But this is not the case. For, while containment is readily definable in the language of Leibniz's coincidence calculus by means of the formula *A coincides with AB*, there is no obvious way of defining coincidence in the language of Leibniz's containment calculus. Now, this is not to say that coincidence cannot be characterized in terms of containment. Indeed, as we have seen, Leibniz characterizes coincidence as mutual containment, appealing to the law that *A coincides with B* just in case both *A is B* and *B is A*.[13] This law of antisymmetry, however, does not constitute a definition of coincidence in the object language of Leibniz's containment calculus. For, crucially, this language does not include any primitive propositional operators for forming complex propositions from simpler ones. In particular, the language does not include any primitive operators for forming conditionals, disjunctions, or conjunctions.[14] Consequently, while both *A is B* and *B is A* are well-formed propositions of the language of Leibniz's containment calculus, this language does not include any primitive operator '&' by which to form their conjunction, (*A is B*) & (*B is A*).[15] Absent such a conjunction operator, or any other propositional operators that might be used to define conjunction, Leibniz's characterization of coincidence as mutual containment is not directly expressible in the object language of his containment calculus. Hence, there is no obvious way for Leibniz to provide an explicit definition of coincidence in terms of containment.

There is, however, an alternative, more indirect way for Leibniz to reduce the logic of coincidence to that of containment without exceeding the syntactic bounds

[11]See *GI* §198.9.

[12]A VI.4 622.

[13]*GI* §30, §88, and §195; see also A VI.4 813.

[14]In some of his earlier essays, Leibniz seems to employ an object language which includes primitive propositional operators for implication and conjunction (see, e.g., A VI.4 127–31, 142–3, 146–50). In his mature logical writings, however, Leibniz does not include such propositional operators in the object language of his calculi (see, e.g., the coincidence calculi formulated at *GI* §§198–200, A VI.4 816–22, 830–55). In the *Generales inquisitiones*, for example, Leibniz formulates his calculus in an object language in which every proposition is of the form *A coincides with B* (see Castañeda 1976: 483–4, Malink & Vasudevan 2016: 689–96). This decision to abstain from the use of primitive propositional operators is not an arbitrary choice on Leibniz's part, but is an essential part of his broader theoretical ambition to reduce propositional logic to a categorical logic of terms and, in particular, to 'reduce hypothetical to categorical propositions' (A VI.4 992; see also *GI* §75, §137, VI.4 811 n. 6, 862–3). For Leibniz's commitment to the Peripatetic program of reducing propositional to categorical logic, see Barnes 1983: 281 and Malink & Vasudevan 2019.

[15]*Pace* Lenzen (1984: 194–5, 1987: 3), who adopts such a conjunctive definition of coincidence in his reconstruction of Leibniz's containment calculus.

of his calculus. Instead of taking the law of antisymmetry to provide an explicit definition of coincidence, one can simply appeal to the fact that this law licenses the following introduction and elimination rules for coincidence:

$$\frac{A \textit{ is } B \qquad B \textit{ is } A}{A \textit{ coincides with } B}$$

$$\frac{A \textit{ coincides with } B}{A \textit{ is } B} \qquad\qquad \frac{A \textit{ coincides with } B}{B \textit{ is } A}$$

If the object language of a containment calculus in which every proposition is of the form *A is B* is extended so as to include new primitive propositions of the form *A coincides with B*, these introduction and elimination rules would allow us to derive laws of coincidence from the underlying laws of containment.[16] In this way, coincidence could be viewed as a purely nominal relation the meaning of which is determined by its introduction and elimination rules in the calculus.[17] Thus, by adopting this approach, Leibniz would be able to integrate coincidence into his containment calculus without thereby abandoning his commitment to the logical and metaphysical primacy of conceptual containment.

In what follows, we show that, by means of this approach, all the laws of Leibniz's coincidence calculus can be derived from the laws of his containment calculus. We thus argue that the shifting emphasis on containment and coincidence that is manifest in Leibniz's logical writings does not betray any ambivalence or incoherence on his part. Instead, Leibniz's logic of conceptual containment and his logic of coincidence can be viewed as two alternative axiomatizations of a single, unified logical theory. To show this, we will gradually build up a version of Leibniz's containment calculus that is strong enough to derive all the principles of the mature coincidence calculus developed in the *Generales inquisitiones*. We begin by considering the most characteristic principle of Leibniz's coincidence calculus, namely, the rule of substitution.

[16]This strategy for deriving laws of coincidence from underlying laws of containment is adopted, for example, by Schröder (1890: 184–5). There are, in fact, a number of striking parallels between Leibniz's containment calculus and Schröder's algebraic system of logic (see nn. 24, 34, and 38 below).

[17]This presupposes that the introduction and elimination rules for coincidence are conservative in the sense that that they do not allow us to derive any new theorems of the form *A is B* not already derivable in the underlying containment calculus. For, if these rules were not conservative, the newly introduced concept of coincidence would have, as Brandom puts it, 'substantive content . . . that is not already implicit in the contents of the other concepts being employed' (Brandom 1994: 127, 2000: 71). If, on the other hand, coincidence is introduced by means of conservative introduction and elimination rules, then 'no new content is really involved' (Brandom 1994: 127, 2000: 71). In this way, the connective of coincidence can be introduced into a pure containment calculus without the need to rely on 'an antecedent idea of the independent meaning of the connective' (Belnap 1962: 134; see also Dummett 1991: 247). This condition of conservativity is satisfied for all the versions of Leibniz's containment calculus discussed in this paper.

2. The Rule of Substitution

The first principle listed by Leibniz in his final axiomatization of the coincidence calculus presented in the *Generales inquisitiones* is the rule of substitution of coincidents. Leibniz formulates this rule as follows:[18]

> 1st. Coincidents can be substituted for one another. (*Generales inquisitiones* §198.1)

This principle licenses the substitution *salva veritate* of one coincident term for another in any proposition of Leibniz's calculus. In the *Generales inquisitiones*, Leibniz writes '$A = B$' to indicate that the terms A and B coincide. Using this notation, the rule of substitution can be formulated as follows, where the turnstile '$\vdash$' indicates derivability in Leibniz's calculus:

> Rule of Substitution: For any terms A and B and any proposition φ, if φ^* is the result of substituting A for an occurrence of B, or vice versa, in φ, then:
>
> $$A = B, \varphi \vdash \varphi^*$$

While this rule of substitution plays a prominent role in Leibniz's coincidence calculus, it does not play any role in the various containment calculi developed in his earlier logical writings. One reason for this is that the language of Leibniz's containment calculi does not include any propositions of the form $A = B$. If, on the other hand, this language were to be extended by adding coincidence as a new primitive relation between terms, the rule of substitution might then be derivable by means of the introduction and elimination rules for coincidence. In order to provide such a derivation of the rule of substitution, it suffices to establish that the underlying containment calculus licenses free substitution under the relation of mutual containment. If we write '$A \supset B$' to indicate that A contains B, the requisite rule of substitution under mutual containment can be formulated as follows:[19]

> Rule of Substitution for Containment: For any terms A, B, C, and D, if $C^* \supset D^*$ is the result of substituting A for an occurrence of B in the proposition $C \supset D$, then:
>
> $$A \supset B, B \supset A, C \supset D \vdash C^* \supset D^*$$

Unlike the rule of substitution of coincidents, this rule of substitution for containment is expressible in the language of Leibniz's containment calculi. Nevertheless, Leibniz does not include the latter rule among the principles of any of these calculi. This is perhaps not surprising since rules for substitutional reasoning appear more natural when formulated for equivalence relations such as coincidence than they

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[18] The rule of substitution also appears as the first principle listed by Leibniz in many subsequent versions of the coincidence calculus (see, e.g., A VI.4 810, 816, 831, 846). For further statements of the rule of substitution, see *GI* §9, §19, A VI.4 626, 746.

[19] It is easy to verify that, given the introduction and elimination rules for coincidence, this rule of substitution for containment implies the general rule of substitution for the extended language of Leibniz's containment calculus, i.e.: $A = B, \varphi \vdash \varphi^*$, where φ is any proposition of the form $C \supset D$ or $C = D$.

do when formulated for non-symmetric relations such as containment. Thus, were Leibniz to posit the above rule of substitution for containment as a principle of his containment calculus, this would seem rather ad hoc and unmotivated.

At the same time, however, Leibniz clearly took the rule of substitution for containment to be valid. In fact, he often observes the validity of this rule in his essays on the logic of containment in the late 1670s. Crucially, however, in these early essays Leibniz does not simply posit substitution for containment as a primitive rule of inference, but instead presents it as a theorem to be derived from more fundamental principles of containment.[20] For example, in his essay *Elementa ad calculum condendum*, written around 1679, Leibniz derives the rule of substitution for containment as follows:

> There is only one fundamental inference: A *is* B and B *is* C, therefore A *is* C. Hence, if A *is* B and B *is* A, the one can be substituted in place of the other *salva veritate*. Take the proposition B *is* C; it will then be possible to substitute A *is* C, since A *is* B. Next, take the proposition D *is* B; it will then be possible to substitute D *is* A, since B *is* A. Evidently, since A *is* B, A can be substituted in place of a subject B. But also since B *is* A, it will be possible to substitute A in place of a predicate B. (*Elementa ad calculum condendum*, A VI.4 154)

In the first sentence of this passage, Leibniz states the following law of containment:

$$A \supset B, B \supset C \vdash A \supset C$$

This is the most basic principle of Leibniz's logic of containment, and it appears as a primitive rule of inference in all the versions of his containment calculi.[21] Since Leibniz identifies propositions of the form $A \supset B$ with the universal affirmative propositions appearing in Aristotelian syllogisms, this rule amounts to the traditional syllogistic mood Barbara.[22] In the above passage, Leibniz appeals to the rule of Barbara to show that, if $A \supset B$ and $B \supset A$, then either of the terms A and B 'can be substituted in place of the other *salva veritate*'. His proof of this claim proceeds by considering two cases: first, the case in which B occurs as the subject term of a containment proposition, as in $B \supset C$; and, second, the case in which B occurs as the predicate term, as in $D \supset B$. In each of these cases, Leibniz justifies the substitution of A for B by an application of Barbara, using the premises $A \supset B$ and $B \supset A$, respectively. Obviously, the same argument shows that B can be substituted for A when the latter appears as either the subject or the predicate term of a containment proposition.[23]

It is clear from this argument that Leibniz aims to justify the rule of substitution for containment by appeal to what he takes to be the more fundamental principle of Barbara.[24] In fact, the rule of Barbara can itself be viewed as a partial rule of substitution whereby the subject term of a proposition $A \supset B$ can be replaced by

[20]See, e.g., A VI.4 154, 275, 284, 294.

[21]See, e.g., A VI.4 144, 154, 275, 281, 293, 497, 551.

[22]See, e.g., A VI.4 280–1, *GI* §47, §124, §129, §§190–1.

[23]See A VI.4 282.

[24]For a similar proof of the rule of substitution based on Barbara in a containment calculus, see Schröder 1890: 186.

any term that contains it, and the predicate term by any term it contains. For this reason, Leibniz refers to Barbara as a rule of 'one-sided substitution' (*substitutio unilateralis*).[25] Thus, in Leibniz's containment calculi, Barbara takes the place of substitution as the most prominent and characteristic rule of inference. This is diametrically opposed to the approach adopted by Leibniz in his mature logical writings, in which he derives Barbara as a theorem in his coincidence calculus by means of a straightforward substitutional proof.[26]

In the passage just quoted, Leibniz establishes the rule of substitution for the specific case in which the substituted term occurs as either the subject or the predicate of the containment proposition in which the substitution takes place. If these were the only two syntactic contexts in which a term could occur in the language of Leibniz's calculus, the proof given by Leibniz in the passage based on the rule of Barbara would succeed in establishing the rule of substitution for containment in full generality. In richer syntactic settings, however, in which these are not the only two contexts in which a term can occur in a proposition, Barbara alone does not suffice to establish the rule of substitution. This is because the latter rule, unlike Barbara, is sensitive to the variety of syntactic contexts in which a term may occur.

To illustrate this last point, consider the syntax of the containment calculus developed by Leibniz in the *Elementa ad calculum condendum*. While the only propositions in the language of this calculus are those of the form $A \supset B$, Leibniz allows the terms A and B to be syntactically complex. In particular, he admits complex terms which are constructed out of simpler ones by the logical operation of composition. In the language of Leibniz's calculus, composition is expressed by concatenating the expressions signifying the terms to be composed. For example, the complex term *rational animal* is the result of composing the simpler terms *rational* and *animal*. More generally, if A and B are terms, then AB is the term which results from composing them. One of the basic laws posited by Leibniz governing the logic of such composite terms is $AB \supset A$.[27] In a proposition of this form, the term B does not occur as either the subject or the predicate of the proposition, but rather as a component of the composite subject term AB. Hence, Leibniz's proof of substitution based on the rule of Barbara does not license substitutions for the term B in the proposition $AB \supset A$. For example, given the premises $B \supset C$ and $C \supset B$, Barbara alone does not allow us to infer $AC \supset A$ from $AB \supset A$. In order to justify such substitutions for components of composite terms, one must appeal to additional principles pertaining to composite terms in the containment calculus. In sum, Leibniz's proof of substitution based on the rule of Barbara is only complete for a simple language of containment in which terms have no syntactic complexity in the sense that no term can occur as a constituent of

[25] A VI.4 809; cf. A VI.4 143–5, 154, 275, 672.

[26] In his essay *Principia calculi rationalis*, for example, Leibniz provides the following substitutional proof of Barbara based on his definition of $A \supset B$ as $A = AB$ (C 229–30). Suppose $A = AB$ and $B = BC$ (i.e., $A \supset B$ and $B \supset C$). Substituting BC for B in the former proposition yields $A = ABC$. Substituting AB for A in this last proposition yields $A = AC$ (i.e., $A \supset C$). For similar substitutional proofs of Barbara, see *GI* §19 and A VI.4 813. A substitutional proof of Barbara along these lines is also given by Jevons (1869: 29–30).

[27] In the *Elementa ad calculum condendum*, this law is stated at A VI.4 154. For other statements of this law, see, e.g., A VI.4 148, 149, 150, 151, 274, 280, 292, 813.

another. For richer languages in which terms do exhibit such syntactic complexity, however, this proof fails to establish the rule of substitution in full generality.

These considerations illustrate the fact that, as the syntactic complexity of terms increases, the rule of substitution undergoes a corresponding increase in deductive power, since it thereby comes to license substitutions into a wider variety of syntactic contexts. Hence, in order to derive the rule of substitution for containment in increasingly complex syntactic settings, the underlying logic of containment needs to be correspondingly stronger. Exactly which additional principles are needed will depend on the specific kind of syntactic complexity exhibited by the terms of the language. In the *Elementa ad calculum condendum*, the only syntactic operation for forming complex terms is that of composition; in his mature logical writings, however, Leibniz posits additional syntactic operations for forming complex terms. Specifically, the language of the calculus developed by Leibniz in the *Generales inquisitiones* includes the following three operations for forming complex terms:

> Composition: If A and B are terms, then AB is a composite term. For example, *rational animal* is the composite of *rational* and *animal*.
>
> Privation: If A is a term, then *non-A* is a privative term. For example, *non-animal* is the privative of *animal*.
>
> Propositional terms: If φ is a proposition, then $\ulcorner\varphi\urcorner$ is a propositional term. For example, ⌜*Man is animal*⌝ is the propositional term generated from the proposition *Man is animal*.

In what follows, we discuss each of these three operations in turn.

3. Composition in the Containment Calculus

In a series of essays written in the late 1670s, Leibniz explores the logic of containment for a language which includes composite terms of the form AB. The syntax of the language employed by Leibniz in these essays can be characterized as follows:

Definition 1. Given a nonempty set of primitive expressions referred to as simple terms, the terms and propositions of the language $\mathcal{L}_c$ are defined as follows:

(1) Every simple term is a term.
(2) If A and B are terms, then AB is a term.
(3) If A and B are terms, then $A \supset B$ is a proposition.

Given this definition of the language $\mathcal{L}_c$, what logical principles regarding composite terms are needed to derive the rule of substitution for containment? While Leibniz does not explicitly address this question, he does describe a calculus for the language $\mathcal{L}_c$ which is strong enough to derive the rule of substitution. This containment calculus is presented in an essay titled *De calculo analytico generali*, written around 1679. In the concluding sections of the essay, Leibniz enumerates the principles of the calculus as follows:

> Axioms:
> (1) *Every A is A*
> (2) *Every AB is A*

(3) If A *is* B, then A *is* B.
If A *is* B and B *is* C, then A *is* B. . . .
(4) If A *is* B and *Every* B *is* C, then A *is* C. . . .
(7) If A *is* B and the same A *is* C, then the same A *is* BC.[28]

(*De calculo analytico generali*, A VI.4 149–50)

While the principles stated by Leibniz in (1), (2), (4), and (7) pertain specifically to the relation of containment, the two principles stated in (3) describe more general, structural features of the relation of derivability in the calculus. The first of these two principles asserts that the relation of derivability is reflexive, i.e.: $\varphi \vdash \varphi$. The second asserts a rule of weakening to the effect that $\varphi, \psi \vdash \varphi$.[29] Throughout his logical writings, Leibniz also tacitly assumes the structural rule of cut, which licenses the construction of complex derivations through the consecutive application of rules of inference.[30] Given the structural rules of cut and weakening, it follows that the derivability relation is monotonic.[31] Hence, Leibniz's containment calculus obeys the standard structural rules of reflexivity, monotonicity, and cut that are typically assumed in the definition of a calculus in modern logic:

Definition 2. A calculus in a language $\mathcal{L}$ is a relation $\vdash$ between sets of propositions of $\mathcal{L}$ and propositions of $\mathcal{L}$, such that:

Reflexivity:	$\{\varphi\} \vdash \varphi$
Monotonicity:	If $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$.
Cut:	If $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Delta \vdash \varphi$, then $\Gamma \cup \Delta \vdash \psi$.

Here, $\Gamma \vdash \varphi$ indicates that the set of propositions Γ stands in the relation $\vdash$ to the proposition φ. In what follows, we use $\vdash \varphi$ as shorthand for $\emptyset \vdash \varphi$, and $\psi_1, \dots, \psi_n \vdash \varphi$ as shorthand for $\{\psi_1, \dots, \psi_n\} \vdash \varphi$.

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[28]We have omitted items (5) and (6) from this list since they do not contain principles of the calculus, but theorems which Leibniz derives from the principles stated in (1)–(4).

[29]In item (3), Leibniz asserts only a restricted version of weakening in which the propositions φ and ψ have a term in common. This restriction, however, is likely inadvertent, since Leibniz asserts the rule of weakening in full generality immediately before the list of principles quoted above: 'A *is* B and C *is* D, therefore A *is* B' (A VI.4 149).

[30]This is clear, for example, from the way in which Leibniz derives the theorem $A = B, B = C, C = D \vdash A = D$ in his coincidence calculus by two consecutive applications of $A = B, B = C \vdash A = C$ (A VI.4 831). In this derivation, Leibniz employs the following instance of cut: if $A = C, C = D \vdash A = D$ and $A = B, B = C \vdash A = C$, then $A = B, B = C, C = D \vdash A = D$. See also the application of the rule of cut at A VI.4 837.

[31]Monotonicity asserts that, if $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$. The structural rules of cut and weakening imply this principle for the case in which Γ and Δ are finite sets of propositions. To see this, we first establish monotonicity for the case in which Γ is empty and Δ is the singleton set $\{\psi\}$, i.e.: if $\vdash \varphi$, then $\psi \vdash \varphi$. By weakening, we have $\varphi, \psi \vdash \varphi$. Hence, given $\vdash \varphi$, it follows by cut that $\psi \vdash \varphi$. Next, we establish monotonicity for the case in which Γ is non-empty and Δ is the singleton set $\{\psi\}$, i.e.: if $\Gamma \vdash \varphi$ and $\Gamma \neq \emptyset$, then $\Gamma \cup \{\psi\} \vdash \varphi$. Let χ be any proposition in Γ. By weakening, we have $\chi, \psi \vdash \chi$. Hence, given $\Gamma \vdash \varphi$, it follows by cut that $\Gamma \cup \{\psi\} \vdash \varphi$. The finite version of monotonicity, in which Γ and Δ are any finite sets of propositions, follows immediately from these two cases. Thus, Leibniz's containment calculus is finitely monotonic. For the sake of simplicity, we will also assume that the calculus is monotonic under the addition of infinitely many premises, although this assumption plays no essential role in what follows and can be freely omitted.

In addition to the structural principles stated by Leibniz in (3), his list of axioms includes a number of principles pertaining specifically to containment and composition. Taken together, these principles define the following containment calculus, $\vdash_c$:

Definition 3. The calculus $\vdash_c$ is the smallest calculus in the language $\mathcal{L}_c$ such that:

(C1) $\vdash_c A \supset A$

(C2) $\vdash_c AB \supset A$

(C3) $\vdash_c AB \supset B$

(C4) $A \supset B, B \supset C \vdash_c A \supset C$

(C5) $A \supset B, A \supset C \vdash_c A \supset BC$

Of these principles, C1, C2, C4, and C5 are explicitly stated by Leibniz in items (1), (2), (4), and (7) of his list of axioms.[32] While C3 does not appear in this list, Leibniz does take this principle for granted in his calculus.[33] For example, he appeals to C3 in his proof of the main theorem established in the *De calculo analytico generali*, which appears in item (8) of Leibniz's list:

> (8) If *A is C* and *B is D*, then *AB is CD*. (*De calculo analytico generali*, A VI.4 150)

Leibniz's proof of this theorem proceeds as follows:

Theorem 4. $A \supset C, B \supset D \vdash_c AB \supset CD$

Proof. By C2, $\vdash_c AB \supset A$. Hence, by C4, $A \supset C \vdash_c AB \supset C$. Now, by C3, $\vdash_c AB \supset B$. Hence, by C4, $B \supset D \vdash_c AB \supset D$. Thus, by C5, we have $A \supset C, B \supset D \vdash_c AB \supset CD$. □

This theorem can be used as a lemma to prove the rule of substitution for containment in the calculus $\vdash_c$. In particular, the theorem allows us to establish the following preliminary to the rule of substitution:

Theorem 5. *For any terms A, B, C of $\mathcal{L}_c$, if C^* is the result of substituting B for an occurrence of A in C, then both:*

$$A \supset B, B \supset A \vdash_c C \supset C^*$$
$$A \supset B, B \supset A \vdash_c C^* \supset C$$

Proof. First, suppose that A is the term C. Then $C \supset C^*$ is the proposition $A \supset B$, and $C^* \supset C$ is the proposition $B \supset A$. Hence, the claim follows by the reflexivity and monotonicity of $\vdash_c$.

Now, suppose that A is a proper subterm of C. Then C is a term of the form DE, where either D or E contains the occurrence of A for which B is substituted to obtain C^*. First, suppose that D contains this occurrence of A. Then C^* is the

[32] In his formulation of these principles, Leibniz takes the phrases '*A is B*' and '*Every A is B*' to be equivalent (see n. 22). This can be seen from the proof given by Leibniz in item (6) of his list, where he takes *Every AB is A* to entail *AB is A*, and, conversely, takes *AB is C* to entail *Every AB is C* (A VI.4 150).

[33] Moreover, C3 is explicitly stated by Leibniz at A VI.4 274–5, 281, *GI* §38, §39, §46.

term D^*E, where D^* is the result of substituting B for an occurrence of A in D. For induction, we assume that the claim holds for substitutions in D, so that both:

$$A \supset B, B \supset A \vdash_c D \supset D^*$$
$$A \supset B, B \supset A \vdash_c D^* \supset D$$

Since, by C1, $\vdash_c E \supset E$, it follows, by Theorem 4, that both:

$$A \supset B, B \supset A \vdash_c DE \supset D^*E$$
$$A \supset B, B \supset A \vdash_c D^*E \supset DE$$

The same argument applies to the case in which E contains the occurrence of A for which B is substituted. This completes the proof. □

Given this theorem, the rule of substitution for containment in the calculus $\vdash_c$ can be established as follows:[34]

Theorem 6. *For any terms A, B, C, D of $\mathcal{L}_c$, if $C^* \supset D^*$ is the result of substituting A for an occurrence of B in the proposition $C \supset D$, then:*

$$A \supset B, B \supset A, C \supset D \vdash_c C^* \supset D^*$$

Proof. Since $C^* \supset D^*$ is the result of replacing a single occurrence of A by B in $C \supset D$, either C^* is identical with C or D^* is identical with D. In the first case, by Theorem 5, we have $A \supset B, B \supset A \vdash_c D \supset D^*$, and so, by C4, it follows that $A \supset B, B \supset A, C \supset D \vdash_c C \supset D^*$. In the second case, by Theorem 5, we have $A \supset B, B \supset A \vdash_c C^* \supset C$, and so, by C4, it follows that $A \supset B, B \supset A, C \supset D \vdash_c C^* \supset D$. □

Thus, the containment calculus $\vdash_c$ developed by Leibniz in the *De calculo analytico generali* is strong enough to derive the rule of substitution for containment in the language $\mathcal{L}_c$. Consequently, if this language is extended by adding a new primitive relation symbol '=' governed by the introduction and elimination rules for coincidence, the resulting calculus is strong enough to validate the general rule of substitution for coincidence. In other words, given $A = B$, the term B can be substituted for A, or vice versa, *salva veritate* in any proposition of the extended language.

The operation of composition axiomatized by the calculus $\vdash_c$ gives rise to an algebra of terms that allows for free substitution under the relation of mutual containment. It can be shown that the specific type of algebraic structure determined by the principles of the calculus is that of a semilattice. More precisely, the calculus $\vdash_c$ is sound and complete with respect to the class of semilattices, when composition is interpreted as the meet operation and containment as the order relation in the

[34] In fact, in order to establish Theorem 6, it is not necessary to appeal to Theorem 4 but only to the following weaker version of this theorem:

$$A \supset C, C \supset A, B \supset D, D \supset B \vdash_c AB \supset CD$$

Given the introduction and elimination rules for coincidence, this is equivalent to the following law asserting that coincidence is a congruence relation with respect to the operation of composition:

$$A = C, B = D \vdash_c AB = CD$$

Schröder (1890: 270) appeals to this latter law to establish a variant of Theorem 6 in his containment calculus.

algebra.[35] Thus, the containment calculus developed by Leibniz in the *De calculo analytico generali* constitutes an axiomatization of the theory of semilattices.

4. Privation in the Containment Calculus

In addition to composite terms, the calculus developed by Leibniz in the *Generales inquisitiones* also includes privative terms. In particular, Leibniz assumes that, for any term A, there is a corresponding privative term *non-A*. In what follows, we write '$\overline{A}$' to designate the privative term *non-A*.

Privative terms are largely absent from Leibniz's writings on the logic of containment in the late 1670s.[36] For example, in the *De calculo analytico generali* Leibniz deliberately excludes privative terms from consideration on the grounds that doing so will allow him to 'set aside many perplexities' (*multas perplexitates abscindemus*).[37] One complication that is raised by the introduction of privative terms into the language of Leibniz's containment calculus is that they create new syntactic contexts into which terms can be substituted. Thus, in order to validate the rule of substitution, the underlying containment calculus must now include additional principles that license the substitution of one mutually containing term for another in these new syntactic contexts. For example, if A and B contain one another, the principles of the calculus must license the substitution of B for A in propositions such as $\overline{A} \supset C$. As it turns out, the only additional principle that is needed to license such substitutions is the following law of contraposition:

$$A \supset B \vdash \overline{B} \supset \overline{A}$$

To show that this law suffices to derive the rule of substitution in a language that includes not only composite but also privative terms, we first specify the syntax of such an enriched language, $\mathcal{L}_{\mathrm{cp}}$:

Definition 7. Given a nonempty set of primitive expressions referred to as simple terms, the terms and propositions of the language $\mathcal{L}_{\mathrm{cp}}$ are defined as follows:

(1) Every simple term is a term.
(2) If A and B are terms, then AB is a term.
(3) If A is a term, then $\overline{A}$ is a term.
(4) If A and B are terms, then $A \supset B$ is a proposition.

In order to establish the rule of substitution in the language $\mathcal{L}_{\mathrm{cp}}$, the key step is to extend Theorem 5 to cover the case of privative terms. To this end, let $\vdash_{\mathrm{cp}}$ be a calculus in $\mathcal{L}_{\mathrm{cp}}$ obeying all the principles of $\vdash_{\mathrm{c}}$ as well as the above law of contraposition. The proof of the relevant theorem then proceeds as follows:

Theorem 8. *For any terms A, B, C of $\mathcal{L}_{\mathrm{cp}}$, if C^* is the result of substituting B for an occurrence of A in C, then both:*

$$A \supset B, B \supset A \vdash_{\mathrm{cp}} C \supset C^*$$
$$A \supset B, B \supset A \vdash_{\mathrm{cp}} C^* \supset C$$

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[35] For a proof of this claim, see Theorem 20 in the appendix below. Swoyer (1994: 14–20 and 29–30) establishes a similar completeness result for the coincidence calculus developed by Leibniz in the *Specimen calculi coincidentium et inexistentium*, written around 1686 (A VI.4 830–45).

[36] Leibniz does on a few separate occasions discuss privative terms in his earlier logical writings (e.g., A VI.4 218, 224, 253, 292–7, 622). The first systematic treatment of these terms, however, appears in the *Generales inquisitiones*.

[37] A VI.4 146.

Proof. First, suppose that A is the term C. Then $C \supset C^*$ is the proposition $A \supset B$, and $C^* \supset C$ is the proposition $B \supset A$. Hence, the claim follows from the reflexivity and monotonicity of $\vdash_{\text{cp}}$.

Now, suppose that A is a proper subterm of C. Then either (i) C is a term of the form DE, where either D or E contains the occurrence of A for which B is substituted to obtain C^*; or (ii) C is a term of the form $\overline{D}$, where D contains the occurrence of A for which B is substituted to obtain C^*. Case (i) is treated in exactly the same way as in the proof of Theorem 5.

In case (ii), C^* is the term $\overline{D^*}$, where D^* is the result of substituting B for an occurrence of A in D. For induction, we assume that the claim holds for substitutions in D, so that both:

$$A \supset B, B \supset A \vdash_{\text{cp}} D \supset D^*$$
$$A \supset B, B \supset A \vdash_{\text{cp}} D^* \supset D$$

By contraposition, it follows that both:

$$A \supset B, B \supset A \vdash_{\text{cp}} \overline{D^*} \supset \overline{D}$$
$$A \supset B, B \supset A \vdash_{\text{cp}} \overline{D} \supset \overline{D^*}$$

This completes the proof. □

Given this theorem, the rule of substitution for the calculus $\vdash_{\text{cp}}$ follows straightforwardly:

Theorem 9. *For any terms A, B, C, D of $\mathcal{L}_{\text{cp}}$, if $C^* \supset D^*$ is the result of substituting A for an occurrence of B in the proposition $C \supset D$, then:*

$$A \supset B, B \supset A, C \supset D \vdash_{\text{cp}} C^* \supset D^*$$

Proof. The proof is the same as that given for Theorem 6. □

Thus, provided that the containment calculus $\vdash_{\text{cp}}$ validates the law of contraposition, it licenses free substitution under mutual containment into any syntactic context of the language $\mathcal{L}_{\text{cp}}$.[38] Consequently, just as before, if the language $\mathcal{L}_{\text{cp}}$ is extended by adding a new primitive relation symbol '=' governed by the introduction and elimination rules for coincidence, the resulting calculus is strong enough to validate the general rule of substitution for coincidence.

It is clear that Leibniz endorses the law of contraposition. He affirms this law on numerous occasions throughout his logical writings.[39] For example, in the *Generales inquisitiones* he writes:

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[38]In fact, in order to establish Theorem 9, it is not necessary to appeal to the law of contraposition but only to the following weaker version of this law:

$$A \supset B, B \supset A \vdash_{\text{cp}} \overline{B} \supset \overline{A}$$

Given the introduction and elimination rules for coincidence, this is equivalent to the following law asserting that coincidence is a congruence relation with respect to the operation of privation:

$$A = B \vdash_{\text{cp}} \overline{A} = \overline{B}$$

Schröder (1890: 306) appeals to this latter law to establish a variant of Theorem 9 in his containment calculus.

[39]See, e.g., *GI* §77, §95, §189.5, §200, A VI.4 224, 813, C 422. In some cases, Leibniz does not posit the law of contraposition as a principle of his calculus, but instead undertakes to derive it from more basic principles (see Lenzen 1986: 13–14 and 27–32, 1988: 63–4).

If *A is B*, then *non-B is non-A*. (*Generales inquisitiones* §93)

In addition to contraposition, Leibniz posits a number of other principles pertaining to privative terms in his mature logical writings. One of these principles is a law of double privation to the effect that a term coincides with the privative of its privative.[40] In the language $\mathcal{L}_{\text{cp}}$, this law is captured by the following two principles:

$$\vdash_{\text{cp}} \overline{\overline{A}} \supset A \qquad\qquad \vdash_{\text{cp}} A \supset \overline{\overline{A}}$$

A further principle pertaining to privative terms that plays an important role in the *Generales inquisitiones* is the following:

> If I say *AB is not*, it is the same as if I were to say *A contains non-B*. (*Generales inquisitiones* §200)

In this passage, Leibniz asserts that the proposition $A \supset \overline{B}$ is equivalent to a proposition that he expresses by the phrase '*AB is not*'. He takes this latter proposition to assert that the composite term AB is a 'non-being' (*non-ens*) or, equivalently, that the term AB is 'false' (*falsum*).[41] In what follows, we write '$\mathbf{F}(A)$' for the proposition expressing that the term A is false. In this notation, the principle stated by Leibniz in the passage just quoted asserts that the propositions $\mathbf{F}(AB)$ and $A \supset \overline{B}$ are equivalent, i.e.:[42]

$$\mathbf{F}(AB) \dashv\vdash A \supset \overline{B}$$

Now, this equivalence is not directly expressible in the language $\mathcal{L}_{\text{cp}}$, since this language does not include any primitive means for expressing propositions of the form $\mathbf{F}(A)$. Nevertheless, Leibniz gives some indication as to how these propositions can be expressed in $\mathcal{L}_{\text{cp}}$ when he asserts that a false term is simply one which 'contains a contradiction' (*Generales inquisitiones* §57). Thus he writes:

> That term is false which contains opposite terms, *A non-A*. (*Generales inquisitiones* §194; similarly, §198.4)

> A proposition is that which states what term is or is not contained in another. Hence, a proposition can also affirm that a term is false, if it says that *Y non-Y* is contained in it. (*Generales inquisitiones* §195)

According to these passages, a term A is false just in case it contains some contradictory term of the form $B\overline{B}$. Note that this characterization of falsehood involves an existential quantification over contradictory terms. Since the language $\mathcal{L}_{\text{cp}}$ does

[40]See, e.g., *GI* §2, §96, §171.4, §189.2, §198.3, A VI.4 218, 624, 740, 807, 811, 814, 877, 931, 935, 939, C 230, 235, 421.

[41]See *GI* §55, §193–4, §197, A VI.4 774 n. 47, 807, 813. In the *Generales inquisitiones*, Leibniz uses a number of phrases interchangeably to expresses the non-being, or falsehood, of a given term *A*, e.g.: '*A is not*' (*GI* §§149–50, §§199–200), '*A is not a thing*' (*GI* §§149–55, §171.8), '*A is a non-being*' (*GI* §32b, §55, A VI.4 810 n. 5, 813, 875, 930, 935), '*A is false*' (*GI* §55, §§58–9, §189.3, §§193–5, §197, §198.4, A VI.4 939), '*A is impossible*' (*GI* §32b, §§33–4, §55, §128, A VI.4 749 n. 11, 807, 875, 930, 935, 939).

[42]For similar statements of this equivalence, see A VI.4 862–3 and C 237.

not include any means of expressing existential quantification, Leibniz's characterization of falsehood is not directly expressible in this language.[43] Nonetheless, it can be shown to entail an explicit definition of falsehood in the language $\mathcal{L}_{\text{cp}}$, given the principles of the calculus $\vdash_{\text{cp}}$ stated so far. This is because these principles entail that a term A contains some contradictory term of the form $B\overline{B}$ just in case it contains its own privative, $\overline{A}$. This is an immediate consequence of the following two theorems:

Theorem 10. $A \supset \overline{A} \vdash_{\text{cp}} A \supset A\overline{A}$

Proof. This follows by C1 and C5. □

Theorem 11. $A \supset B\overline{B} \vdash_{\text{cp}} A \supset \overline{A}$

Proof. By C2 and C4, we have $A \supset B\overline{B} \vdash_{\text{cp}} A \supset B$. So, by contraposition:

$$A \supset B\overline{B} \vdash_{\text{cp}} \overline{B} \supset \overline{A}$$

But, by C3 and C4:

$$A \supset B\overline{B} \vdash_{\text{cp}} A \supset \overline{B}$$

Hence, by C4, $A \supset B\overline{B} \vdash_{\text{cp}} A \supset \overline{A}$. □

Since, according to Leibniz's characterization of falsehood, a false term is one which contains some contradictory term of the form $B\overline{B}$, these two theorems yield the following definition of falsehood:

$$\mathbf{F}(A) \dashv\vdash A \supset \overline{A}$$

With this definition in hand, the principle stated by Leibniz in §200 of the *Generales inquisitiones* can now be formulated in the language $\mathcal{L}_{\text{cp}}$. This principle, recall, asserts that $\mathbf{F}(AB)$ is equivalent to $A \supset \overline{B}$. Since $\mathbf{F}(AB)$ is definable as $AB \supset \overline{AB}$, this principle can be formulated in $\mathcal{L}_{\text{cp}}$ as follows:

$$AB \supset \overline{AB} \dashv\vdash_{\text{cp}} A \supset \overline{B}$$

All told, then, the containment calculus $\vdash_{\text{cp}}$ can be obtained from $\vdash_{\text{c}}$ by adding to it the law of contraposition, double privation, and Leibniz's principle asserting the equivalence of $\mathbf{F}(AB)$ and $A \supset \overline{B}$. As it turns out, the right-to-left direction of this equivalence is already derivable from contraposition and the principles of $\vdash_{\text{c}}$.[44] Moreover, given this equivalence, one direction of the principle of double privation, $A \supset \overline{\overline{A}}$, is already provable in the calculus.[45] Thus, without any loss of deductive power, the calculus $\vdash_{\text{cp}}$ can be defined as follows:

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[43] In the *Generales inquisitiones*, Leibniz explores the possibility of expressing existential quantification in the language of his calculus by means of what he calls 'indefinite letters' (§§16–31; see Lenzen 1984*b*: 7–13, 2004: 47–50, Hailperin 2004: 329). However, the introduction of indefinite letters into the calculus gives rise to complications analogous to those which arise in connection with the elimination of existential quantifiers in modern quantificational logic (see, e.g., *GI* §§21–31). Leibniz was aware of some of these complications and was never entirely satisfied with the use of indefinite letters, indicating that it would be preferable to omit them from the language of his calculus (see *GI* §162 in conjunction with §128 and A VI.4 766 n. 35; cf. Schupp 1993: 153, 168, 182–3). In the absence of indefinite letters, the language of Leibniz's calculus lacks the resources to express existential quantification.

[44] See the proof of Theorem 21.i in the appendix.

[45] See Theorem 21.iii in the appendix.

Definition 12. The calculus $\vdash_{\text{cp}}$ is the smallest calculus in the language $\mathcal{L}_{\text{cp}}$ such that:

(C1) $\vdash_{\text{cp}} A \supset A$

(C2) $\vdash_{\text{cp}} AB \supset A$

(C3) $\vdash_{\text{cp}} AB \supset B$

(C4) $A \supset B, B \supset C \vdash_{\text{cp}} A \supset C$

(C5) $A \supset B, A \supset C \vdash_{\text{cp}} A \supset BC$

(CP1) $A \supset B \vdash_{\text{cp}} \overline{B} \supset \overline{A}$

(CP2) $\vdash_{\text{cp}} \overline{\overline{A}} \supset A$

(CP3) $AB \supset \overline{AB} \vdash_{\text{cp}} A \supset \overline{B}$

It should be acknowledged that Leibniz does not formulate this definition of the containment calculus $\vdash_{\text{cp}}$ in any of his logical writings. Nonetheless, this calculus captures all the laws of conceptual containment endorsed by Leibniz in the *Generales inquisitiones* that pertain to the operations of composition and privation.

These two operations, as axiomatized by the calculus $\vdash_{\text{cp}}$, give rise to an algebra of terms that allows for free substitution under the relation of mutual containment. It can be shown that the specific type of algebraic structure determined by the principles of the calculus is that of a Boolean algebra. More precisely, the calculus $\vdash_{\text{cp}}$ is sound and complete with respect to the class of Boolean algebras, when composition is interpreted as the meet operation, privation as the complement operation, and containment as the order relation in the algebra.[46] Thus, the containment calculus $\vdash_{\text{cp}}$ constitutes an axiomatization of the theory of Boolean algebras.

5. Propositional Terms in the Containment Calculus

In addition to composition and privation, the language of the calculus developed by Leibniz in the *Generales inquisitiones* includes a third operation for forming complex terms. By means of this operation, a new term can be generated from any given proposition, in accordance with Leibniz's view that 'every proposition can be conceived of as a term'.[47] Leibniz characterizes this operation as follows:

> If the proposition *A is B* is considered as a term, as we have explained that it can be, there arises an abstract term, namely *A's being B*. And if from the proposition *A is B* the proposition *C is D* follows, then from this there comes about a new proposition of the following kind: *A's being B is* (or *contains*) *C's being D*; or, in other words, *the B-ness of A contains the D-ness of C*, or *the B-ness of A is the D-ness of C*. (*Generales inquisitiones* §138)

[46]For a proof of this claim, see Theorem 25 in the appendix. The Boolean completeness of Leibniz's containment calculus was first established by Lenzen (1984*a*: 200–2), albeit with respect to a somewhat different axiomatization of the calculus than that given in Definition 12.

[47]See *GI* §75, §109, §197. When a proposition is conceived of as a term, Leibniz describes the proposition as giving rise to a 'new term' (*terminus novus*, §197 and §198.7; *contra* Parkinson 1966: 86 n. 2).

According to this passage, a proposition such as $A \supset B$ gives rise to a new abstract term, which Leibniz signifies by the phrases '*A's being B*' or '*the B-ness of A*'.[48] In what follows, we refer to such terms as propositional terms.[49] We use corner quotes to denote the operation mapping a proposition to the corresponding propositional term. Thus, $\ulcorner A \supset B \urcorner$ is the propositional term generated from the proposition $A \supset B$. In this notation, the complex proposition mentioned by Leibniz in the passage just quoted is written:

$$\ulcorner A \supset B \urcorner \supset \ulcorner C \supset D \urcorner$$

If the language $\mathcal{L}_{\text{cp}}$ is extended so as to include propositional terms, the result is the following language, $\mathcal{L}_{\supset}$:

Definition 13. Given a nonempty set of primitive expressions referred to as simple terms, the terms and propositions of the language $\mathcal{L}_{\supset}$ are defined as follows:

(1) Every simple term is a term.
(2) If A and B are terms, then AB is a term.
(3) If A is a term, then $\overline{A}$ is a term.
(4) If A and B are terms, then $A \supset B$ is a proposition.
(5) If $A \supset B$ is a proposition, then $\ulcorner A \supset B \urcorner$ is a term.

In the *Generales inquisitiones*, Leibniz posits two main principles pertaining to propositional terms. The first characterizes the relationship between propositional terms and the propositions from which they are generated. Specifically, this principle describes what it means for one propositional term to be contained in another, and is formulated by Leibniz as follows:

> For a proposition to follow from a proposition is nothing other than for the consequent to be contained in the antecedent as a term in a term. By this method we reduce consequences to propositions, and propositions to terms. (*Generales inquisitiones* §198.8)

According to this passage, one proposition follows from another just in case the propositional term corresponding to the former is contained in the propositional term corresponding to the latter.[50] When Leibniz says that one proposition follows from another, he means that the latter is derivable from the former in his calculus.[51] Thus, the principle of propositional containment stated by Leibniz in the passage

[48]See *GI* §§138–42 and A VI.4 740.

[49]The label 'propositional term' is borrowed from Sommers 1982: 156, 1993: 172–3, Barnes 1983: 315, and Swoyer 1995: 110–11. Leibniz does not use this label, but instead uses 'complex term' to refer to propositional terms and Boolean compounds thereof (see *GI* §61, §65, §75; cf. A VI.4 528–9).

[50]Similarly, Leibniz writes that 'whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows' (§189.6). See also §138, VI4A 809, 811 n. 6, 863; cf. Swoyer 1995: 110.

[51]To justify the claim that one proposition follows (*sequitur*) from another, Leibniz often provides a derivation of the former from the latter in his calculus (e.g., §49, §§52–4, §100). Moreover, there is no indication that Leibniz entertained the possibility that a proposition might follow from another despite not being derivable from it in his calculus. On the contrary, he asserts the completeness of the principles of his calculus when he states that 'whatever cannot be demonstrated from these principles does not follow by virtue of logical form' (§189.7).

just quoted can be formulated as follows, where φ and ψ are any propositions:

$$\varphi \vdash \psi \qquad \text{if and only if} \qquad \vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$$

In the coincidence calculus of the *Generales inquisitiones*, this biconditional entails the following stronger version of the principle, where Γ is any set of propositions:[52]

$$\Gamma \cup \{\varphi\} \vdash \psi \qquad \text{if and only if} \qquad \Gamma \vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$$

In Leibniz's coincidence calculus, this principle plays a role analogous to that of a deduction theorem in modern proof systems: it allows facts concerning the inferential relations between propositions to be expressed by propositions in the language of the calculus itself. Thus, by means of this principle we can, as Leibniz puts it, 'reduce consequences to propositions'.

In order to reproduce the full deductive power of the coincidence calculus developed in the *Generales inquisitiones*, Leibniz's containment calculus must validate the stronger version of the principle of propositional containment. In the coincidence calculus, as we just noted, this stronger version of the principle can be derived from the weaker version. This derivation, however, cannot be reproduced in the containment calculus, because it relies on substitutions of one coincident term for another within a propositional term. In the coincidence calculus, the legitimacy of such substitutions is taken for granted as part of the general rule of substitution licensing the free substitution of coincident terms in any syntactic context of the language, including those that occur within propositional terms.[53] By contrast, in the containment calculus the legitimacy of such substitutions cannot be taken for granted. Consequently, rather than deriving the stronger version of propositional containment from the weaker version, we will include the stronger version among the principles of the containment calculus.

As it turns out, once the stronger version of the principle has been posited, this suffices to derive the rule of substitution for the language $\mathcal{L}_{\supset}$. The key step in the proof of this rule is to extend Theorems 5 and 8 to cover the case of propositional terms. To this end, let the unadorned turnstile '$\vdash$' designate a calculus in the language $\mathcal{L}_{\supset}$ that obeys all the principles of $\vdash_{\text{cp}}$ as well as the stronger version of propositional containment. The proof of the relevant theorem then proceeds as follows:

[52]See Theorem 4.53 in Malink & Vasudevan 2016: 738.

[53]In an earlier paper, we claimed that the validity of substitutions within propositional terms can be justified by the weak principle of propositional containment asserting that $\varphi \vdash \psi$ just in case $\vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$ (Malink & Vasudevan 2016: 697 n. 41 and 702 n. 66). This claim, however, is not correct. In order to establish the validity of substitutions within propositional terms, one in fact needs the following two-premise version of the principle of propositional containment: $\varphi, \psi \vdash \chi$ just in case $\varphi \vdash \ulcorner\psi\urcorner \supset \ulcorner\chi\urcorner$. Now, there is independent textual evidence that Leibniz endorses this two-premise version of the principle (see Malink & Vasudevan 2019: Section 3). Thus, Leibniz is committed to the validity of substitutions within propositional terms. This comports well with the fact that, in his various statements of the rule of substitution in the *Generales inquisitiones* and elsewhere, Leibniz gives no indication that this rule is to be restricted to syntactic contexts that do not occur within propositional terms (see, e.g., *GI* §198.1, A VI.4 746, 810, 816, 831, 846).

Theorem 14. *For any terms A, B, C of $\mathcal{L}_{\supset}$, if C^* is the result of substituting B for an occurrence of A in C, then both:*

$$A \supset B, B \supset A \vdash C \supset C^*$$
$$A \supset B, B \supset A \vdash C^* \supset C$$

Proof. First, suppose that A is the term C. Then $C \supset C^*$ is the proposition $A \supset B$ and $C^* \supset C$ is the proposition $B \supset A$. Hence, the claim follows from the reflexivity and monotonicity of $\vdash$.

Now, suppose that A is a proper subterm of C. Then one of the following three cases holds: (i) C is a term of the form DE, where either D or E contains the occurrence of A for which B is substituted to obtain C^*; or (ii) C is a term of the form $\overline{D}$, where D contains the occurrence of A for which B is substituted to obtain C^*; or (iii) C is a term of the form $\ulcorner D \supset E \urcorner$, where either D or E contains the occurrence of A for which B is substituted to obtain C^*. Cases (i) and (ii) are treated in exactly the same way as in the proofs of Theorems 5 and 8, respectively.

In case (iii), first suppose that D contains the occurrence of A for which B is substituted. Then C^* is the term $\ulcorner D^* \supset E \urcorner$, where D^* is the result of substituting B for an occurrence of A in D. For induction, we assume that the claim holds for substitutions in D, so that both:

$$A \supset B, B \supset A \vdash D \supset D^*$$
$$A \supset B, B \supset A \vdash D^* \supset D$$

Given C4, the first of these claims implies:

$$A \supset B, B \supset A, D^* \supset E \vdash D \supset E$$

Hence, by the stronger version of propositional containment, we have:

$$A \supset B, B \supset A \vdash \ulcorner D^* \supset E \urcorner \supset \ulcorner D \supset E \urcorner$$

Likewise, given C4, the second claim implies:

$$A \supset B, B \supset A, D \supset E \vdash D^* \supset E$$

So, again, by the stronger version of propositional containment, we have:

$$A \supset B, B \supset A \vdash \ulcorner D \supset E \urcorner \supset \ulcorner D^* \supset E \urcorner$$

The same argument applies to the case in which E contains the occurrence of A for which B is substituted. This completes the proof. □

Given this theorem, the rule of substitution for containment in the calculus $\vdash$ follows straightforwardly:

Theorem 15. *For any terms A, B, C, D of $\mathcal{L}_{\supset}$, if $C^* \supset D^*$ is the result of substituting A for an occurrence of B in the proposition $C \supset D$, then:*

$$A \supset B, B \supset A, C \supset D \vdash C^* \supset D^*$$

Proof. The proof is the same as that given for Theorem 6. □

Thus, provided that the containment calculus $\vdash$ validates the stronger version of propositional containment, it licenses free substitution under mutual containment into any syntactic context of the language $\mathcal{L}_{\supset}$. Consequently, just as before, if the language $\mathcal{L}_{\supset}$ is extended by adding a new primitive relation symbol '$=$' governed by the introduction and elimination rules for coincidence, the resulting calculus is strong enough to validate the general rule of substitution for coincidence.

In addition to the principle of propositional containment, there is one more principle pertaining to propositional terms that is posited by Leibniz in the *Generales inquisitiones*. Leibniz formulates this principle as follows:[54]

> If B is a proposition, *non-B* is the same as *B is false*, or, *B's being false*. (*Generales inquisitiones* §32a, A VI.4 753 n. 18)

In this passage, Leibniz describes the effect of applying the operation of privation to propositional terms.[55] Specifically, he asserts that, if B is a propositional term, then its privative, *non-B*, coincides with the propositional term generated from the proposition *B is false*. In our notation, this principle states that the privative term $\overline{\ulcorner\varphi\urcorner}$ coincides with the propositional term $\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner$ generated from the proposition $\mathbf{F}(\ulcorner\varphi\urcorner)$. Since, as we have seen, $\mathbf{F}(A)$ is definable as $A \supset \overline{A}$, the principle states that $\overline{\ulcorner\varphi\urcorner}$ coincides with the propositional term $\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\varphi\urcorner}\urcorner$. In the language $\mathcal{L}_\supset$, this coincidence is captured by the following two laws:

$$\vdash \overline{\ulcorner\varphi\urcorner} \supset \ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\varphi\urcorner}\urcorner \qquad\qquad \vdash \ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\varphi\urcorner}\urcorner \supset \overline{\ulcorner\varphi\urcorner}$$

In what follows, we refer to these laws collectively as the principle of propositional privation.

The principle of propositional privation has a number of important consequences. For one, this principle allows us to simulate in Leibniz's calculus the propositional operation of classical conjunction. To see this, let φ and ψ be any propositions of $\mathcal{L}_\supset$, and let $\varphi \,\&\, \psi$ be the following proposition:

$$\mathbf{F}\left(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\right)$$

The principles of Leibniz's calculus then imply that the operation &, defined in this way, satisfies the classical laws of conjunction-introduction and -elimination. Specifically, if $\vdash$ is a calculus in $\mathcal{L}_\supset$ that obeys all the principles of $\vdash_{\mathrm{cp}}$ as well as the principles of propositional containment and propositional privation, we have:[56]

$$\begin{aligned} \varphi, \psi &\vdash \varphi \,\&\, \psi \\ \varphi \,\&\, \psi &\vdash \varphi \\ \varphi \,\&\, \psi &\vdash \psi \end{aligned}$$

Hence, although the language of Leibniz's containment calculus lacks any primitive propositional operators for forming conjunctions or other complex propositions, the calculus is strong enough to reproduce classical conjunctive reasoning.

The availability of classical conjunction in Leibniz's containment calculus allows us to solve the problem, discussed at the beginning of the paper, of how to define coincidence in a language in which every proposition is of the form $A \supset B$. For, given the above definition of conjunction, Leibniz's characterization of coincidence

[54]For a similar statement of this principle, see A VI.4 809.

[55]On the one hand, the fact that Leibniz applies the term-operation of privation to B indicates that B is a term. Moreover, the fact that he characterizes B as a proposition strongly suggests that B is, more specifically, a propositional term.

[56]See Theorem 29 in the appendix.

as mutual containment is expressible in the language $\mathcal{L}_{\supset}$ by the proposition $(A \supset B) \mathbin{\&} (B \supset A)$, i.e.:

$$\mathbf{F}\left(\ulcorner \ulcorner A \supset B \urcorner \supset \overline{\ulcorner B \supset A \urcorner} \urcorner\right)$$

It is an immediate consequence of the classical laws of conjunction stated above that the relation of coincidence defined in this way satisfies the introduction and elimination rules for coincidence. Thus, unlike the containment calculi $\vdash_{\mathrm{c}}$ and $\vdash_{\mathrm{cp}}$, in which coincidence had to be defined implicitly by means of its introduction and elimination rules, the full containment calculus $\vdash$ is strong enough to allow for an explicit definition of coincidence. This explicit definition is made possible by Leibniz's device of propositional terms and the two principles governing their operation, namely, the principles of propositional containment and propositional privation.

All told, then, the full containment calculus $\vdash$ can be obtained from $\vdash_{\mathrm{cp}}$ by adding to it the principles of propositional containment and propositional privation. As it turns out, the principle of propositional containment can be somewhat weakened in the resulting calculus $\vdash$ without any loss of deductive power. Specifically, the strong version of this principle stated above can be replaced with the following weaker version, in which the set of premises Γ consists of a single proposition:[57]

$$\varphi, \psi \vdash \chi \qquad \text{if and only if} \qquad \varphi \vdash \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner$$

Thus, positing only this weaker version of propositional containment, the calculus $\vdash$ can be defined as follows:

Definition 16. The calculus $\vdash$ is the smallest calculus in the language $\mathcal{L}_{\supset}$ such that:

(C1) $\vdash A \supset A$

(C2) $\vdash AB \supset A$

(C3) $\vdash AB \supset B$

(C4) $A \supset B, B \supset C \vdash A \supset C$

(C5) $A \supset B, A \supset C \vdash A \supset BC$

(CP1) $A \supset B \vdash \overline{B} \supset \overline{A}$

(CP2) $\vdash \overline{\overline{A}} \supset A$

(CP3) $AB \supset \overline{AB} \vdash A \supset \overline{B}$

(PT1) $\vdash \overline{\ulcorner \varphi \urcorner} \supset \ulcorner \ulcorner \varphi \urcorner \supset \overline{\ulcorner \varphi \urcorner} \urcorner$

(PT2) $\vdash \ulcorner \ulcorner \varphi \urcorner \supset \overline{\ulcorner \varphi \urcorner} \urcorner \supset \overline{\ulcorner \varphi \urcorner}$

(PT3) $\varphi, \psi \vdash \chi$ iff $\varphi \vdash \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner$

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[57] This weaker version of the principle of propositional containment suffices to establish the classical laws of conjunction in the calculus $\vdash$ (see the proof of Theorem 29 in the appendix). Given these laws, it can be shown that this weaker version of the principle entails the strong version for those cases in which Γ is a finite set of propositions. For, given the rule of conjunction-introduction, all the propositions in the finite set Γ can be conjoined into a single proposition, to which the weaker version of the principle can be applied. This conjunctive proposition can then be unpacked again by the rule of conjunction-elimination. The extension to cases in which Γ is infinite follows from monotonicity and the fact that a proposition is derivable from an infinite set of premises in the calculus $\vdash$ just in case it is derivable from some finite subset of this set.

As in the case of the calculi $\vdash_{\rm c}$ and $\vdash_{\rm cp}$, the full containment calculus $\vdash$ gives rise to an algebra of terms that allows for free substitution under the relation of mutual containment. The specific type of algebraic structure determined by the principles of the calculus $\vdash$ is what we call an auto-Boolean algebra. An auto-Boolean algebra is a Boolean algebra equipped with an additional binary operation $\circ$ defined as follows:[58]

$$x \circ y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Here, 1 and 0 are the top and bottom elements of the Boolean algebra, respectively. It can be shown that the calculus $\vdash$ is sound and complete with respect to the class of auto-Boolean algebras, when containment is interpreted as the order relation in the algebra, composition as the meet operation, privation as the complement operation, and the operation mapping A and B to $\ulcorner A \supset B \urcorner$ as the auto-Boolean operation mapping x and y to the element $x \circ (x \wedge y)$.[59] In this auto-Boolean semantics of the language $\mathcal{L}_{\supset}$, a propositional term $\ulcorner \varphi \urcorner$ designates the top element of the Boolean algebra if the proposition φ is true, and the bottom element of the algebra if φ is false.

We have now completed our survey of the fundamental laws of conceptual containment posited by Leibniz in his logical writings. These laws guarantee that the relation of containment is a preorder, i.e., that it is reflexive and transitive. Moreover, they specify how this relation interacts with the three operations of composition, privation, and forming propositional terms. Taken together, these laws constitute the containment calculus $\vdash$ in the language $\mathcal{L}_{\supset}$. This calculus, it turns out, has all the expressive and deductive power of the coincidence calculus developed by Leibniz in the *Generales inquisitiones*.[60] More precisely, if the relation of coincidence is expressed in the language $\mathcal{L}_{\supset}$ by the formula $(A \supset B) \,\&\, (B \supset A)$, where the operation & is defined in the manner just described, then all the theorems of Leibniz's coincidence calculus are derivable in the containment calculus $\vdash$.[61] Conversely, if containment is reduced to coincidence by defining $A \supset B$ as $A = AB$, then all the theorems of the containment calculus $\vdash$ are derivable in Leibniz's coincidence calculus.[62] Thus, the containment calculus $\vdash$ and the coincidence calculus developed by Leibniz in the *Generales inquisitiones* can be viewed as alternative axiomatizations, in different but equally expressive languages, of one and the same logical theory.

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[58]Up to a change of signature, auto-Boolean algebras just are what are known as simple monadic algebras (cf. Halmos 1962: 40–8, Goldblatt 2006: 14). These are Boolean algebras equipped with an additional unary operation f defined by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

An auto-Boolean algebra is a simple monadic algebra in which the operation f is defined by $f(x) = (x \circ 0) \circ 0$. Conversely, a simple monadic algebra is an auto-Boolean algebra in which the operation $\circ$ is defined by $x \circ y = (f((x' \wedge y) \vee (y' \wedge x)))'$.

[59]For a proof of this claim, see Theorem 44 in the appendix.

[60]This coincidence calculus is specified in Definition 36 below. For a more detailed exposition of this calculus, see Malink & Vasudevan 2016: 696–706.

[61]For a proof of this claim, see Theorem 37 below.

[62]This can be seen by comparing the principles of the containment calculus $\vdash$ listed in Definition 16 with the laws of the coincidence calculus stated in Theorems 4.15–4.26 and 4.53 in Malink & Vasudevan 2016: 723–38.

6. The Two Roots of Leibniz's Logic

While Leibniz's containment calculus and his coincidence calculus are equal in their expressive and deductive power, they also differ in a number of important respects. As we have seen, one such difference concerns the way in which substitutional reasoning is implemented in the two calculi. In the coincidence calculus, a rule of substitution licensing the free substitution of one coincident term for another is posited as a primitive rule of inference. Indeed, this rule is usually stated by Leibniz as the first principle of his coincidence calculus, highlighting the fact that substitution is the characteristic mode of reasoning in this calculus.[63] By contrast, in the containment calculus, substitutional reasoning is not implemented by means of a primitive rule of inference, but is licensed by a derived rule of substitution for containment. The derivation of this rule proceeds by reducing substitutions of mutually containing terms to applications of the syllogistic mood Barbara. Thus, whereas substitutional reasoning is characteristic of the coincidence calculus, the characteristic mode of reasoning in the containment calculus is Barbara. As Leibniz puts it in his derivation of the rule of substitution for containment in the *Elementa ad calculum condendum*, 'there is only one fundamental inference: *A is B* and *B is C*, therefore *A is C*'.[64]

In his logical writings from the late 1670s Leibniz gives priority to conceptual containment over coincidence, while in his mature logical writings from the mid-1680s he tends to adopt the opposite approach. According to which of these two approaches he adopts, he treats either the rule of Barbara or that of substitution as the most basic mode of reasoning. This shifting emphasis between containment and Barbara, on the one hand, and coincidence and substitution, on the other, reflects two distinct approaches to the study of logic that play a formative role in Leibniz's thought.

The first of these approaches is the syllogistic approach to logic which originated with Aristotle's theory of the categorical syllogism and constituted the dominant paradigm in logic throughout the scholastic and early modern periods. As is well known, Leibniz had a fascination with the traditional theory of the syllogism that began in his early youth with his study of the works of scholastic logicians:[65]

> When I was not yet twelve years old, I filled pages with remarkable exercises in logic; I sought to exceed the subtleties of the scholastics, and neither Zabarella nor Ruvio nor Toledo caused me much delay. (*Guilielmi Pacidii initia et specimina scientiae generalis*, A VI.4 494)

Leibniz's mastery of the technicalities of syllogistic logic is already apparent in one of his earliest philosophical works, the *Dissertatio de arte combinatoria*.[66] Over the span of his career, Leibniz wrote numerous essays on the theory of the categorical syllogism, and his admiration for this theory did not wane even in his later years. Thus, for example, in the *Nouveaux Essais* he writes:

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[63]See n. 18 above.

[64]A VI.4 154.

[65]See also Leibniz's autobiographical note published in Pertz 1847: 167–8. Cf. Couturat 1901: 33–5.

[66]See A VI.1 179–87.

> I hold the invention of the syllogistic form to be one of the most beautiful inventions of the human mind, and indeed one of the most notable. It is a sort of universal mathematics, the importance of which is too little known. (*Nouveaux Essais*, A VI.6 478)

The traditional theory of the categorical syllogism thus had a profound influence on the shape of Leibniz's logic. As we have noted, Leibniz took the universal affirmative propositions that appear in syllogisms of the form Barbara to express the conceptual containment of one term in another.[67] Thus, the syllogistic approach to logic is especially pronounced in those of his logical writings in which priority is given to the relation of conceptual containment and the rule of Barbara.

On the other hand, those writings in which Leibniz gives priority to coincidence and the rule of substitution are informed by a different approach to logic. This second approach derives, not from Leibniz's study of the traditional theory of the syllogism, but instead from his investigations into the foundations of mathematics. In particular, Leibniz's emphasis on substitutional reasoning in his coincidence calculus is motivated by the central role that such reasoning plays in equational systems of arithmetic and geometry. Since their development in Greek antiquity, both arithmetic and geometry have made essential use of the technique of substituting equals by equals.[68] Despite this fact, ancient systems of mathematics did not include among their principles any general rule of substitution that would provide a direct justification of this technique.[69] For example, no explicit statement of a rule of substitution appears among the principles of Euclid's *Elements*.[70] Instead, Euclid only posits certain corollaries of this rule in his list of 'common notions', such as:

CN1: Things which are equal to the same thing are also equal to one another.
CN2: If equals are added to equals, the wholes are equal.
CN3: If equals are subtracted from equals, the remainders are equal.

In his own investigations into the foundations of geometry, Leibniz did not follow Euclid in positing these common notions as separate principles. Instead, he sought to derive them all from a single rule licensing the free substitution of equals by equals.[71] For example, in his personal copy of Euclid's *Elements* Leibniz added the following marginal note immediately after the list of common notions, which are there referred to as 'axioms':[72]

> Those things are equal which are indiscernible in magnitude, or which can be substituted for one another *salva magnitudine*. From this alone all axioms are demonstrated. (quoted in De Risi 2016a: 31 n. 17)

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[67] See the references given in n. 22 above.

[68] See Netz 1999: 189–97. Netz argues that in Greek mathematics 'the concept of substitution *salva veritate* is the key to most proofs' (1999: 190).

[69] See Netz 1999: 191.

[70] By contrast, an explicit statement of the rule of substitution was included in some 17th-century axiomatizations of Euclid's *Elements* (see De Risi 2016b: 618–24).

[71] For Leibniz's desire to provide demonstrations of Euclid's common notions, see De Risi 2016a: 25 n. 7, 31 n. 17, 33 n. 19.

[72] See also A II.1 769, GM 7 274.

In this note, Leibniz does not explain how such demonstrations of the common notions are to proceed. However, he supplies the desired demonstrations elsewhere, in an essay titled *Demonstratio axiomatum Euclidis*, written in 1679. Leibniz begins this essay by positing a definition of equality which licenses the free substitution of equals by equals. He then proceeds to use this rule of substitution to derive all of Euclid's common notions. For example, he derives the first of these common notions as follows:

> Definition 1. Those things are equal which are indiscernible in magnitude, or which can be substituted for one another *salva magnitudine*. ...
> Axiom 1. Those things which are equal to the same thing are equal to each other. (1) *A is equal to B* and (2) *B is equal to C*. Therefore, (3) *A is equal to C*. For, since *A is equal to B* per 1, it follows by the definition of equals that *A* can be substituted for *B* in proposition 2, from which proposition 3 will result. (*Demonstratio axiomatum Euclidis*, A VI.4 165–6)

In this passage, Leibniz formulates Euclid's first common notion as follows: *A is equal to B* and *B is equal to C*, therefore, *A is equal to C*. Rather than positing this as an indemonstrable principle, Leibniz derives it by means of the rule of substitution licensed by his definition of equality. He gives similar proofs for the remaining common notions posited by Euclid. Such substitutional proofs of the common notions appear not only in the *Demonstratio axiomatum Euclidis*, but are given by Leibniz in a number of other essays as well.[73] Thus, it is clear that Leibniz regarded the substitution of equals by equals as a primitive mode of mathematical reasoning that plays a foundational role in arithmetic and geometry.

Leibniz's substitutional proofs of Euclid's common notions had a direct impact on the shape of the coincidence calculi developed by him in his logical writings of the mid-1680s. Indeed, in many of these essays, Leibniz reproduces the same patterns of substitutional reasoning that he employed in the late 1670s in his derivations of Euclid's common notions. Consider, for example, the derivation of the transitivity of coincidence given by Leibniz in his essay *Specimen calculi coincidentium et inexistentium*, written around 1686:[74]

> Definition 1. Those things are the same or coincident of which either can be substituted for the other anywhere *salva veritate*. ...
> Proposition 3. If $A = B$ and $B = C$, then $A = C$. Those things which are the same as a single third thing are the same as each other. For, if in the statement $A = B$ (which is true by hypothesis) C is substituted in place of B (by Definition 1, since $B = C$), the proposition $A = C$ will be true.[75] (*Specimen calculi coincidentium et inexistentium*, A VI.4 831)

[73]See, e.g., A VI.4 165-7, 506–7, GM 5 156, GM 7 77–80.

[74]For similar substitutional proofs of the transitivity of coincidence, see A VI.4 750, 815, 816, 849.

[75]In our translation of this passage, we print '=' in place of Leibniz's symbol '∞', the former being the symbol that Leibniz uses in the *Generales inquisitiones* to express the relation of coincidence between terms.

Leibniz's reasoning in this passage follows exactly the same pattern as his proof of Euclid's first common notion quoted above. Unlike in this earlier proof, however, in the present passage the schematic letters 'A', 'B', and 'C' do not stand for mathematical objects. Instead, these letters can be taken to stand for terms, or concepts, such as *human*, *animal*, and *omniscient*.[76] On this interpretation, the symbol '=' does not express the relation of equality in magnitude between mathematical objects, but instead expresses the relation of coincidence between terms.

It is clear, then, that the basic structure of the coincidence calculi developed by Leibniz in his later logical writings is modeled closely on his equational axiomatization of Euclid's common notions. Thus, it is these later logical writings which display most clearly the influence that Leibniz's work in the foundations of mathematics had on the shape of his logical theory. At the time when Leibniz provided his axiomatization of Euclid's common notions in 1679, his logical calculi had not yet taken the form of a coincidence calculus based on the rule of substitution. Instead, his logical writings from this earlier period still embody the traditional syllogistic approach to the subject, and thus treat as primitive the relation of conceptual containment and the rule of Barbara. At the same time, through his work in the foundations of mathematics, Leibniz was made acutely aware of the power of substitutional reasoning. This realization would have naturally led him to attempt to validate substitutional reasoning in his logic of conceptual containment. As we have seen, Leibniz succeeds in deriving a rule of substitution for the elementary case in which the terms of his containment calculus have no internal syntactic structure. The derivation of the rule of substitution, however, becomes more intricate as the syntactic structure of the terms increases in complexity. With the introduction of each new operation for forming complex terms, substitutions must be licensed into a wider variety of syntactic contexts. By the time the language of Leibniz's calculus reaches its full syntactic complexity in the *Generales inquisitiones*, Leibniz forgoes any attempt to derive the rule of substitution, preferring instead to develop a coincidence calculus in which substitution is simply posited as a primitive rule of inference. In this way, Leibniz transitions from an approach to logic informed by the traditional theory of the categorical syllogism to a substitutional approach modeled on the paradigm of equational systems of mathematics.

The substitutional approach to logic exemplified in Leibniz's mature logical writings of the mid-1680s prefigures in many important respects the algebraic approach to logic which rose to prominence in the second half of the 19th century through the work of logicians such as Boole and Jevons. Jevons, in particular, regarded the rule of substitution as 'the one supreme rule of inference'.[77] In his treatise *The Substitution of Similars: The True Principle of Reasoning*, he writes:[78]

[76]The coincidence calculus developed by Leibniz in the *Specimen calculi coincidentium et inexistentium* is intended to be an abstract calculus that admits of multiple interpretations. In particular, Leibniz intends the calculus to be applicable to any algebraic structure which satisfies the laws of commutativity and idempotence, $AB = BA$ and $AA = A$ (A VI.4 834). As Leibniz explains, the law of idempotence excludes arithmetical interpretations of the calculus (A VI.4 834; see also *GI* §129, A VI.4 512 and 811). At the same time, Leibniz points out that the calculus is applicable to the domain of terms, or concepts (*notiones*), when '=' is taken to expresses the relation of coincidence, and the concatenation of letters is taken to express the operation of composition on terms (see Swoyer 1994: 7 and 18, Lenzen 2000: 79–82, Mugnai 2017: 175–6).

[77]Jevons 1887: 17.

[78]Similarly, Jevons writes: 'it is not difficult to show that all forms of reasoning consist in repeated employment of the universal process of the *substitution of equals*' (Jevons 1869: 23–4).

> That most familiar process in mathematical reasoning, of substituting one member of an equation for the other, appears to be the type of all reasoning, and we may fitly name this all-important process the substitution of equals. (Jevons 1869: 20)

Jevons notes that the idea of a logical calculus based on the rule of substitution 'can be traced back to no less a philosopher than Leibnitz'.[79] He expresses admiration for Leibniz's derivation of the laws of coincidence by means of the rule of substitution, praising in particular the substitutional derivations given by Leibniz in his essay *Non inelegans specimen demonstrandi in abstractis*.[80] This emphasis on coincidence and substitutional reasoning, Jevons maintains, 'anticipates the modern views of logic'.[81] At the same time, Jevons laments the fact that Leibniz ignores the rule of substitution in those of his logical writings in which he adopts a more traditional, syllogistic approach to the subject:

> When he [Leibniz] proceeds to explain the syllogism, as in the paper *Definitiones logicae*, he gives up substitution altogether, and falls back upon the notion of inclusion of class in class .... He proceeds to make out certain rules of the syllogism involving the distinction of subject and predicate, and in no important respect better than the old rules of the syllogism. (Jevons 1887: xix)

The essay *Definitiones logicae* referred to by Jevons in this passage was written in the 1690s, when Leibniz had already completed his major essays on the logic of coincidence.[82] Contrary to what Jevons suggests, however, the syllogistic approach adopted by Leibniz in this essay does not represent a 'falling back' into an older and obsolete way of doing logic. Nor does it indicate an abandonment by Leibniz of the more modern, algebraic approach to the subject based on the rule of substitution. Instead, starting with the *Generales inquisitiones* in 1686, Leibniz's mature work in logic is simultaneously informed by both approaches. On the one hand, the calculus developed by Leibniz in the *Generales inquisitiones* is a coincidence calculus designed to take full advantage of the power of substitutional reasoning. At the same time, as we have seen above, throughout his mature logical writings Leibniz remains firmly committed to the conceptual containment theory of truth.[83] Indeed, just before presenting the final axiomatization of his coincidence calculus in §§198–200 of the *Generales inquisitiones*, Leibniz defines the relation of coincidence by analyzing it as mutual containment between terms (§195). This suggests that, despite Leibniz's recognition in his mature logical writings of the technical advantages that come with treating coincidence as a primitive relation in his calculus, he continues throughout to endorse the logical and metaphysical primacy of conceptual containment.

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[79]Jevons 1887: xvi.

[80]Jevons 1887: xvii–xviii. The derivations of the laws of coincidence given by Leibniz in this essay follow exactly the same pattern as his derivations of these laws in the *Specimen calculi coincidentium et inexistentium* referred to above (cp. A VI.4 831–7 with A VI.4 846–50).

[81]Jevons 1887: xviii.

[82]This essay appears at GP 7 208–10. It also appears under the title *De syllogismo categorico ex inclusione exclusioneve terminorum* in a preliminary volume of the Academy edition (*Vorausedition* A VI.5 1120). The editors of this volume date the essay between 1690 and 1696.

[83]See Section 1 above.

In this respect, Leibniz's views regarding the relative priority of containment and coincidence differ from those of Jevons. Instead, they are more in agreement with the position advanced by Peirce in his writings on algebraic logic. According to Peirce, the relation of containment expressed by universal affirmative propositions is more fundamental than the relation of coincidence expressed by algebraic equalities:

> There is a difference of opinion among logicians as to whether $\supset$ or $=$ is the simpler relation. But in my paper on the *Logic of Relatives*, I have strictly demonstrated that the preference must be given to $\supset$ in this respect.[84] (Peirce 1880: 21 n. 1; with notational adjustments)

Thus Peirce rejects the view of logicians, such as Jevons, who maintain that $=$ is a more fundamental relation than $\supset$. On Peirce's view, the laws that govern the relation of coincidence derive their justification, not from a primitive rule of substitution, but from underlying laws of containment such as Barbara. Peirce affirms this position in a letter to Jevons written in 1870:

> I have shown rigidly that according to admitted principles, the conception of $=$ is compounded of those of $\supset$ and $\subset$ (or $\leqq$ and $\geqq$). This being so the substitution-syllogism
>
> $$\frac{A{=}B \qquad B{=}C}{A{=}C}$$
>
> is a compound of the two
>
> $$\frac{A \supset B \qquad B \supset C}{A \supset C} \quad \text{and} \quad \frac{A \subset B \qquad B \subset C}{A \subset C}$$
>
> and the logician in analyzing inferences ought to represent it so. (Peirce 1984: 446; with notational adjustments)

The view espoused by Peirce in this passage agrees with that adopted by Leibniz in the exposition of his containment calculi, according to which coincidence is to be defined as mutual containment and the laws of coincidence are to be derived from the underlying laws of containment. While Peirce acknowledges that, in certain contexts, it may be more convenient for a logician to deal directly with propositions expressing coincidence, in his view this does not take away from the fact that, in the logical and metaphysical order of things, containment is the more fundamental relation:

> It frequently happens that it is more convenient to treat the propositions $A \supset B$ and $B \supset A$ together in their form $A = B$; but it also frequently happens that it is more convenient to treat them separately. ... In logic, our great object is to analyze all the operations of reason and reduce them to their ultimate elements; and to

[84] The 'demonstration' referred to in this passage appears at Peirce 1873: 318 n. 1.

> make a calculus of reasoning is a subsidiary object. Accordingly, it is more philosophical to use the copula $\supset$, apart from all considerations of convenience. Besides, this copula is intimately related to our natural logical and metaphysical ideas; and it is one of the chief purposes of logic to show what validity those ideas have. (Peirce 1880: 21 n. 1; with notational adjustments)

Like Peirce, Leibniz maintains that, from a logical and metaphysical perspective, the relation of containment is more fundamental than that of coincidence. At the same time, when developing his logical calculi, Leibniz often finds it more convenient to treat coincidence as a primitive relation and to define containment in terms of coincidence rather than the other way around. The apparent tension between these two approaches is indicative of the two distinct roots of Leibniz's logic: the syllogistic and the equational. Far from being incoherent, however, the two approaches fit together harmoniously within Leibniz's overarching conception of logic. For, as we have seen in this paper, Leibniz's logic of conceptual containment and his logic of coincidence are, in effect, two alternative axiomatizations of one and the same logical theory.

Appendix: Algebraic Semantics for Leibniz's Containment Calculi

In this appendix, we provide an algebraic semantics for the containment calculi $\vdash_{\mathrm{c}}$, $\vdash_{\mathrm{cp}}$, and $\vdash$ (introduced in Definitions 3, 12, and 16 above). The three main results established in this appendix are:

- The calculus $\vdash_{\mathrm{c}}$ is sound and complete with respect to the class of semilattices (Theorem 20).
- The calculus $\vdash_{\mathrm{cp}}$ is sound and complete with respect to the class of Boolean algebras (Theorem 25).
- The calculus $\vdash$ is sound and complete with respect to the class of auto-Boolean algebras (Theorem 44).

Semilattice Semantics for $\vdash_{\mathrm{c}}$. In this section, we establish that the calculus $\vdash_{\mathrm{c}}$ is sound and complete with respect to the class of semilattice interpretations of the language $\mathcal{L}_{\mathrm{c}}$. As a preliminary step, we first note the following elementary fact about the calculus $\vdash_{\mathrm{c}}$:

Theorem 17. *For any terms A, B of $\mathcal{L}_{\mathrm{c}}$:*

(i) $\vdash_{\mathrm{c}} A \supset AA$
(ii) $\vdash_{\mathrm{c}} AB \supset BA$

Proof. For (i): by C5, we have $A \supset A \vdash_{\mathrm{c}} A \supset AA$. Hence, by C1, $\vdash_{\mathrm{c}} A \supset AA$.

For (ii): by C2 and C3, we have $\vdash_{\mathrm{cp}} AB \supset A$ and $\vdash_{\mathrm{cp}} AB \supset B$. Hence, by C5, $\vdash_{\mathrm{cp}} AB \supset BA$. □

We now introduce an algebraic semantics for the language $\mathcal{L}_{\mathrm{c}}$:

Definition 18. An interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_{\mathrm{c}}$ consists of (i) an algebraic structure $\mathfrak{A} = \langle \mathbb{A}, \wedge \rangle$, where $\mathbb{A}$ is a nonempty set and $\wedge$ is a binary operation on $\mathbb{A}$; and (ii) a function μ mapping each term of $\mathcal{L}_{\mathrm{c}}$ to an element of $\mathbb{A}$ such that:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$

A proposition $A \supset B$ of $\mathcal{L}_c$ is satisfied in an interpretation $(\mathfrak{A}, \mu)$ iff $\mu(A) = \mu(A) \wedge \mu(B)$. We write $(\mathfrak{A}, \mu) \models A \supset B$ to indicate that $(\mathfrak{A}, \mu)$ satisfies $A \supset B$.

Definition 19. An interpretation $(\langle \mathbb{A}, \wedge \rangle, \mu)$ of $\mathcal{L}_c$ is a semilattice interpretation if $\langle \mathbb{A}, \wedge \rangle$ is a semilattice. If Γ is a set of propositions of $\mathcal{L}_c$ and φ a proposition of $\mathcal{L}_c$, we write $\Gamma \models_{sl} \varphi$ to indicate that, for any semilattice interpretation $(\mathfrak{A}, \mu)$: if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.

The following completeness proof employs the standard technique introduced by Birkhoff (1935) in his proof of the algebraic completeness of equational calculi.

Theorem 20. *For any set of propositions Γ of $\mathcal{L}_c$ and any proposition φ of $\mathcal{L}_c$: $\Gamma \vdash_c \varphi$ iff $\Gamma \models_{sl} \varphi$.*

Proof. The left-to-right direction follows straightforwardly from the fact that the principles C1–C5 (as well as the general properties of calculi stated in Definition 2) are satisfied in every semilattice interpretation.

For the right-to-left direction, let Γ be a set of propositions of $\mathcal{L}_c$, and let $\equiv$ be the binary relation between terms of $\mathcal{L}_c$ defined by:

$$A \equiv B \quad \text{iff} \quad \Gamma \vdash_c A \supset B \text{ and } \Gamma \vdash_c B \supset A$$

Clearly, if $A \equiv B$, then $B \equiv A$. Also, by C1, we have $A \equiv A$. By C4, we have: if $A \equiv B$ and $B \equiv C$, then $A \equiv C$. Hence, $\equiv$ is an equivalence relation on the set of all terms. Moreover, by Theorem 4, we have: if $A \equiv B$ and $C \equiv D$, then $AC \equiv BD$. Thus, $\equiv$ is a congruence relation on the algebra of terms formed by the operation of composition.

Now, let μ be the function mapping each term to its equivalence class under $\equiv$, and let $\mathbb{T}$ be the set $\{\mu(A) : A \text{ is a term of } \mathcal{L}_c\}$. Since $\equiv$ is a congruence relation with respect to the operation of composition, there is a binary operation $\wedge$ on $\mathbb{T}$ such that, for any terms A and B:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$

This equation implies:

$$\begin{aligned} (\mu(A) \wedge \mu(B)) \wedge \mu(C) &= \mu(AB) \wedge \mu(C) \\ &= \mu(ABC) \\ &= \mu(A) \wedge \mu(BC) \quad = \quad \mu(A) \wedge (\mu(B) \wedge \mu(C)) \end{aligned}$$

In other words, the operation $\wedge$ is associative. Now, by C2 and Theorem 17.i, we have $\vdash_c AA \supset A$ and $\vdash_c A \supset AA$. It follows that $\mu(AA) = \mu(A)$. Hence, $\wedge$ is idempotent. Moreover, by Theorem 17.ii, we have $\vdash_c AB \supset BA$ and $\vdash_c BA \supset AB$. It follows that $\mu(AB) = \mu(BA)$. Hence, $\wedge$ is commutative. All told, then, $\wedge$ is an idempotent, associative, commutative operation on $\mathbb{T}$. Hence, $\mathfrak{T} = \langle \mathbb{T}, \wedge \rangle$ is a semilattice, and $(\mathfrak{T}, \mu)$ is a semilattice interpretation.

We now show that:

$$(\mathfrak{T}, \mu) \models A \supset B \quad \text{iff} \quad \Gamma \vdash_c A \supset B$$

First, suppose that $(\mathfrak{T}, \mu) \models A \supset B$. Then, $\mu(A) = \mu(A) \wedge \mu(B) = \mu(AB)$, i.e., $A \equiv AB$. This implies that $\Gamma \vdash_c A \supset AB$, and so, since $\vdash_c AB \supset B$, it follows, by C4, that $\Gamma \vdash_c A \supset B$. Next, suppose that $\Gamma \vdash_c A \supset B$. Then, since $\vdash_c A \supset A$, by C5, we have $\Gamma \vdash_c A \supset AB$. But, since, by C2, $\vdash_c AB \supset A$, it follows that $A \equiv AB$, i.e., $\mu(A) = \mu(AB) = \mu(A) \wedge \mu(B)$. But this means that $(\mathfrak{T}, \mu) \models A \supset B$.

Now, suppose $\Gamma \nvdash_{\mathrm{c}} \varphi$. Then, $(\mathfrak{T}, \mu) \not\models \varphi$. But since $\Gamma \vdash_{\mathrm{c}} \psi$ for all $\psi \in \Gamma$, it follows that $(\mathfrak{T}, \mu) \models \psi$ for all $\psi \in \Gamma$. Hence, since $(\mathfrak{T}, \mu)$ is a semilattice interpretation, $\Gamma \not\models_{\mathrm{sl}} \varphi$. □

Boolean Semantics for $\vdash_{\mathrm{cp}}$. In this section, we establish that the calculus $\vdash_{\mathrm{cp}}$ is sound and complete with respect to the class of Boolean interpretations of the language $\mathcal{L}_{\mathrm{cp}}$. As a preliminary step, we first note a few elementary facts about the calculus $\vdash_{\mathrm{cp}}$.

Theorem 21. *For any terms A, B of $\mathcal{L}_{\mathrm{cp}}$:*

(i) $A \supset \overline{B} \vdash_{\mathrm{cp}} AB \supset \overline{AB}$
(ii) $A \supset \overline{B} \vdash_{\mathrm{cp}} B \supset \overline{A}$
(iii) $\vdash_{\mathrm{cp}} A \supset \overline{\overline{A}}$
(iv) $A \supset B \dashv\vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$
(v) $A \supset \overline{A} \vdash_{\mathrm{cp}} A \supset B$
(vi) $\vdash_{\mathrm{cp}} A\overline{A} \supset B$

Proof. For (i): by C3 and CP1, we have $\vdash_{\mathrm{cp}} \overline{B} \supset \overline{AB}$. Also, by C2 and C4, $A \supset \overline{B} \vdash_{\mathrm{cp}} AB \supset \overline{B}$. Hence, by C4, $A \supset \overline{B} \vdash_{\mathrm{cp}} AB \supset \overline{AB}$.

For (ii): by Theorem 17.ii and CP1, we have $\vdash_{\mathrm{cp}} \overline{AB} \supset \overline{BA}$. Now, by (i) above and C4, we have $A \supset \overline{B} \vdash_{\mathrm{cp}} BA \supset \overline{BA}$. Hence, by CP3, $A \supset \overline{B} \vdash_{\mathrm{cp}} B \supset \overline{A}$.

For (iii): by (ii) above, we have $\overline{A} \supset \overline{A} \vdash_{\mathrm{cp}} A \supset \overline{\overline{A}}$. Hence, by C1, $\vdash_{\mathrm{cp}} A \supset \overline{\overline{A}}$.

For (iv): by CP3 and (i) above, $A \supset \overline{\overline{B}} \dashv\vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$. But by (iii) above, CP2, and C4, we have $A \supset \overline{\overline{B}} \dashv\vdash_{\mathrm{cp}} A \supset B$. Hence, $A \supset B \dashv\vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$.

For (v): by C2 and CP1, we have both $\vdash_{\mathrm{cp}} A\overline{B} \supset A$ and $\vdash_{\mathrm{cp}} \overline{A} \supset \overline{A\overline{B}}$. So, by C4, $A \supset \overline{A} \vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$. Hence, by (iv) above, $A \supset \overline{A} \vdash_{\mathrm{cp}} A \supset B$.

For (vi): by (iv) above, $A \supset A \vdash_{\mathrm{cp}} A\overline{A} \supset \overline{A\overline{A}}$. So, by C1, $\vdash_{\mathrm{cp}} A\overline{A} \supset \overline{A\overline{A}}$. Hence, by (v) above, $\vdash_{\mathrm{cp}} A\overline{A} \supset B$. □

Theorem 22. *For any terms A, B, C of $\mathcal{L}_{\mathrm{cp}}$:*

(i) $A\overline{B} \supset C\overline{C}, C\overline{C} \supset A\overline{B} \vdash_{\mathrm{cp}} A \supset B$
(ii) $A \supset B \vdash_{\mathrm{cp}} A\overline{B} \supset C\overline{C}$
(iii) $A \supset B \vdash_{\mathrm{cp}} C\overline{C} \supset A\overline{B}$

Proof. For (i): by Theorem 21.vi, we have $\vdash_{\mathrm{cp}} C\overline{C} \supset \overline{A\overline{B}}$. So, by C4, $A\overline{B} \supset C\overline{C} \vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$. But, by Theorem 21.iv, this implies $A\overline{B} \supset C\overline{C} \vdash_{\mathrm{cp}} A \supset B$. Hence, $A\overline{B} \supset C\overline{C}, C\overline{C} \supset A\overline{B} \vdash_{\mathrm{cp}} A \supset B$.

For (ii): by Theorem 21.iv, we have $A \supset B \vdash_{\mathrm{cp}} A\overline{B} \supset \overline{A\overline{B}}$. Hence, by Theorem 21.v, $A \supset B \vdash_{\mathrm{cp}} A\overline{B} \supset C\overline{C}$.

Lastly, (iii) follows by Theorem 21.vi. □

We now introduce an algebraic semantics for the language $\mathcal{L}_{\mathrm{cp}}$:

Definition 23. An interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_{\mathrm{cp}}$ consists of (i) an algebraic structure $\mathfrak{A} = \langle \mathbb{A}, \wedge, ' \rangle$, where $\mathbb{A}$ is a nonempty set, $\wedge$ is a binary operation on $\mathbb{A}$, and $'$ is a unary operation on $\mathbb{A}$; and (ii) a function μ mapping each term of $\mathcal{L}_{\mathrm{cp}}$ to an element of $\mathbb{A}$ such that:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$
$$\mu\left(\overline{A}\right) = \mu(A)'$$

A proposition $A \supset B$ of $\mathcal{L}_{\mathrm{cp}}$ is satisfied in an interpretation $(\mathfrak{A}, \mu)$ iff $\mu(A) = \mu(A) \wedge \mu(B)$. We write $(\mathfrak{A}, \mu) \models A \supset B$ to indicate that $(\mathfrak{A}, \mu)$ satisfies $A \supset B$.

Definition 24. An interpretation $(\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ of $\mathcal{L}_{\mathrm{cp}}$ is a Boolean interpretation if $\langle \mathbb{A}, \wedge, ' \rangle$ is a Boolean algebra. If Γ is a set of propositions of $\mathcal{L}_{\mathrm{cp}}$ and φ a proposition of $\mathcal{L}_{\mathrm{cp}}$, we write $\Gamma \models_{\mathrm{ba}} \varphi$ to indicate that, for any Boolean interpretation $(\mathfrak{A}, \mu)$: if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.

The following completeness proof relies on a concise axiomatization of the theory of Boolean algebras discovered by Byrne (1946). In the language $\mathcal{L}_{\mathrm{cp}}$, the main principle in this axiomatization is given by Theorem 22.

Theorem 25. *For any set of propositions Γ of $\mathcal{L}_{\mathrm{cp}}$ and any proposition φ of $\mathcal{L}_{\mathrm{cp}}$: $\Gamma \vdash_{\mathrm{cp}} \varphi$ iff $\Gamma \models_{\mathrm{ba}} \varphi$.*

Proof. The left-to-right direction follows straightforwardly from the fact that the principles C1–C5 and CP1–CP3 are satisfied in every Boolean interpretation.

For the right-to-left direction, let Γ be a set of propositions of $\mathcal{L}_{\mathrm{cp}}$, and let $\equiv$ be the binary relation between terms of $\mathcal{L}_{\mathrm{cp}}$ defined by:

$$A \equiv B \quad \text{iff} \quad \Gamma \vdash_{\mathrm{cp}} A \supset B \text{ and } \Gamma \vdash_{\mathrm{cp}} B \supset A$$

Following the reasoning in the proof of Theorem 20, $\equiv$ is an equivalence relation on the set of terms of $\mathcal{L}_{\mathrm{cp}}$. Moreover, by Theorem 4 and CP1, $\equiv$ is a congruence relation on the algebra of terms formed by the operations of composition and privation.

Now, let μ be the function mapping each term to its equivalence class under $\equiv$, and let $\mathbb{T}$ be the set $\{\mu(A) : A \text{ is a term of } \mathcal{L}_{\mathrm{cp}}\}$. Since $\equiv$ is a congruence relation with respect to the operations of composition and privation, there is a binary operation $\wedge$ on $\mathbb{T}$ and a unary operation $'$ on $\mathbb{T}$ such that, for any terms A and B:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$
$$\mu\left(\overline{A}\right) = \mu(A)'$$

The operation $\wedge$ is associative and commutative (see the proof of Theorem 20). Moreover, by Theorem 22, we have for any terms A, B, C of $\mathcal{L}_{\mathrm{cp}}$:

$$\mu(A) \wedge \mu(B)' = \mu(C) \wedge \mu(C)' \quad \text{iff} \quad \mu(A) = \mu(A) \wedge \mu(B)$$

Byrne (1946: 269-271) has shown that this latter condition, in conjunction with the associativity and commutativity of $\wedge$, implies that $\mathfrak{T} = \langle \mathbb{T}, \wedge, ' \rangle$ is a Boolean algebra. Hence $(\mathfrak{T}, \mu)$ is a Boolean interpretation.

Following the reasoning in the proof of Theorem 20, we have:

$$(\mathfrak{T}, \mu) \models A \supset B \quad \text{iff} \quad \Gamma \vdash_{\mathrm{cp}} A \supset B$$

Now, suppose $\Gamma \nvdash_{\mathrm{cp}} \varphi$. Then, $(\mathfrak{T}, \mu) \not\models \varphi$. But since $\Gamma \vdash_{\mathrm{cp}} \psi$ for all $\psi \in \Gamma$, it follows that $(\mathfrak{T}, \mu) \models \psi$ for all $\psi \in \Gamma$. Hence, since $(\mathfrak{T}, \mu)$ is a Boolean interpretation, $\Gamma \not\models_{\mathrm{ba}} \varphi$. □

Auto-Boolean Semantics for $\vdash$. In this section, we establish that the containment calculus $\vdash$ is sound and complete with respect to the class of auto-Boolean interpretations of the language $\mathcal{L}_{\supset}$. The proof of completeness will rely on the auto-Boolean completeness of the coincidence calculus developed by Leibniz's in the *Generales inquisitiones*. We begin by establishing some preliminary facts about the containment calculus $\vdash$.

Definition 26. For any term A of $\mathcal{L}_{\supset}$, we write $\mathbf{F}(A)$ for the proposition $A \supset \overline{A}$.

Theorem 27. *For any propositions φ, ψ, χ of $\mathcal{L}_{\supset}$:*

(i) *If* $\varphi, \psi \vdash \chi$, *then* $\varphi, \mathbf{F}(\ulcorner\chi\urcorner) \vdash \mathbf{F}(\ulcorner\psi\urcorner)$

(ii) $\varphi \vdash \psi$ *iff* $\vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$

(iii) *If* $\varphi \vdash \psi$, *then* $\mathbf{F}(\ulcorner\psi\urcorner) \vdash \mathbf{F}(\ulcorner\varphi\urcorner)$

Proof. For (i): suppose that $\varphi, \psi \vdash \chi$. By PT3, we have $\varphi \vdash \ulcorner\psi\urcorner \supset \ulcorner\chi\urcorner$. Hence, by CP1, $\varphi \vdash \overline{\ulcorner\chi\urcorner} \supset \overline{\ulcorner\psi\urcorner}$. By C4, PT1, and PT2, it follows that $\varphi \vdash \ulcorner\mathbf{F}(\ulcorner\chi\urcorner)\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner$. Hence, by PT3, $\varphi, \mathbf{F}(\ulcorner\chi\urcorner) \vdash \mathbf{F}(\ulcorner\psi\urcorner)$.

For (ii): by PT3, $A \supset A, \varphi \vdash \psi$ iff $A \supset A \vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$. Hence, by C1, $\varphi \vdash \psi$ iff $\vdash \ulcorner\varphi\urcorner \supset \ulcorner\psi\urcorner$.

For (iii): suppose that $\varphi \vdash \psi$. Then, $A \supset A, \varphi \vdash \psi$. By (i) above, we have $A \supset A, \mathbf{F}(\ulcorner\psi\urcorner) \vdash \mathbf{F}(\ulcorner\varphi\urcorner)$. Hence, by C1, $\mathbf{F}(\ulcorner\psi\urcorner) \vdash \mathbf{F}(\ulcorner\varphi\urcorner)$. □

Theorem 28. *For any proposition φ of $\mathcal{L}_{\supset}$:*

(i) $\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner) \vdash \varphi$

(ii) $\varphi \vdash \mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)$

Proof. For (i): by PT2, we have $\vdash \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner \supset \overline{\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner}$. Moreover, by PT1 and CP1, we have $\vdash \overline{\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner} \supset \overline{\overline{\ulcorner\varphi\urcorner}}$. Hence, by C4, $\vdash \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner \supset \overline{\overline{\ulcorner\varphi\urcorner}}$. By CP2 and C4, it follows that $\vdash \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner \supset \ulcorner\varphi\urcorner$. Hence, by Theorem 27.ii, $\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner) \vdash \varphi$.

For (ii): by PT1, we have $\vdash \overline{\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner} \supset \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner$. Moreover, by PT2 and CP1, we have $\vdash \overline{\overline{\ulcorner\varphi\urcorner}} \supset \overline{\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner}$. Hence, by C4, $\vdash \overline{\overline{\ulcorner\varphi\urcorner}} \supset \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner$. By Theorem 21.iii and C4, it follows that $\vdash \ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)\urcorner$. Hence, by Theorem 27.ii, $\varphi \vdash \mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)$. □

Theorem 29. *For any propositions φ, ψ of $\mathcal{L}_{\supset}$:*

(i) $\varphi, \psi \vdash \mathbf{F}\left(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\right)$

(ii) $\mathbf{F}\left(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\right) \vdash \varphi$

(iii) $\mathbf{F}\left(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\right) \vdash \psi$

Proof. For (i): since $\ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner \vdash \ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner$, by PT3 we have $\varphi, \ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner \vdash \mathbf{F}(\ulcorner\psi\urcorner)$. By C4 and PT1, it follows that $\varphi, \ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner} \vdash \mathbf{F}(\ulcorner\psi\urcorner)$. Hence, by Theorem 27.i, $\varphi, \mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner) \vdash \mathbf{F}\left(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\right)$. The desired result follows by Theorem 28.ii.

For (ii): since $\varphi, \psi \vdash \varphi$, by Theorem 27.i we have $\varphi, \mathbf{F}(\ulcorner\varphi\urcorner) \vdash \mathbf{F}(\ulcorner\psi\urcorner)$. By PT3, it follows that $\mathbf{F}(\ulcorner\varphi\urcorner) \vdash \ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner$. Hence, by PT2 and C4, $\mathbf{F}(\ulcorner\varphi\urcorner) \vdash \ulcorner\varphi\urcorner \supset$

$\overline{\ulcorner\psi\urcorner}$. By Theorem 27.iii, it follows that $\mathbf{F}\Big(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\Big) \vdash \mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\varphi\urcorner)\urcorner)$. The desired result follows by Theorem 28.i.

For (iii): since $\varphi, \mathbf{F}(\ulcorner\psi\urcorner) \vdash \mathbf{F}(\ulcorner\psi\urcorner)$, by PT3 we have $\mathbf{F}(\ulcorner\psi\urcorner) \vdash \ulcorner\varphi\urcorner \supset \ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner$. Hence, by C4 and PT2, $\mathbf{F}(\ulcorner\psi\urcorner) \vdash \ulcorner\varphi\urcorner \supset \overline{\overline{\ulcorner\psi\urcorner}}$. By Theorem 27.iii, it follows that $\mathbf{F}\Big(\ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\psi\urcorner}\urcorner\Big) \vdash \mathbf{F}(\ulcorner\mathbf{F}(\ulcorner\psi\urcorner)\urcorner)$. The desired result follows by Theorem 28.i. □

Definition 30. For any terms A, B of $\mathcal{L}_\supset$, we write $A \approx B$ for the proposition:

$$\mathbf{F}\Big(\ulcorner\ulcorner A \supset B\urcorner \supset \overline{\ulcorner B \supset A\urcorner}\urcorner\Big)$$

Theorem 31. *For any terms A, B of $\mathcal{L}_\supset$:*

(i) $A \supset B \dashv\vdash A \approx AB$
(ii) $\vdash AA \approx A$
(iii) $\vdash AB \approx BA$
(iv) $\vdash \overline{\overline{A}} \approx A$

Proof. For (i): by C2, we have $\vdash AB \supset A$, and so $A \supset B \vdash AB \supset A$. Moreover, by C1 and C5, we have $A \supset B \vdash A \supset AB$. Hence, by Theorem 29.i, $A \supset B \vdash A \approx AB$. For the converse, by Theorem 29.ii, we have $A \approx AB \vdash A \supset AB$. Hence, by C3 and C4, $A \approx AB \vdash A \supset B$.

For (ii): by C2, we have $\vdash AA \supset A$. Moreover, by Theorem 17.i, we have $\vdash A \supset AA$. Hence, by Theorem 29.i, $\vdash AA \approx A$.

For (iii): by Theorem 17.ii, we have both $\vdash AB \supset BA$ and $\vdash BA \supset AB$. Hence, by Theorem 29.i, $\vdash AB \approx BA$.

For (iv): by CP2, we have $\vdash \overline{\overline{A}} \supset A$. Moreover, by Theorem 21.iii, we have $\vdash A \supset \overline{\overline{A}}$. Hence, by Theorem 29.i, $\vdash \overline{\overline{A}} \approx A$. □

In the remainder of this section, we establish that the calculus $\vdash$ is complete with respect to the class of auto-Boolean algebras. This completeness proof will rely on the auto-Boolean completeness of the coincidence calculus developed by Leibniz in the *Generales inquisitiones*. The crucial step in the proof will be to show that all the theorems of the latter calculus can be reproduced in the containment calculus $\vdash$. To this end, we first specify the language $\mathcal{L}_=$ of Leibniz's coincidence calculus:

Definition 32. The terms and propositions of the language $\mathcal{L}_=$ are defined as follows:

(1) Every simple term of the language $\mathcal{L}_\supset$ is a term.
(2) If A and B are terms, then AB is a term.
(3) If A is a term, then $\overline{A}$ is a term.
(4) If A and B are terms, then $A = B$ is a proposition.
(5) If $A = B$ is a proposition, then $\ulcorner A = B\urcorner$ is a term.

The following two definitions describe how the terms and propositions of $\mathcal{L}_\supset$ can be translated into the language $\mathcal{L}_=$ and vice versa:

Definition 33. The function σ maps every term or proposition of $\mathcal{L}_\supset$ to a term or proposition of $\mathcal{L}_=$ as follows:

(1) If A is a simple term of $\mathcal{L}_\supset$, then $\sigma(A)$ is the term A.
(2) $\sigma(AB)$ is the term $\sigma(A)\sigma(B)$.
(3) $\sigma(\overline{A})$ is the term $\overline{\sigma(A)}$.

(4) $\sigma(A \supset B)$ is the proposition $\sigma(A) = \sigma(A)\sigma(B)$.
(5) $\sigma(\ulcorner\varphi\urcorner)$ is the term $\ulcorner\sigma(\varphi)\urcorner$.

If Γ is a set of propositions of $\mathcal{L}_\supset$, we write $\sigma(\Gamma)$ for the set $\{\sigma(\varphi) : \varphi \in \Gamma\}$.

Definition 34. The function τ maps every term or proposition of $\mathcal{L}_=$ to a term or proposition of $\mathcal{L}_\supset$ as follows:

(1) If A is a simple term of $\mathcal{L}_\supset$, then $\tau(A)$ is the term A.
(2) $\tau(AB)$ is the term $\tau(A)\tau(B)$.
(3) $\tau(\overline{A})$ is the term $\overline{\tau(A)}$.
(4) $\tau(A = B)$ is the proposition $\tau(A) \approx \tau(B)$.
(5) $\tau(\ulcorner\varphi\urcorner)$ is the term $\ulcorner\tau(\varphi)\urcorner$.

If Γ is a set of propositions of $\mathcal{L}_=$, we write $\tau(\Gamma)$ for the set $\{\tau(\varphi) : \varphi \in \Gamma\}$.

Theorem 35. *For any term A and any proposition φ of $\mathcal{L}_\supset$:*

(i) $\vdash A \supset \tau(\sigma(A))$ *and* $\vdash \tau(\sigma(A)) \supset A$
(ii) $\varphi \dashv\vdash \tau(\sigma(\varphi))$

Proof. The proof proceeds by mutual induction on the structure of the terms and propositions of $\mathcal{L}_\supset$ (see Definition 13). First, suppose that A is a simple term of $\mathcal{L}_\supset$. Then $\tau(\sigma(A))$ just is the term A, and so the claim follows by C1.

Next, suppose that the claim holds for the terms A and B. By Theorem 4, we have:

$$\vdash AB \supset \tau(\sigma(A))\tau(\sigma(B))$$
$$\vdash \tau(\sigma(A))\tau(\sigma(B)) \supset AB$$

But since $\tau(\sigma(A))\tau(\sigma(B))$ is the term $\tau(\sigma(AB))$, the claim holds for the term AB. Moreover, by CP1, we have:

$$\vdash \overline{A} \supset \overline{\tau(\sigma(A))}$$
$$\vdash \overline{\tau(\sigma(A))} \supset \overline{A}$$

But since $\overline{\tau(\sigma(A))}$ is the term $\tau(\sigma(\overline{A}))$, the claim holds for the term $\overline{A}$.

Next, given that the claim holds for the terms A and B, it follows by the rule of substitution for containment established in Theorem 15 that:

$$A \approx AB \dashv\vdash \tau(\sigma(A)) \approx \tau(\sigma(A))\tau(\sigma(B))$$

Hence, by Theorem 31.i, we have:

$$A \supset B \dashv\vdash \tau(\sigma(A)) \approx \tau(\sigma(A))\tau(\sigma(B))$$

But since $\tau(\sigma(A)) \approx \tau(\sigma(A))\tau(\sigma(B))$ is the proposition $\tau(\sigma(A \supset B))$, the claim holds for the proposition $A \supset B$.

Finally, suppose that the claim holds for the proposition φ. By Theorem 27.ii, we have:

$$\vdash \ulcorner\varphi\urcorner \supset \ulcorner\tau(\sigma(\varphi))\urcorner$$
$$\vdash \ulcorner\tau(\sigma(\varphi))\urcorner \supset \ulcorner\varphi\urcorner$$

But since $\ulcorner\tau(\sigma(\varphi))\urcorner$ is the term $\tau(\sigma(\ulcorner\varphi\urcorner))$, the claim holds for the term $\ulcorner\varphi\urcorner$. This completes the induction. □

We now introduce the coincidence calculus, $\Vdash$, developed by Leibniz in the *Generales inquisitiones* (as reconstructed in Malink & Vasudevan 2016: 696–706):

Definition 36. The calculus $\Vdash$ is the smallest calculus in the language $\mathcal{L}_=$ such that:

(P1) $\Vdash AA = A$

(P2) $\Vdash AB = BA$

(P3) $\Vdash \overline{\overline{A}} = A$

(P4) $A\overline{B} = A\overline{B}\,\overline{A\overline{B}} \dashv\Vdash A = AB$

(P5) $\Vdash \overline{\ulcorner\varphi\urcorner} = \ulcorner \ulcorner\varphi\urcorner = \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner$

(P6) $\varphi \Vdash \psi$ iff $\Vdash \ulcorner\varphi\urcorner = \ulcorner\varphi\urcorner\ulcorner\psi\urcorner$

(P7) $A = B, \varphi \Vdash \varphi^*$, where φ^* is the result of substituting B for an occurrence of A, or vice versa, in φ.

The following result states that all theorems of the coincidence calculus $\Vdash$ can be reproduced under the translation τ in the containment calculus $\vdash$:

Theorem 37. *For any set of propositions Γ of $\mathcal{L}_=$ and any proposition φ of $\mathcal{L}_=$: if $\Gamma \Vdash \varphi$, then $\tau(\Gamma) \vdash \tau(\varphi)$.*

Proof. The proof proceeds by induction on the definition of the coincidence calculus $\Vdash$ (Definition 36). For P1–P6, it suffices to show that for any terms A, B of $\mathcal{L}_\supset$ and any propositions φ, ψ of $\mathcal{L}_\supset$:

(i) $\vdash AA \approx A$

(ii) $\vdash AB \approx BA$

(iii) $\vdash \overline{\overline{A}} \approx A$

(iv) $A\overline{B} \approx A\overline{B}\,\overline{A\overline{B}} \dashv\vdash A \approx AB$

(v) $\vdash \overline{\ulcorner\varphi\urcorner} \approx \ulcorner\ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner$

(vi) $\varphi \vdash \psi$ iff $\vdash \ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\ulcorner\psi\urcorner$

Claims (i)–(iii) are stated in Theorem 31.ii–iv. Claim (iv) follows by Theorems 21.iv and 31.i. For claim (v), by Theorems 31.i and 27.ii, we have:

$$\vdash \ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\varphi\urcorner}\urcorner \supset \ulcorner\ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner$$
$$\vdash \ulcorner\ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner \supset \ulcorner\ulcorner\varphi\urcorner \supset \overline{\ulcorner\varphi\urcorner}\urcorner$$

Hence, by PT1, PT2, and C4, we have:

$$\vdash \overline{\ulcorner\varphi\urcorner} \supset \ulcorner\ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner$$
$$\vdash \ulcorner\ulcorner\varphi\urcorner \approx \ulcorner\varphi\urcorner\overline{\ulcorner\varphi\urcorner}\urcorner \supset \overline{\ulcorner\varphi\urcorner}$$

Claim (v) then follows by Theorem 29.i.

Claim (vi) follows by Theorems 27.ii and 31.i.

Finally, for P7 we must show that, for any terms A, B of $\mathcal{L}_=$ and any proposition φ of $\mathcal{L}_=$: if φ^* is the result of substituting B for an occurrence of A, or vice versa, in φ, then $\tau(A) \approx \tau(B), \tau(\varphi) \vdash \tau(\varphi^*)$. To see this, we first note that, given Definition 34, if φ^* is the result of substituting A for an occurrence of B, or vice versa, in φ, then $\tau(\varphi^*)$ is the result of substituting $\tau(A)$ for one or more occurrences of $\tau(B)$, or vice versa, in $\tau(\varphi)$. Moreover, it follows by Theorems 15 and 29.ii–iii that, if $\tau(\varphi^*)$ is the result of substituting $\tau(A)$ for one or more occurrences of $\tau(B)$, or vice versa, in $\tau(\varphi)$, then $\tau(A) \approx \tau(B), \tau(\varphi) \vdash \tau(\varphi^*)$. This completes the induction. □

We now introduce an algebraic semantics for the language $\mathcal{L}_=$:

Definition 38. An interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_=$ consists of (i) an algebraic structure $\mathfrak{A} = \langle \mathbb{A}, \wedge, ' \rangle$, where $\mathbb{A}$ is a nonempty set, $\wedge$ is a binary operation on $\mathbb{A}$, and $'$ is a unary operation on $\mathbb{A}$; and (ii) a function μ mapping each term of $\mathcal{L}_=$ to an element of $\mathbb{A}$ such that:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$
$$\mu\left(\overline{A}\right) = \mu(A)'$$

A proposition $A = B$ of $\mathcal{L}_=$ is satisfied in an interpretation $(\mathfrak{A}, \mu)$ iff $\mu(A) = \mu(B)$. We write $(\mathfrak{A}, \mu) \Vdash\!\models A = B$ to indicate that $(\mathfrak{A}, \mu)$ satisfies $A = B$.

Definition 39. An interpretation $(\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ of $\mathcal{L}_=$ is an auto-Boolean interpretation iff $\langle \mathbb{A}, \wedge, ' \rangle$ is a Boolean algebra such that, for any terms A, B of $\mathcal{L}_=$:

$$\mu\left(\ulcorner A = B \urcorner\right) = \begin{cases} 1 & \text{if } \mu(A) = \mu(B) \\ 0 & \text{otherwise} \end{cases}$$

Here, 1 and 0 are the top and bottom elements of the Boolean algebra $\langle \mathbb{A}, \wedge, ' \rangle$, respectively. If Γ is a set of propositions of $\mathcal{L}_=$ and φ a proposition of $\mathcal{L}_=$, we write $\Gamma \Vdash\!\models \varphi$ to indicate that, for any auto-Boolean interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_=$: if $(\mathfrak{A}, \mu) \Vdash\!\models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \Vdash\!\models \varphi$.

The following theorem asserts that the coincidence calculus $\Vdash$ is complete with respect to the class of auto-Boolean interpretations of $\mathcal{L}_=$:

Theorem 40. *For any set of propositions Γ of $\mathcal{L}_=$ and any proposition φ of $\mathcal{L}_=$: if $\Gamma \Vdash\!\models \varphi$, then $\Gamma \Vdash \varphi$.*

The proof of this completeness theorem is given in Malink & Vasudevan 2016: 744–8 (Theorem 4.94).

With this completeness theorem for the coincidence calculus $\Vdash$ in hand, we are now in a position to establish the auto-Boolean completeness of the cointainment calculus $\vdash$. To this end, we first introduce an algebraic semantics for the language $\mathcal{L}_\supset$:

Definition 41. An interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_\supset$ consists of (i) an algebraic structure $\mathfrak{A} = \langle \mathbb{A}, \wedge, ' \rangle$, where $\mathbb{A}$ is a nonempty set, $\wedge$ is a binary operation on $\mathbb{A}$, and $'$ is a unary operation on $\mathbb{A}$; and (ii) a function μ mapping each term of $\mathcal{L}_\supset$ to an element of $\mathbb{A}$ such that:

$$\mu(AB) = \mu(A) \wedge \mu(B)$$
$$\mu\left(\overline{A}\right) = \mu(A)'$$

A proposition $A \supset B$ of $\mathcal{L}_\supset$ is satisfied in an interpretation $(\mathfrak{A}, \mu)$ iff $\mu(A) = \mu(A) \wedge \mu(B)$. We write $(\mathfrak{A}, \mu) \models A \supset B$ to indicate that $(\mathfrak{A}, \mu)$ satisfies $A \supset B$.

Definition 42. An interpretation $(\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ of $\mathcal{L}_\supset$ is an auto-Boolean interpretation iff $\langle \mathbb{A}, \wedge, ' \rangle$ is a Boolean algebra such that, for any terms A, B of $\mathcal{L}_\supset$:

$$\mu\left(\ulcorner A \supset B \urcorner\right) = \begin{cases} 1 & \text{if } \mu(A) = \mu(A) \wedge \mu(B) \\ 0 & \text{otherwise} \end{cases}$$

Here, 1 and 0 are the top and bottom elements of the Boolean algebra $\langle \mathbb{A}, \wedge, ' \rangle$, respectively. If Γ is a set of propositions of $\mathcal{L}_\supset$ and φ a proposition of $\mathcal{L}_\supset$, we write $\Gamma \models \varphi$ to indicate that, for any auto-Boolean interpretation $(\mathfrak{A}, \mu)$ of $\mathcal{L}_\supset$: if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.

Theorem 43. *For any set of propositions Γ of $\mathcal{L}_\supset$ and any proposition φ of $\mathcal{L}_\supset$: if $\Gamma \models \varphi$, then $\sigma(\Gamma) \Vdash\!\models \sigma(\varphi)$.*

Proof. Let $(\mathfrak{A}, \mu) = (\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ be an auto-Boolean interpretation of $\mathcal{L}_=$, and let μ^* be the function mapping each term of $\mathcal{L}_\supset$ to an element of $\mathbb{A}$ defined by:

$$\mu^*(A) = \mu(\sigma(A))$$

We then have:

$$\begin{aligned}
\mu^*(\ulcorner A \supset B \urcorner) &= \mu(\sigma(\ulcorner A \supset B \urcorner)) \\
&= \mu(\ulcorner \sigma(A \supset B) \urcorner) \\
&= \mu(\ulcorner \sigma(A) = \sigma(A)\sigma(B) \urcorner) \\
&= \begin{cases} 1 & \text{if } \mu(\sigma(A)) = \mu(\sigma(A)\sigma(B)) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \mu(\sigma(A)) = \mu(\sigma(A)) \wedge \mu(\sigma(B)) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \mu^*(A) = \mu^*(A) \wedge \mu^*(B) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Hence, by Definition 42, $(\mathfrak{A}, \mu^*)$ is an auto-Boolean interpretation of $\mathcal{L}_\supset$.

Now, suppose that $\Gamma \models \varphi$ and that $(\mathfrak{A}, \mu) \Vdash\!\models \psi$ for all $\psi \in \sigma(\Gamma)$. By Definition 41 and the definition of μ^*, we have for any proposition ψ of $\mathcal{L}_\supset$: $(\mathfrak{A}, \mu) \Vdash\!\models \sigma(\psi)$ iff $(\mathfrak{A}, \mu^*) \models \psi$. Hence, $(\mathfrak{A}, \mu^*) \models \psi$ for all $\psi \in \Gamma$. But since $(\mathfrak{A}, \mu^*)$ is an auto-Boolean interpretation of $\mathcal{L}_\supset$, it follows that $(\mathfrak{A}, \mu^*) \models \varphi$. Thus, by the above biconditional, $(\mathfrak{A}, \mu) \Vdash\!\models \sigma(\varphi)$. So, if $(\mathfrak{A}, \mu) \Vdash\!\models \psi$ for all $\psi \in \sigma(\Gamma)$, then $(\mathfrak{A}, \mu) \Vdash\!\models \sigma(\varphi)$. But since $(\mathfrak{A}, \mu)$ was an arbitrary auto-Boolean interpretation of $\mathcal{L}_=$, this means that $\sigma(\Gamma) \Vdash\!\models \sigma(\varphi)$. □

The following theorem asserts that the containment calculus $\vdash$ is sound and complete with respect to the class of auto-Boolean interpretations of $\mathcal{L}_\supset$:

Theorem 44. *For any set of propositions Γ of $\mathcal{L}_\supset$ and any proposition φ of $\mathcal{L}_\supset$: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.*

Proof. The left-to-right direction follows straightforwardly from the fact that the principles C1–C5, CP1–CP3, and PT1–PT3 are satisfied in every auto-Boolean interpretation.

For the right-to-left direction, suppose that $\Gamma \models \varphi$. Then, by Theorem 43, $\sigma(\Gamma) \Vdash\!\models \sigma(\varphi)$. By Theorem 40, $\sigma(\Gamma) \Vdash \sigma(\varphi)$. By Theorem 37, $\tau(\sigma(\Gamma)) \vdash \tau(\sigma(\varphi))$. Hence, by Theorem 35.ii, it follows that $\Gamma \vdash \varphi$. □

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