

Dynamic Campaign Spending*

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Abstract

We build a model of electoral campaigning in which two office-motivated candidates each allocate a budget over time to affect their relative popularity, which evolves as a mean-reverting stochastic process. In equilibrium the ratio of spending by each candidate equals the ratio of their available budgets in every period. This result holds under a wide range of specifications of the model. We characterize the path of spending over time as a function of the parameters of the popularity process. We then use this relationship to recover estimates of the decay rate in the popularity process for U.S. elections from 2000-2014 and find substantial weekly decay rates well above 50%, consistent with the estimates obtained using different approaches by the literature on political advertising.

1 Introduction

It is now well-established that political advertising has positive effects on support for the advertising candidate, but that these effects decay rapidly over time. In a famous field experiment conducted during the 2006 Texas gubernatorial election, for example, Gerber et al. (2011) find that the effects of political advertising on television almost completely disappear a week after the ads are aired. Similarly, examining survey data Hill et al. (2013) find the weekly decay rate in political advertising in subnational U.S.

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elections to be between 70% and 95%. These decay rates for political advertising are substantially higher than decay rates for non-political ads.¹

Given these high decay rates for political advertising, the question we ask in this paper is: How should strategic candidates optimally time their spending on political advertising (and other persuasion efforts) in the run-up to the election?

To answer this question, we build a simple dynamic allocation model in which two candidates, 1 and 2, allocate their stock of available resources across a finite number of periods to influence the movement of their relative popularity, and eventually win the election.² The candidates begin the game with one being possibly more popular than the other. At each moment in time, relative popularity may go up, meaning that candidate 1’s popularity increases relative to candidate 2’s popularity; or it may go down. Relative popularity evolves between periods according to a (possibly) mean-reverting Brownian motion—the Ornstein-Uhlenbeck process—so that the next period’s starting level of relative popularity is normally distributed with a fixed variance and a mean that is the weighted average of the current level of relative popularity and the long-run mean of the process. In the baseline specification of the model, we assume that the long-run mean of the process that governs the evolution of relative popularity between consecutive periods depends on the candidates’ spending decisions through the ratio of their spending levels. At the final date, an election takes place and the more popular candidate wins office. Money left over has no value, so the game is zero-sum.

The solution to the optimal spending decision rests on a key result, which we call the “equal spending ratio result:” at every history, the two candidates spend the same fraction of their remaining budgets. This result is robust across various extensions and alternative specifications of the baseline model. This includes extensions in which (i) the long run mean of the popularity process is affected not by the ratio of the candidates’ spending levels, but by differences in (nonlinear) transformations of their

¹For example, Dubé et al. (2005) study advertising carry-over in the frozen food industry, where firms build a capital of “goodwill” through ads, which decays over time. They report a half life of 6 weeks in the effect of advertising, which corresponds to a weekly decay rate of about 12%. See also Leone (1995) and Tellis et al. (2005) for other studies in the marketing literature. See DellaVigna and Gentzkow (2010), Kalla and Broockman (2018), Jacobson (2015) and the references in these papers for the state of current knowledge on the effects of political advertising, and persuasion more generally.

²A key premise of our model is that advertising can influence elections. For recent evidence on this, see Spenkuch and Toniatti (2018) who leverage a natural experiment to show that ads affect vote shares but (surprisingly) do not affect aggregate turnout, and Martin (2014) who estimates the persuasive and informative channels of TV ads, and finds evidence for both channels with the persuasive channel being twice as large as the informative channel.

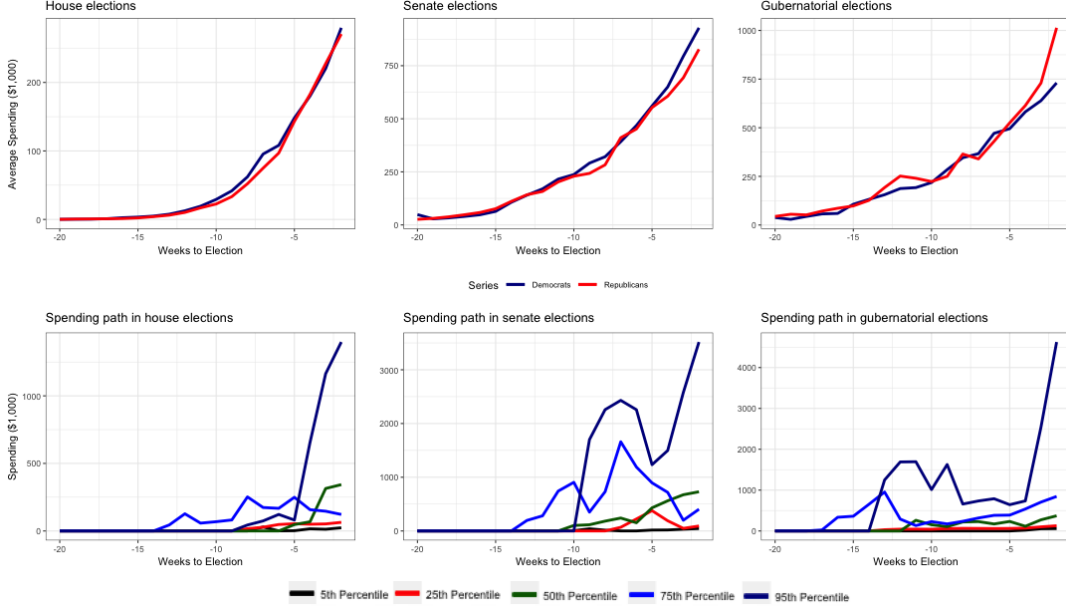


Figure 1: Upper figures are average spending paths by Democrats and Republicans on TV ads in “competitive” House, Senate and gubernatorial races in the period 2000-2014. These are elections in which both candidates spent a positive amount; see Section 5.1 for the source of these data, and more details. Bottom figures are spending paths for 5th, 25th, 50th, 75th, and 95th percentile candidates in terms of total money spent in the corresponding elections of the upper panel.

levels of spending, (ii) the candidates’ available budgets evolve over time in response to relative popularity, and (iii) electoral competition is over multiple districts.

For our baseline model, the equal spending ratio result facilitates a clean characterization of the unique equilibrium path of spending over time as a function of the popularity process. The equilibrium ratio of spending by either candidate in any two consecutive periods equals $e^{\lambda\Delta}$, where λ is the speed of mean reversion of the popularity process, and Δ the time interval between periods. This implies that when $\lambda = 0$ (the case of no mean-reversion) the candidates spread their resources evenly across periods. When $\lambda > 0$, popularity leads tend to decay between consecutive periods at the rate $1 - e^{-\lambda\Delta}$, and in this case, candidates increase their spending over time. For higher values of λ they spend more towards the end of the race and less in the early stages. This establishes a one-to-one relationship between the decay rate and the equilibrium spending path, holding fixed the time interval between periods of action.

The fact that spending increases over time when popularity leads tend to decay rationalizes the pattern of spending in real-life elections. Figure 1 shows the pattern of TV ad spending over time for candidates in U.S. House, Senate and gubernatorial

elections over the period 2000-2014. The upper figures show that the average spending patterns for Democrats and Republicans in these races are nearly identical, suggesting that the equal spending ratio result holds “on average.” The lower figures show that candidates tend to increase their spending over time ahead of the election date, ramping it up in the final weeks, especially in contests that see the highest spending levels.

These patterns are not only qualitatively consistent with the predictions of our model, they also appear to be quantitatively consistent. To show this, we use the one-to-one relationship between the decay rate and the shape of the equilibrium spending path to recover the implied decay rate—i.e., the decay rate that best fits the patterns of spending observed in the data. We find that spending patterns are remarkably consistent with the high estimates of the decay rate coming out of the prior work mentioned above. In House elections, for example, our point estimate for the average weekly decay rate of a polling lead is 88%. In Senate and gubernatorial elections, these are 74% and 73%. We also compare these estimates from spending data to direct estimates of the decay rate from polling averages, despite polling data being very sparse. We find that the two estimates are very close, though decay rates estimated from polling data are typically a few percentage points higher than the ones recovered from spending data.

Our paper relates to the prior literature on campaigning, which typically focuses on other aspects of the contest. Kawai and Sunada (2015), for example, build on the work of Erikson and Palfrey (1993, 2000) to estimate a model of fund-raising and campaigning in which the inter-temporal resource allocation decisions that candidates make are across different elections rather than across periods in the run-up to a particular election. de Roos and Sarafidis (2018) explain how candidates that have won past races may enjoy “momentum,” which results from a complementarity between prior electoral success and current spending.³ Meirowitz (2008) studies a static model to show how asymmetries in the cost of effort can explain the incumbency advantage. Polborn and David (2004) and Skaperdas and Grofman (1995) also examine static campaigning models in which candidates must choose between positive or negative advertising.⁴ Iaryczower et al. (2017) estimate a model in which campaign spending weakens electoral accountability assuming

³Other dynamic models of electoral campaigns in which candidates enjoy momentum—such as Callander (2007), Knight and Schiff (2010), Ali and Kartik (2012)—are models of *sequential voting*.

⁴Other static models of campaigning include Prat (2002) and Coate (2004), who investigate how one-shot campaign advertising financed by interest groups can affect elections and voter welfare, and Krassa and Polborn (2010) who study a model in which candidates compete on the level of effort that they apply to different policy areas. Prato and Wolton (2018) study the effects of reputation and partisan imbalances on the electoral outcome.

that the cost of spending is exogenous rather than subject to an inter-temporal budget constraint. Garcia-Jimeno and Yildirim (2017) estimate a dynamic model of campaigning in which candidates decide how to target their campaigns taking into account the strategic role of the media in communicating with voters. Gul and Pesendorfer (2012) study a model of campaigning in which candidates provide information to voters over time, and face the strategic timing decision of when to stop.

Our paper also relates to the literature on dynamic contests (see Konrad et al., 2009, and Vojnović, 2016, for reviews of this literature). In this literature, Gross and Wagner (1950) study a continuous Blotto game; Harris and Vickers (1985, 1987), Klumpp and Polborn (2006) and Konrad and Kovenock (2009) study models of races; and Glazer and Hassin (2000) and Hinnesaar (2018) study sequential contests. Ours is the first paper, to our knowledge, that studies a dynamic strategic allocation problem.

2 Model

Consider the following complete information *dynamic campaigning game* between two candidates, $i = 1, 2$, ahead of an election. Time runs continuously from 0 to T and candidates take actions at times in $\mathcal{T} := \{0, \Delta, 2\Delta, \dots, (N-1)\Delta\}$, with $\Delta := T/N$ being the time interval between consecutive actions. We identify these times with N discrete periods indexed by $n \in \{0, \dots, N-1\}$. For all $t \in [0, T]$, we use $\underline{t} := \max\{\tau \in \mathcal{T} : \tau \leq t\}$ to denote the last time that the candidates took actions.

At the start of the game the candidates are endowed with positive resource stocks, $X_0 \geq 0$ and $Y_0 \geq 0$ respectively for candidates 1 and 2.⁵ Candidates allocate their resources across periods to influence changes in their relative *popularity*. Relative popularity at time t is measured by a continuous random variable $Z_t \in \mathbb{R}$ whose realization at time t is denoted by z_t . We will interpret this as a measure of candidate 1's lead in the polls. If $z_t > 0$, then candidate 1 is ahead of candidate 2. If $z_t < 0$, then candidate 2 is ahead; and if $z_t = 0$, it is a dead heat. We assume that at the beginning of the game, relative popularity is equal to $z_0 \in \mathbb{R}$.

At any time $t \in \mathcal{T}$, the candidates simultaneously decide how much of their resource stock to invest in influencing their future relative popularity. Candidate 1's investment

⁵Although candidates raise funds over time, our assumption that they start with a fixed stock is tantamount to assuming that they can forecast how much will be available to them. In fact, some large donors make pledges early on and disburse their funds as they are needed over time. Nevertheless, in Section 4.2 we relax this assumption and consider an extension of the model in which the candidates' resources evolve over time in response to the candidates' relative popularity.

is denoted x_t while candidate 2's is denoted y_t . The size of the resource stock that is available to candidate 1 at time $t \in \mathcal{T}$ is denoted $X_t = X_0 - \sum_{\tau \in \{t' \in \mathcal{T}: t' < t\}} x_\tau$ and that available to candidate 2 is $Y_t = Y_0 - \sum_{\tau \in \{t' \in \mathcal{T}: t' < t\}} y_\tau$. At every time $t \in \mathcal{T}$, budget constraints must be satisfied, so $x_t \leq X_t$ and $y_t \leq Y_t$.

Throughout, we will maintain the assumption that for all times t , the evolution of popularity is governed by the following Brownian motion:

$$dZ_t = (q(x_t/y_t) - \lambda Z_t) dt + \sigma dW_t \quad (1)$$

where $\lambda \geq 0$ and $\sigma > 0$ are parameters and $q(\cdot)$ is a strictly increasing, strictly concave function on $[0, \infty)$. Thus, the drift of popularity depends on the ratio of investments through the function $q(\cdot)$, and it may be mean-reverting if $\lambda > 0$.⁶

Finally, we assume that the *winner* of the election collects a payoff of 1 while the *loser* collects a payoff of 0. For analytical convenience, we make the assumption that if either candidate $i = 1, 2$ invests an amount equal to 0 at any time in \mathcal{T} , then the game ends immediately. If $j \neq i$ invested a positive amount at that time, then j is the winner while if j also invested 0 at that time, then each candidate wins with equal probability.⁷ If both candidates invest a positive amount at every time $t \in \mathcal{T}$, then the game only ends at time T , with candidate 1 winning if $z_T > 0$, losing if $z_T < 0$, and both candidates winning with equal probability if $z_T = 0$. In other words, if the game does not end before time T , then the winner is the candidate that is more popular at time T , and if they are equally popular they win with equal probability.

3 Analysis

Since the game is in continuous time, strategies must be measurable with respect to the filtration generated by W_t . However, since candidates take actions only at discrete times, we will forgo this additional formalism and treat the game as a game in discrete

⁶If $\lambda = 0$ the process governing the evolution of popularity in the interval between two consecutive times in \mathcal{T} is a standard Brownian motion—the continuous time limit of the random walk in which popularity goes up with probability $\frac{1}{2} + q(x_t/y_t)\sqrt{\Delta}$ and goes down with complementary probability. If $\lambda > 0$, instead, popularity evolves in this interval according to the Ornstein-Uhlenbeck process, under which the leading candidate's lead has a tendency to decay.

⁷These assumptions close the model since q is undefined if the denominator of its argument is 0. The assumptions also guarantee that Z_t follows an Itô process at every history. This model can be considered the limiting case of two different models. One is a model in which the marginal return to investing an ϵ amount of resources starting at 0 goes to infinity. The other is one in which candidates have to spend a minimum amount ϵ in each period to sustain the campaign, and ϵ goes to 0.

time. By our assumption about the popularity process in (1), the distribution of $Z_{t+\Delta}$ at any time $t \in \mathcal{T}$, conditional on (x_t, y_t, z_t) , is normal with constant variance and a mean that is a weighted sum of $q(x_t/y_t)$ and z_t ; specifically,

$$Z_{t+\Delta} \mid (x_t, y_t, z_t) \sim \begin{cases} \mathcal{N}(q(x_t/y_t)\Delta + z_t, \sigma^2\Delta) & \text{if } \lambda = 0 \\ \mathcal{N}((1 - e^{-\lambda\Delta})q(x_t/y_t)/\lambda + e^{-\lambda\Delta}z_t, \sigma^2(1 - e^{-2\lambda\Delta})/2\lambda) & \text{if } \lambda > 0 \end{cases} \quad (2)$$

where $\mathcal{N}(\cdot, \cdot)$ denotes the normal distribution whose first component is mean and second is variance. Note that the mean and variance of $Z_{t+\Delta}$ in the $\lambda = 0$ case correspond to the limits as $\lambda \rightarrow 0$ of the mean and variance in the $\lambda > 0$ case.

The model is therefore strategically equivalent to a discrete time model in which relative popularity is a state variable that transitions over discrete periods, and in each period it is normally distributed with a constant variance and a mean that depends on the popularity in the last period and on the ratio of candidates' spending.

With this, our equilibrium concept is subgame perfect Nash equilibrium (SPE) in pure strategies. We will refer to this concept succinctly as “equilibrium.”

In the remainder of this section, we establish results on the paths of spending and popularity over time. We begin with a key observation, established in Section 3.1 below, that facilitates the analysis: on the equilibrium path of play, the ratio of the candidates' spending, x_t/y_t , is constant across all periods $t \in \mathcal{T}$.

3.1 Equal Spending Ratios

We refer to the ratio of a candidate's current spending to current budget as that candidate's *spending ratio*. For candidate 1 this is x_t/X_t and for candidate 2 it is y_t/Y_t . We will show that on the equilibrium path, these two ratios equal each other at every time t that the candidates make spending decisions.

Consider any time $t \in \mathcal{T}$ at which the game has not ended and candidates have to make their investment decisions. If $t = (N - 1)\Delta$, then both candidates will spend their remaining budgets, i.e. $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Therefore, both candidates' spending ratios equal 1.

Now suppose that $t < (N - 1)\Delta$ and assume that the stock of resources available to the two candidates are $X_t, Y_t > 0$.⁸ Also, suppose that after the candidates choose their spending levels x_t and y_t , the probability that candidate 1 will win the election at time T

⁸Recall that if either X_t or Y_t equal 0, the game will end at time t : either both candidates have no money to spend, or the one with a positive budget will spend any positive amount and win.

when evaluated at time $t+\Delta$ depends on $X_{t+\Delta} = X_t - x_t$ and $Y_{t+\Delta} = Y_t - y_t$ only through the ratio $(X_t - x_t)/(Y_t - y_t)$. Denote this probability by $\pi_t((X_t - x_t)/(Y_t - y_t), z_{t+\Delta})$. Further, let $F(z_{t+\Delta}|x_t/y_t, z_t)$ denote the c.d.f. of $Z_{t+\Delta}$ conditional on (x_t, y_t, z_t) , and let $f(z_{t+\Delta}|x_t/y_t, z_t)$ denote the associated p.d.f. (Recall that these are normal distributions that depend on x_t and y_t only through the ratio x_t/y_t .)

If both candidates spend a positive amount in every period, candidate 1's expected payoff at time t is given by

$$\Pi_t(x_t, y_t|X_t, Y_t, z_t) = \int \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right) dF(z_{t+\Delta}|x_t/y_t, z_t)$$

and candidate 2's expected payoff is $1 - \Pi_t(x_t, y_t|X_t, Y_t, z_t)$. The pair of necessary first order conditions for interior equilibrium values of x_t and y_t are

$$\begin{aligned} \frac{1}{y_t} \int \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right) \frac{\partial f(z_{t+\Delta}|x_t/y_t, z_t)}{\partial(x_t/y_t)} dz_{t+\Delta} &= \\ &= \frac{1}{Y_t - y_t} \int \frac{\partial \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right)}{\partial\left(\frac{X_t - x_t}{Y_t - y_t}\right)} dF(z_{t+\Delta}|x_t/y_t, z_t); \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{x_t}{(y_t)^2} \int \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right) \frac{\partial f(z_{t+\Delta}|x_t/y_t, z_t)}{\partial(x_t/y_t)} dz_{t+\Delta} &= \\ &= \frac{X_t - x_t}{(Y_t - y_t)^2} \int \frac{\partial \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right)}{\partial\left(\frac{X_t - x_t}{Y_t - y_t}\right)} dF(z_{t+\Delta}|x_t/y_t, z_t). \end{aligned} \quad (4)$$

Taking the ratios of the respective left and right hand sides of these equations implies that $x_t/y_t = (X_t - x_t)/(Y_t - y_t)$, or $x_t/y_t = X_t/Y_t$. This observation suggests that our supposition that the remaining budgets $X_t - x_t$ and $Y_t - y_t$ affect continuation payoffs only through their ratio, can be established by induction provided that the second order conditions are satisfied. The main steps in the proof of the following proposition involve establishing these facts. This and all other proofs appear in the Appendix.⁹

Proposition 1. *There exists an essentially unique equilibrium. If $X_t, Y_t > 0$ are the remaining budgets of candidates 1 and 2 at any time $t \in \mathcal{T}$, then in all equilibria,*

$$x_t/X_t = y_t/Y_t.$$

⁹The word ‘‘essentially’’ appears in the proposition below only because the equilibrium is not unique at histories at which either $X_t = 0 < Y_t$ or $X_t > 0 = Y_t$ — histories that do not arise on the path of play. In these cases, the candidate with a positive resource stock may spend any amount in period t and win. Apart from this trivial source of multiplicity, the equilibrium is unique.

The model described so far satisfies two conditions, each one of which is sufficient for the equal spending ratio result of Proposition 1, and which serve as the basis for the generalizations we provide in Section 4 below. The first condition is that there exists a homothetic function $p(x_t, y_t)$ whose ratio of partials with respect to x_t and y_t respectively is invertible, such that for all $t \in \mathcal{T}$ we can write

$$Z_{t+\Delta} = (1 - e^{-\lambda\Delta}) p(x_t, y_t) + e^{-\lambda\Delta} Z_t + \varepsilon_t, \quad (5)$$

where ε_t is a mean-zero normally distributed random variable. This makes the term that depends on (x_t, y_t) linearly separable from the stochastic terms (Z_t, ε_t) . We establish the sufficiency of this condition for the equal spending ratio result in Section 4.1.

The second condition is that the distribution of Z_T given $((x_\tau, y_\tau, z_\tau)_{\tau \leq t-\Delta}, z_t)$ depends on $(x_\tau, y_\tau)_{\tau \geq t}$ only through the ratios $(x_\tau/y_\tau)_{\tau \geq t}$. When this is the case, if $(x_\tau^*, y_\tau^*)_{\tau \geq t}$ is an equilibrium in the continuation game in which the candidates' remaining budgets are $X_t, Y_t > 0$ then $(\theta x_\tau^*, \theta y_\tau^*)_{\tau \geq t}$ must be an equilibrium when the budgets are $\theta X_t, \theta Y_t$, for all $\theta > 0$.¹⁰ This observation serves as the basis for the generalizations of the baseline model that we present in Sections 4.2 and 4.3.

3.2 Equilibrium Spending and Popularity Paths

An immediate corollary of Proposition 1 is a characterization of the process governing the evolution of relative popularity on the equilibrium path.

Corollary 1. *On the equilibrium path, relative popularity follows the process*

$$dZ_t = (q(X_0/Y_0) - \lambda Z_t) dt + \sigma dW_t \quad (6)$$

If $\lambda = 0$, this is a Brownian motion with constant drift $q(X_0/Y_0)$. If $\lambda > 0$, it is the Ornstein-Uhlenbeck process with long-term mean $q(X_0/Y_0)/\lambda$ and speed of reversion λ .

Therefore, when $\lambda > 0$ popularity leads have a tendency to decay towards zero. The instantaneous volatility of the process is σ and the stationary variance is $\sigma^2/2\lambda$.

¹⁰If this were not the case, we could find $(\tilde{x}_\tau)_{\tau \geq t}$ that gives a higher probability of winning to candidate 1 given $(\theta y_\tau^*)_{\tau \geq t}$. Because Z_T is determined by $(x_\tau/y_\tau)_{\tau \geq t}$, this would imply that the distribution of Z_T given $(\tilde{x}_\tau/\theta y_\tau^*)_{\tau \geq t}$ is more favorable to candidate 1 than the distribution given $(\theta x_\tau^*/\theta y_\tau^*)_{\tau \geq t} = (x_\tau^*/y_\tau^*)_{\tau \geq t}$. Because $(\tilde{x}_\tau/\theta y_\tau^*)_{\tau \geq t}$ is a feasible continuation spending path when the budgets are (X_t, Y_t) , this would contradict the optimality of $(x_\tau^*)_{\tau \geq t}$ when candidate 2 plays $(y_\tau^*)_{\tau \geq t}$.

Proposition 1 also enables us to solve, in closed form, for the equilibrium spending ratio at each history.

Proposition 2. *Let $t \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in equilibrium, spending ratios depend only on calendar time, the time interval between consecutive actions, and the speed of reversion λ . In particular,*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \begin{cases} \Delta/(T-t) & \text{if } \lambda = 0 \\ \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}} & \text{if } \lambda > 0 \end{cases}$$

which is continuous at $\lambda = 0$.

Proposition 2 implies that the fraction of their initial budget that each candidate spends in each period $n\Delta$ is the same for both candidates, and so is the ratio of spending in consecutive periods $n\Delta$ and $(n+1)\Delta$; we define these quantities as dependent on n and λ to be, respectively,

$$\gamma_\lambda(n) := \frac{x_{n\Delta}}{X_0} = \frac{y_{n\Delta}}{Y_0} \quad \text{and} \quad r_n(\lambda) := \frac{x_{(n+1)\Delta}}{x_{n\Delta}} = \frac{y_{(n+1)\Delta}}{y_{n\Delta}} \quad (7)$$

If $\lambda = 0$, then Proposition 2 implies that the candidates will spend a fraction $\gamma_0(n) = 1/N$ of their available resources in each period $n\Delta$, and the ratio of spending in consecutive periods is $r_n(0) = 1$. The $\lambda > 0$ case is handled in the following proposition.

Proposition 3. *Fix the number of periods N , total time $T = N\Delta$, and consider the case in which $\lambda > 0$. Then, for all n ,*

$$\gamma_\lambda(n) = \frac{e^{\lambda\Delta} - 1}{e^{\lambda N\Delta} - 1} e^{\lambda\Delta n} \quad \text{and} \quad r_n(\lambda) = e^{\lambda\Delta}.$$

Since $r_n(\lambda)$ is increasing in λ , the shape of $\gamma_\lambda(n)$ is clear: it is increasing in n , and as λ grows it becomes higher for higher values of n and lower for lower values. Figure 2 depicts these properties by plotting $\gamma_\lambda(n)$ for different values of λ . The key property is that as the speed of reversion increases, candidates save even more of their resources for the final stages of the campaign.

The intuition behind these results is straightforward. When $\lambda = 0$, popularity advantages do not decay at all, and candidates equate the marginal benefit of spending against the marginal (opportunity) cost by spending evenly over time. As λ increases, then the marginal benefit of spending early drops since any popularity advantage produced by an

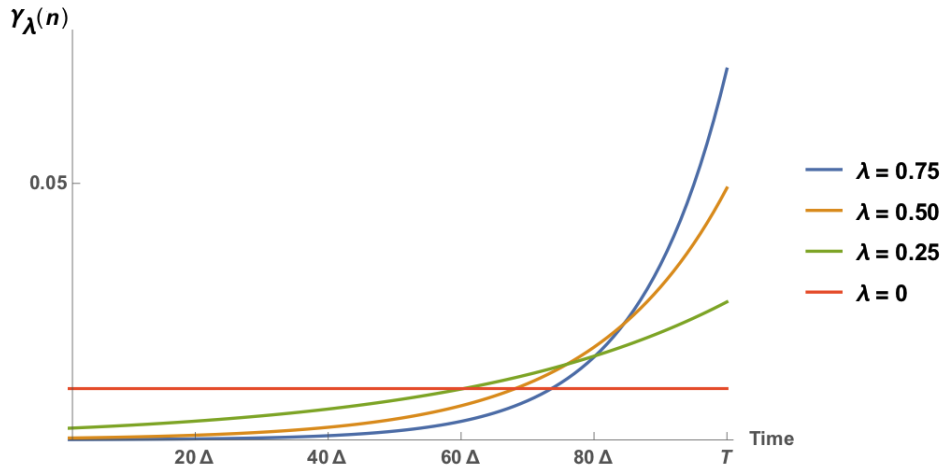


Figure 2: The fraction $\gamma_\lambda(n)$ of initial budget that the candidates spend over time, for $N = 100$ and various values of λ .

early investment has a tendency to decay, where this tendency is greater the greater is λ . In particular, if λ is high then any advantage in popularity that a candidate builds early on is harder to grow or even maintain. This means that candidates have an incentive to invest less in the early stages and more in the later stages of the campaign.

Finally, we can write a clean closed-form expression for the fraction of a candidate's initial budget cumulatively spent at time t by taking the continuous time limit as $\Delta \rightarrow 0$, fixing T . We have

$$\lim_{\Delta \rightarrow 0} \sum_{n\Delta \leq t} \gamma_\lambda(n) = \frac{e^{\lambda t} - 1}{e^{\lambda T} - 1}. \quad (8)$$

4 Robustness and Extensions

In this section, we study the robustness of the equal spending ratio result under various generalizations of the baseline model. Throughout the section, we focus on sufficient conditions for the equal spending ratio result to hold, and say that an equilibrium is *interior* if the first order conditions in (3) and (4) are satisfied at the equilibrium.

4.1 Alternative Specifications

Two of the key implications of the baseline model studied above are the equal spending ratio result of Proposition 1 and the implication of Proposition 2 that the spending ratios x_t/X_t and y_t/Y_t are independent of the past history $(z_\tau)_{\tau \leq t}$ of relative popularity. We show that these results are robust across many possible alternative specifications of the law of motion of relative popularity. In particular, suppose that instead of equation (1), relative popularity evolves according to

$$dZ_t = (p(x_t, y_t) - \lambda Z_t) dt + \sigma dW_t$$

for some twice differentiable real valued function p . This generalizes the baseline model by allowing the drift of the process to depend on spending levels rather than simply the spending ratio, but we continue to assume that the effect of spending is additively separable from the current popularity level.¹¹ It turns out that this separability is sufficient for the spending ratios to be independent of the past history of relative popularity. Under this assumption, equation (5) holds, and we have

$$Z_T = (1 - e^{-\lambda\Delta}) \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} p(x_{n\Delta}, y_{n\Delta}) + z_0 e^{-\lambda N\Delta} + \sum_{n=0}^{N-1} e^{-\lambda\Delta(N-1-n)} \varepsilon_{n\Delta}, \quad (9)$$

where $(\varepsilon_\tau)_{\tau \geq 0}$ are i.i.d. normal shocks with mean 0 and variance $\sigma^2(1 - e^{-2\lambda\Delta})/2\lambda$. Hence, an interior equilibrium exists if $p(\cdot, y)$ is quasiconcave for all y and $p(x, \cdot)$ is quasiconvex for all x . The equilibrium spending profile (x_t, y_t) is notably independent of z_t . Moreover, the equal spending ratio result generalizes under the assumption that p is a homothetic function with an invertible ratio of marginals; specifically—

Assumption A. *There is an invertible function $\psi : (0, \infty) \rightarrow \mathbb{R}$ s.t.*

$$\forall x, y > 0, \quad p_x(x, y) = \psi(x/y) p_y(x, y).$$

Proposition 4. *There is a unique equilibrium if $p(\cdot, y)$ is quasiconcave in all y and $p(x, \cdot)$ is quasiconvex in all x , and the equilibrium is interior. In equilibrium, x_t/X_t and y_t/Y_t are independent of the past history $(z_\tau)_{\tau \leq t}$ of relative popularity. Under Assumption A, the equal spending ratio result also holds: $x_t/X_t = y_t/Y_t$ for all $t \in \mathcal{T}$ s.t. $X_t, Y_t > 0$.*

¹¹Using the result in Karatzas and Shreve (1998) equation (6.30), we can write down sufficient conditions to obtain this separability. Details are available upon request.

Assumption A is satisfied, for example, by $p(x, y) = h(\alpha_1\varphi(x) - \alpha_2\varphi(y))$ where h is a differentiable function, α_1 and α_2 are constants, and φ is a function such that $\varphi'(x) = x^\beta$.¹² Also note that given Z_T from equation (9), at any time $t \in \mathcal{T}$ candidate 1 maximizes $\Pr [Z_T \geq 0 \mid z_t, X_t, Y_t]$ under the constraint $\sum_{n=t/\Delta}^{N-1} x_{n\Delta} \leq X_t$, while candidate 2 minimizes this probability under the constraint $\sum_{n=t/\Delta}^{N-1} y_{n\Delta} \leq Y_t$. Using this fact, we can apply the Euler method from consumer theory to solve the equilibrium for this example, provided the first order conditions are sufficient and h is a homogenous function of degree d for some $d \geq 1$.¹³

The candidates' first order conditions with respect to $x_{n\Delta}$ and $y_{n\Delta}$ for each $n < N-1$ are respectively

$$\begin{aligned} e^{-\lambda\Delta(N-1-n)} x_{n\Delta}^\beta h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) &= x_{(N-1)\Delta}^\beta h'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta})) \\ e^{-\lambda\Delta(N-1-n)} y_{n\Delta}^\beta h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) &= y_{(N-1)\Delta}^\beta h'(\alpha_1\varphi(x_{(N-1)\Delta}) - \alpha_2\varphi(y_{(N-1)\Delta})) \end{aligned}$$

Note that we can recover the equal spending ratio result from taking the ratios of these conditions. To find the equilibrium, we equate the left hand sides of candidate 1's first order conditions for two consecutive periods n and $n+1$ to get

$$e^{-\lambda\Delta} x_{n\Delta}^\beta h'(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta})) = x_{(n+1)\Delta}^\beta h'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) \quad (10)$$

Then, we guess that the consecutive period spending ratio, $\tilde{r}_n(\lambda)$, equals some constant r for both candidates, as in the baseline model. If this guess is correct then

$$\begin{aligned} h'(\alpha_1\varphi(x_{(n+1)\Delta}) - \alpha_2\varphi(y_{(n+1)\Delta})) &= h'(r^{1+\beta}(\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta}))) \\ &= r^{(1+\beta)(d-1)} h'((\alpha_1\varphi(x_{n\Delta}) - \alpha_2\varphi(y_{n\Delta}))) \end{aligned}$$

since $\varphi(x) = x^{1+\beta}/(1+\beta)$ and the derivative of a homogenous function of degree d is a homogenous function of degree $d-1$. Therefore, using this in equation (10), the consecutive period spending ratio for candidate 1 is $r = e^{-\lambda\Delta/[(1+\beta)d-1]}$. The same is true also for candidate 2. This verifies our guess that the consecutive period spending

¹²The assumption holds, defining $\psi(x/y) = -(\alpha_1/\alpha_2)(x/y)^\beta$. Also note that this example also nests our baseline model with $\alpha_1 = \alpha_2 = -\beta = 1$ (so that $\varphi = \log$) and an appropriate choice of h .

¹³If h is the identity, for example, the assumptions needed for an interior equilibrium are satisfied for $\beta < 0$ and $\alpha_1, \alpha_2 > 0$.

ratio is constant over time. The equilibrium spending path is therefore characterized by

$$\tilde{r}_n(\lambda) = \frac{x_{(n+1)\Delta}}{x_{n\Delta}} = \frac{y_{(n+1)\Delta}}{y_{n\Delta}} = e^{-\lambda\Delta/[(1+\beta)d-1]}$$

for all $n < N - 1$. This gives us a parametric generalization for the equilibrium spending path from our baseline model.

The generalization shows that our main results are robust to allowing the popularity process to depend on levels of spending rather than just the ratio of candidates' spending, and they are not driven by a specification of the drift in a neighborhood of zero spending.¹⁴ In the example above, if $\beta = -0.5$, say, then the total dollar amounts spent by the candidates matter, and the drift is insensitive to spending levels close to zero. Moreover, for this specification we can accommodate 0 spending by either or both candidates without assuming, as we did in the baseline model, that the game ends immediately if one of them does not spend a positive amount.¹⁵

We conclude this section with some additional remarks about the robustness of the results above. First, the proof of Proposition 4 in the appendix actually shows that the Nash equilibrium of this extension is unique. Second, since the game is zero-sum and the unique equilibrium is in pure strategies, all of our results are also robust to having the candidates move sequentially within a period, with arbitrary (and possibly stochastic) order of moves across periods. Third, since the equilibrium strategies do not depend on realizations of the relative popularity path, the results are also robust to having the candidates imperfectly and asymmetrically observe the realization of the path of popularity. Fourth, the results are also robust to allowing the final payoffs to depend linearly on Z_T (an assumption that encompasses the case where candidates care not just about winning but also about margin of victory) so long as the game remains zero-sum. Finally, since the model of this section is a generalization of the baseline model, all of these observations apply to the baseline model as well.

¹⁴One concern with the baseline specification in which q is a function of the ratio x_t/y_t of candidates' spending, is that the effect of candidate 1 spending \$2 against candidate 2 spending \$1 on relative popularity is the same as candidate 1 spending \$2 million and candidate 2 spending \$1 million, which seems unreasonable. The extension shows that our key results are not driven by this feature.

¹⁵This also shows that we are not artificially forcing the candidates to spend substantial amounts of their resources early by assuming that they lose immediately if they don't.

4.2 Evolving Budgets

Our baseline model assumes that candidates are endowed with a fixed budget at the start of the game (or they can perfectly forecast how much money they will raise), but in reality the amount of money raised may depend on how well the candidates poll over the campaign cycle. To account for this, we present an extension here in which the resources stock also evolves in a way that depends on the evolution of popularity. We retain all the features of the baseline model except the ones described below.

Candidates start with exogenous budgets X_0 and Y_0 as in the baseline model. However, the budgets now evolve according to the following geometric Brownian motions:

$$\frac{dX_t}{X_t} = az_t dt + \sigma_X dW_t^X \quad \text{and} \quad \frac{dY_t}{Y_t} = bz_t dt + \sigma_Y dW_t^Y$$

where a , b , σ_X and σ_Y are constants, and W_t^X and W_t^Y are Wiener processes. None of our results hinge on it, but we also make the assumption for simplicity that dW_t is independent of dW_t^X and of dW_t^Y , while dW_t^X and dW_t^Y have covariance $\rho \geq 0$.

In this setting, if $b < 0 < a$ then donors raise their support for candidate that is leading in the polls and withdraw support from the one that is trailing. If $a < 0 < b$ then donors channel their resources to the underdog. Popularity therefore feeds back into the budget process. The feedback is positive if $a - b > 0$ and negative if $a - b < 0$. We refer to a and b as the feedback parameters.¹⁶

All other features of the model are exactly the same as in the baseline model, including the process (1) governing the evolution of popularity, though we now assume for analytical tractability that

$$q(x/y) = \log(x/y).$$

Proposition 5. *In the model with evolving budgets, for every N , T , and $\lambda > 0$, there exists $-\eta < 0$ such that whenever $a - b \geq -\eta$, there is an essentially unique equilibrium. For all $t \in \mathcal{T}$, if $X_t, Y_t > 0$, then in equilibrium,*

$$x_t/X_t = y_t/Y_t.$$

¹⁶Also, note that dX_t and dY_t may be negative. One interpretation is that X_t and Y_t are expected total budgets available for the remainder of the campaign, where the expectation is formed at time t . Depending on the level of relative popularity, the candidates revise their expected future inflow of funds and adjust their spending choices accordingly.

To understand the condition $a - b \geq -\eta$, note that when $a < 0 < b$, there is a negative feedback between popularity and the budget flow: a candidate's budget increases when she is less popular than her opponent. The condition $a - b \geq -\eta$ puts a bound on how negative this feedback can be. If this condition is not satisfied, candidates may want to *reduce* their popularity as much as they can in the early stages of the campaign to accumulate a larger war chest to use in the later stages. This could undermine the existence of an equilibrium in pure strategies.

One question that we can ask of this extension is how the distribution of spending over time varies with the feedback parameters a and b that determine the rate of flow of candidates' budgets in response to shifts in relative popularity. In the baseline model, when $\lambda > 0$ the difficulty in maintaining an early lead means that there is a disincentive to spend resources early on. This produces the result that spending is increasing over time. However, in this extension, if $b < 0 < a$ then there is a force working in the other direction: spending to build early leads may be advantageous because it results in faster growth of the war chest, which is valuable for the future. The disincentive to spend early is mitigated by this opposing force, and may even be overturned if a is much larger than b , i.e., if donors have a greater tendency to flock to the leading candidate.

We can establish this intuition formally. Recall that $r_n(\lambda)$ defined in the main text gave the ratio of equilibrium spending in consecutive periods, n and $n + 1$. For this extension with evolving budgets, we define the analogous ratio, \tilde{r}_n , which we show in the appendix depends on a and b only through the difference $a - b$ and is the same for both candidates. Specifically,

$$\tilde{r}_n(\lambda, a - b) = \frac{x_{(n+1)\Delta}/X_{(n+1)\Delta}}{x_{n\Delta}/X_{n\Delta}} = \frac{y_{(n+1)\Delta}/Y_{(n+1)\Delta}}{y_{n\Delta}/Y_{n\Delta}}$$

Proposition 6. *Fix the number of periods N , total time $T = N\Delta$, and consider the case in which $\lambda > 0$. Then, for all n , if $a - b$ is sufficiently small then the ratio $\tilde{r}_n(\lambda, a - b)$ of spending in consecutive periods n and $n + 1$ conditional on the history up to period n is (i) greater than 1, (ii) increasing in λ , and (iii) decreasing in $a - b$.*

The baseline model (with $q(x/y) = \log(x/y)$) is the special case of the model with evolving budgets in which the total budget is constant over time: $a = b = \sigma_X = \sigma_Y = 0$. What Proposition 6 says is that starting with this special case, as we increase the

difference $a - b$ from zero, spending plans becomes more balanced over time: there is a greater incentive to spend in earlier periods of the race than there is if $a = b$.¹⁷

4.3 Multi-district Competition

We now provide an extension to address the possibility that the candidates compete in S winner-take-all districts (rather than a single district) and each must win a certain subset of these to win the electoral contest.¹⁸ This extension is general enough to cover the electoral college for U.S. presidential elections, as well as competition between two parties seeking to control a majoritarian legislature composed of representatives from winner-take-all single-member districts, and other such settings.

Relative popularity in each district s is the random variable Z_t^s with realizations z_t^s , and we assume that the joint distribution of the vector $(Z_{t+1}^s)_{s=1}^S$ depends on $(x_t^s/y_t^s, z_t^s)_{s=1}^S$ only. This allows for correlation of relative popularity across districts.

All other structural features are the same as in the baseline model. In particular, to close this version of the model, we assume that if a candidate stop spending money in a particular district, then she loses the election right away if the other candidate is spending a positive amount in all districts and she wins the election with probability $1/2$ if the other candidate does not campaign in at least one district as well.

Proposition 7. *In any equilibrium of this extension, if $X_t, Y_t > 0$ are the remaining budgets of candidates 1 and 2 at any time $t \in \mathcal{T}$, then for all districts s ,*

$$x_t^s/X_t = y_t^s/Y_t.$$

¹⁷It is also worth commenting on the fact that the results of Proposition 6 do not necessarily hold when $a - b$ is very large. We have examples in which $\tilde{r}_n(\lambda, a - b)$ is *increasing* in $a - b$ for large λ , n , and $a - b$. (One such example is $\lambda = 0.8$, $\Delta = 0.9$, and $n = a - b = 10$.) The intuition behind these examples rests on the fact that when the degree of mean reversion is high, then it is important for candidates to build up a large war chest that they can deploy in the final stages of the race. If the election date is distant and $a - b$ is large, then early spending is mostly for the purpose of building up these resources. But spending too much in any one period, especially an early period, is risky: if the resource stock does not grow (or even if it grows but insufficiently) then there is less money, and hence less opportunity, to grow it in the subsequent periods. Since q is concave, the candidates would like to have many attempts to grow the war chest early on, and this is even more the case as the importance of the relative feedback $a - b$ gets large.

¹⁸For example, if the set of districts is $\mathcal{S} = \{1, \dots, S\}$ then consider any electoral rule such that for all partitions of \mathcal{S} of the form $\{\mathcal{S}_1, \mathcal{S}_2\}$, either candidate 1 wins if he wins all the districts in \mathcal{S}_1 or 2 wins if she wins all the districts in \mathcal{S}_2 . The rule should be monotonic in the sense that for any partitions $\{\mathcal{S}_1, \mathcal{S}_2\}$ and $\{\mathcal{S}'_1, \mathcal{S}'_2\}$ if candidate i wins by winning \mathcal{S}_i then i wins by winning $\mathcal{S}'_i \supseteq \mathcal{S}_i$.

The key implication of this result is that the total spending of each of the two candidates across all districts at a given time also respects the equal spending ratio result: if $x_s := \sum_s x_t^s$ is candidate 1's total spending at time t and $y_t := \sum_s y_t^s$ is candidate 2's then the proposition above implies $x_t/X_t = y_t/Y_t$.

5 Quantitative Analysis

The one-to-one relationship between λ and the shape of equilibrium spending path presented in Proposition 3 above, and depicted in Figure 2 can be used to recover estimates of the decay rate in polling leads in elections by fitting the actual pattern of spending to the predicted pattern of spending. Here, we establish an identification result, introduce an estimator for λ , apply it to estimate λ from past electoral spending data, and compare the implied decay rates to estimates of the decay rate for TV ads from past studies. We first describe the data for the elections we study, which include U.S. House, Senate, and gubernatorial elections in the period 2000 to 2014.

5.1 Data

While spending in our model refers to all spending (e.g., TV ads, calls, mailers, door-to-door canvassing visits) that directly affects the candidates' relative popularity, it is not straightforward to separate out this kind of spending from other campaign spending (e.g. fixed costs, or administrative costs) that does not influence relative popularity. That said, in the period that we study, advertising constitutes around 30% of the total expenditures by congressional candidates, and the vast majority of ads bought (around 90%) are TV ads (Albert, 2017). So we collect data only on TV ad spending and proceed under the assumption that any residual spending on the type of campaign activities that directly affect relative popularity is proportional to spending on TV ads.

Our TV ad spending data are from the Wesleyan Media Project and the Wisconsin Advertising Database. For each election in which TV ads were bought, the database contains information about the candidate each ad supports, the date it was aired, and the estimated cost. For the year 2000, the data covers only the 75 largest Designated Market Areas (DMAs), and for years 2002-2004, it covers only the 100 largest DMAs.

The data from 2006 onwards covers all of the 210 DMAs. For 2006, where ad price data are missing, we estimate prices using ad prices in 2008.¹⁹

We aggregate ad spending made on behalf of the two major parties' candidates by week and focus on the 20 weeks leading to election day, though we will drop the final week which is typically incomplete since elections are held on Tuesdays.²⁰ We get 1918 unique House, Senate and gubernatorial elections between 2000 and 2014. We then drop all elections that are clearly not genuine contests to which our model does not apply—i.e., elections in which one of the candidates did not spend anything for at least 18 weeks. This leaves us with 600 House, 167 Senate, and 161 gubernatorial elections. We focus on the last 20 weeks of the race both because TV ad spending is usually zero prior to this period, and because we want to restrict attention to the general election campaign. Nevertheless, there are still some states where primaries are held after the last week of June. So, whenever possible, we restrict attention to ads bought for the general election campaign.²¹ Figure 1 in the introduction plots weekly spending averages from these races, showing that spending over time is generally increasing.

We investigate the main robust prediction of our model that $x_t/X_t - y_t/Y_t$ is constant over time. In the data, we define $x_t/X_t - y_t/Y_t$ as the difference between the weekly spending of the Democratic candidate and the Republican candidate, as a percentage of their remaining budget. Figure 3 plots x_t/X_t against y_t/Y_t , and the density of the difference in the spending ratios over the final twenty weeks for each election. Consistent with our expectations, the differences are small. The absolute difference in spending ratios is less than 0.01 for 76% of our dataset, and less than 0.05 for 88%.²²

¹⁹Federal regulations limit the ability of TV stations to increase ad prices as the election approaches, requiring them to charge political candidates “the lowest unit charge of the station for the same class and amount of time for the same period” (Chapter 5 of Title 47 of the United States Code 315, Subchapter III, Part 1, Section 315, 1934).

²⁰Election day is defined by law as “the first Tuesday after November 1,” so candidates do not have a full week to spend on the last calendar week of the cycle.

²¹The data allow us to do this for elections in 2000, 2012 and 2014. Since in some races primaries end later than the start of twenty weeks from election day, we also conduct the same analysis using data from only the last 12 weeks of campaigns and find that the results are similar; see the appendix.

²²In the appendix, we also investigate whether failures of the equal spending ratio result are driven by the candidate that eventually wins the election spending higher ratios than the one that eventually loses. We find very limited evidence for this.

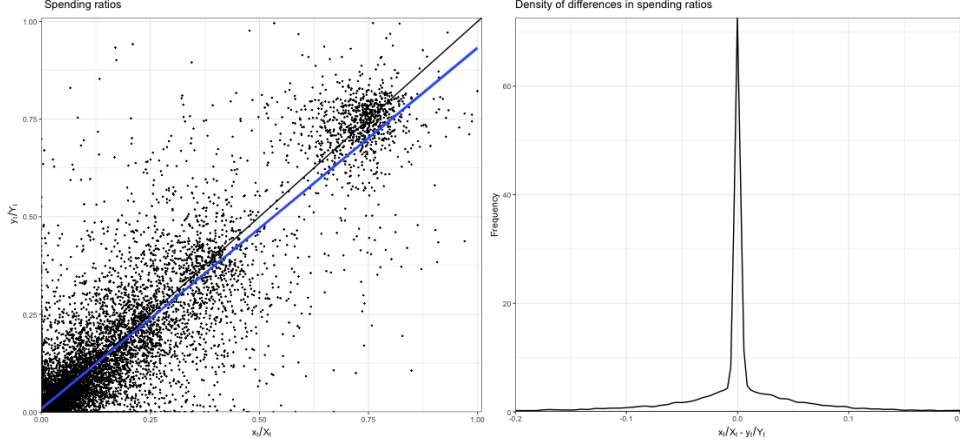


Figure 3: The left figure plots the TV ad spending of the Democratic candidate (x_t/X_t) and the Republican candidate (y_t/Y_t) for each week in our dataset. The black line is the 45 degree line, and the blue line is the fitted regression line. The right figure depicts the density of the difference in spending ratios for each week.

5.2 Estimating Decay Rates from Spending Data

We begin by establishing an identification result that shows that we can empirically identify λ for an arbitrary choice of Δ .

Proposition 8. *Let Γ^Δ denote the game of our baseline model, and consider a modified game $\Gamma^{\bar{\Delta}}$ in which all other parameters are the same but the candidates take actions more frequently at time periods of length $\bar{\Delta} = \Delta/K$, where K is a positive integer. Let x_t^Δ and y_t^Δ be the equilibrium amounts that the candidates spend in game Γ^Δ at times $t \in \mathcal{T}$ and $x_t^{\bar{\Delta}}$ and $y_t^{\bar{\Delta}}$ be the equilibrium amounts that they spend in game $\Gamma^{\bar{\Delta}}$ at times $t \in \bar{\mathcal{T}} := \{0, \bar{\Delta}, \dots, (N-1)\bar{\Delta}\}$. Then, for all times $t \in \mathcal{T}$,*

$$x_t^\Delta = \sum_{k=0}^{K-1} x_{t+k\bar{\Delta}}^{\bar{\Delta}} \quad \text{and} \quad y_t^\Delta = \sum_{k=0}^{K-1} y_{t+k\bar{\Delta}}^{\bar{\Delta}}$$

The key implication of this proposition is that λ and Δ cannot be separately identified from the data; only their product $\lambda\Delta$ can be identified.

Therefore, for our analysis of spending in the final twenty weeks of each election, we fix $\Delta = 1$ week, set $T = 19$ (recall that we drop the final incomplete week), and estimate λ for these values of Δ and T . We report results on the implied decay rate, where

$$\text{decay rate} = 1 - e^{-\lambda}$$

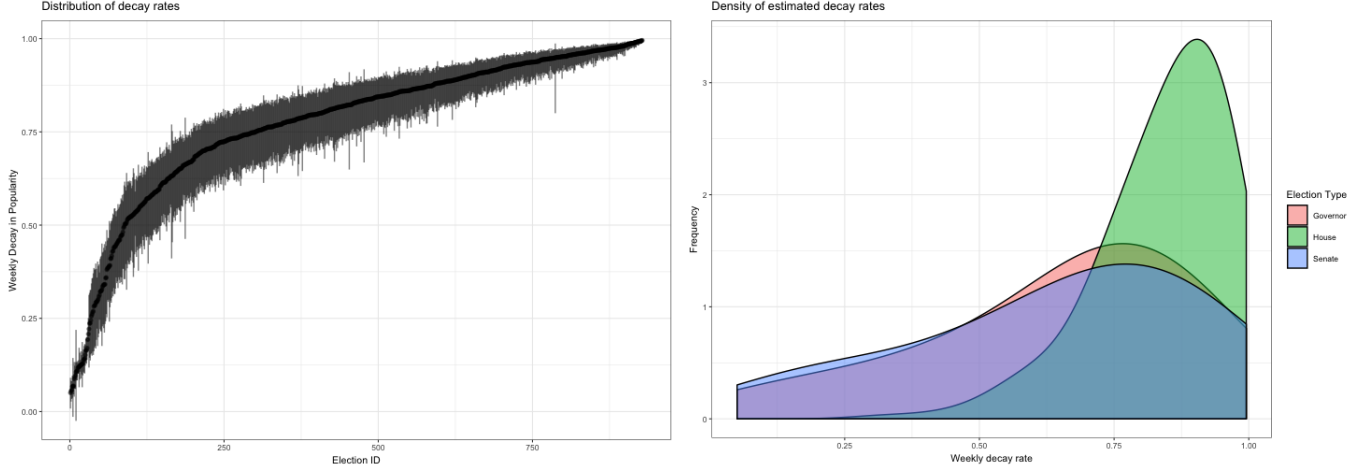


Figure 4: Estimated decay rates for House, Senate and gubernatorial elections along with 95% confidence intervals.

is the percentage decay in a polling lead absent any financial influence of the candidates on the path of relative popularity. We transform the 95% confidence intervals for our estimates of λ to get the exact 95% confidence intervals for the decay rate.

To estimate λ we use a truncated maximum likelihood estimator. Let $\{x_{n\Delta}\}$ denote a path of spending, and assume that we observe in the data $\{\ell(x_{n\Delta})\}$, where

$$\ell(x_{n\Delta}) := \max \{0, \log x_{n\Delta} + \epsilon_{n\Delta}\}$$

where $\epsilon_{n\Delta}$ is i.i.d. mean zero normal measurement error. Proposition 3 shows that $\log x_{n\Delta} = \log \gamma_\lambda(n)X_0$ and Proposition 8 allows us to take $\Delta = 1$, allowing λ to vary, so we can write the likelihood function as

$$L(\lambda, \mu, \sigma_\epsilon) := \prod_{n:\ell(x_{n\Delta})=0} \Phi\left(\frac{-\mu - \lambda n}{\sigma_\epsilon}\right) \prod_{n:\ell(x_{n\Delta})>0} \phi\left(\frac{\ell(x_{n\Delta}) - \mu - \lambda n}{\sigma_\epsilon}\right)$$

where

$$\mu = \log(e^\lambda - 1) - \log(e^{\lambda T} - 1) + \log X_0$$

and Φ and ϕ are the cdf and pdf of the normal distribution with variance σ_ϵ^2 of shocks $\epsilon_{n\Delta}$. The estimator for $(\lambda, \mu, \sigma_\epsilon)$ maximizes the log of this likelihood function. It is well known that under regularity conditions this estimator is consistent and asymptotically normal, which gives us an estimator for the standard error of λ (see Amemiya, 1973).

Figure 4 presents the estimated values of λ across the House, Senate and gubernatorial elections in our sample, as well as the implied decay rates along with 95% confidence intervals. The median estimated λ across House elections is 2.02 (95% CI = [1.57, 2.47]), corresponding to a weekly decay rate of 86% ([78%,91%]). The median estimated λ in Senate elections is 1.23 ([1.00, 1.47]) corresponding to a decay rate of 70% ([63%, 77%]) while the median estimated λ in gubernatorial elections is 1.29 ([0.99, 1.46]) corresponding to a decay rate of 72%([62%, 76%]).

The densities of our point estimates for λ values and decay rates across all three settings, House, Senate, and gubernatorial elections are also depicted in Figure 4. The figure shows that while the distribution of decay rates is remarkably similar across Senate and gubernatorial elections, decay rates for House elections are typically higher.

Finally, as a quantitative exercise, we tabulate in the final weeks of the campaign the cumulative percent of budget that is spent in equilibrium under various decay rates in the range that we recovered, according to the expression we derived in (8). Our tabulation suggests that for typical decay rates equilibrium spending remains low until the final couple weeks, but then ramps up very quickly:

Cumulative equilibrium spending for different weakly decay rates					
weeks to election:	4	3	2	1	0
decay rate = 25%	31.35%	41.94%	56.06%	74.89%	100%
decay rate = 50%	6.25%	12.5%	25%	50%	100%
decay rate = 75%	0.39%	1.56%	6.25%	25%	100%

5.3 Comparisons

How do our estimates of the weekly decay rate compare to other studies and estimation techniques? One alternative approach is to estimate the decay rate directly from polling data. To investigate this approach, we collect polling data from the public version of FiveThirtyEight’s polls database and from HuffPost’s Pollster database. We find, not surprisingly, that polling data for these elections are very sparse; so our estimates are likely to be very noisy, precluding us from doing any meaningful inference.²³ Nevertheless, we implement the approach to compare point estimates across the two methodologies.

²³The sparsity of polling data is an additional reason for why our model’s ability to indirectly recover estimates of the decay rate from spending data is particularly valuable.

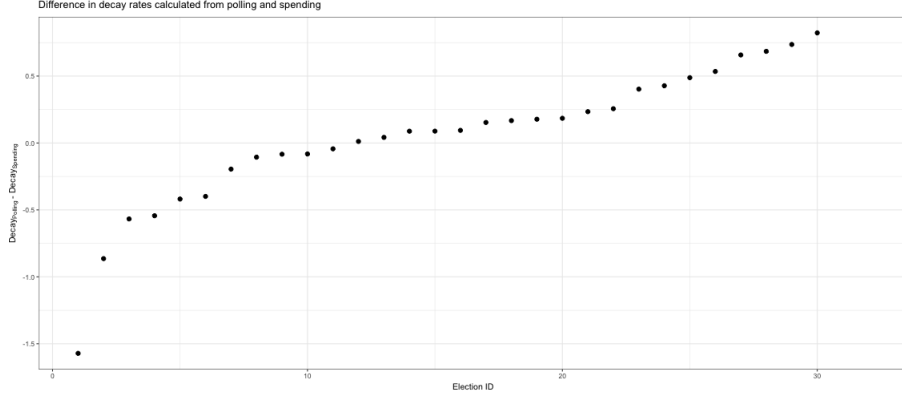


Figure 5: Differences in decay rates estimated from polling data and spending data.

Given equation (2), our model implies that for $\lambda > 0$, relative popularity evolves according to a simple AR(1) process:

$$Z_{(n+1)\Delta} = \beta_0 + \beta_1 Z_{n\Delta} + \epsilon \quad (11)$$

where ϵ is the noise,

$$\beta_0 = (1 - e^{-\lambda})q(X_0/Y_0)/\lambda \quad \text{and} \quad \beta_1 = e^{-\lambda} \quad (12)$$

since again we set $\Delta = 1$ week. Therefore, the weekly decay rate is simply $1 - \beta_1$. For this estimation to work, however, we need at least three weeks of consecutive polling data. Applying this criteria, we get 27 elections from Pollster’s database and 68 elections from FiveThirtyEight’s database, three of which are overlapping. In this case, we use Pollster’s data since Pollster’s polling data are richer for these elections. This gives us a total of 90 elections, all of which are statewide elections. For 60 of these, however, we get point estimates of β_1 that are negative, implying that consecutive period polling is negatively correlated.²⁴ We drop these since $-\log \beta_1$ is undefined for these elections, meaning that it not possible to recover estimates of λ . The median decay rate is 65%, which is close to but lower than the median estimated decay rates across the House, Senate and gubernatorial elections using spending data.

For 30 of the statewide elections, we have both weekly spending data and sufficiently rich weekly polling data, so we can do an election-by-election comparison of the estimated

²⁴This is a large number of elections, though this may be related to the endogenous collection of polling data for elections that are expected to be close.

decay rates using the two different methodologies. Figure 5 shows that point estimates of the decay rate from polling data are more often higher than estimates of the decay rate from spending data, with the average difference in λ being +0.34 and the average difference in decay rates being 4.5 percentage points.

We can also compare our decay rates to decay rates found by other studies. One study by Hill et al. (2013) finds the weekly decay rate to be between 70% and 95% for subnational U.S. elections in 2006, which is consistent with our estimates, though higher than our median. In another famous study, Gerber et al. (2011) conduct a field experiment during the 2006 Texas gubernatorial election, about eleven months prior to election day, and depending on the econometric specification finds the weekly decay rate to be between 25% and 94%.²⁵ For this specific election, we get a point estimate for λ of 3.11, ([2.46, 3.75]), corresponding to a weekly decay rate of 95% ([91%, 97%]), which is in the ballpark—though closer to the higher end—of their estimates.

6 Conclusion

We have proposed a new model of dynamic campaigning, and used it to recover estimates of the decay rate in the popularity process using spending data alone.

Our theoretical contribution raises new questions, however. Since we focused on the strategic choices made by the campaigns, we abstracted away from some important considerations. For example, we left unmodeled the behavior of the voters that generates over-time fluctuations in relative popularity. In addition, we abstracted away from the motivations and choices of the donors, and the effort decisions of the candidates in how much time to allocate to campaigning versus fundraising. These abstractions leave open questions about how to micro-found the behavior of voters and donors, and effort allocation decision for the candidates. We leave these questions to future work.²⁶

Moreover, we have abstracted from the fact that in real life, campaigns may not know what the return to spending is at the various stages of the campaign, what the

²⁵For example, their 3rd order polynomial distributed lag model estimates show that the standing of the advertising candidate increases by 4.07 percentage points in the week that the ad is aired, and the effect goes down to 3.05 percentage points the following week (a 25% decay). In another specification, the first week effect is 6.48%, and goes down to 0.44 % (a 94% decay).

²⁶Bouton et al. (2018) address some of these questions in a static model. They study the strategic choices of donors who try to affect the electoral outcome and show that donor behavior depends on the competitiveness of the election. Similarly, Mattozzi and Michelucci (2017) analyze a two-period dynamic model in which donors decide how much to contribute to each of two possible candidates without knowing ex-ante who is the more likely winner.

decay rate is, as this may be specific to the personal characteristics of their respective candidates, and changes in the political environment, including the “mood” of voters. Real-life campaigns face an optimal experimentation problem whereby they try to learn about their environment through early spending. Our model also abstracted away from the question of how early spending may benefit campaigns by providing them with information about what kinds of campaign strategies seem to work well for their candidate. This is a considerably difficult problem, especially in the face of a fixed election deadline, and the endogeneity of donor interest and available resources. But there is no doubt that well-run campaigns spend to acquire valuable information about how voters are engaging with and responding to the candidates over the course of the campaign. These are interesting and important questions that ought to be addressed by future work.

Appendix

A Proofs

A.1 Proof of Proposition 1

We consider the case of $\lambda > 0$. The $\lambda = 0$ case must be handled separately, but is very similar, so we omit the details.²⁷

We prove by induction that, in any equilibrium, if $X_t, Y_t > 0$, then for all $t \in \mathcal{T}$,

- (i) $x_\tau/y_\tau = X_t/Y_t$ at all times $\tau \geq t$ at which the candidates take actions;
- (ii) if $t < (N - 1)\Delta$, then the distribution of Z_T computed at time $t \in \mathcal{T}$ given z_t is

$$\mathcal{N}\left(p\left(\frac{X_t}{Y_t}\right)(1 - e^{-\lambda(T-t)}) + z_t e^{-\lambda(T-t)}, \frac{\sigma^2(1 - e^{-2\lambda(T-t)})}{2\lambda}\right)$$

The claim is obviously true at $t = (N - 1)\Delta$, since in any equilibrium the candidates’ payoffs depend only on z_T and in the final period they must spend the remainder of their budget.

²⁷ We have continuity at the limit: all of the results for the $\lambda = 0$ case hold as the limits of the $\lambda > 0$ case as $\lambda \rightarrow 0$.

Suppose, for the inductive step, that for all $\tau \geq t + \Delta$, both statements (i) and (ii) above hold. The distribution of $Z_{t+\Delta}$ at time $t \in \mathcal{T}$ given (x_t, y_t, z_t) is

$$\mathcal{N}\left(p\left(\frac{x_t}{y_t}\right)(1 - e^{-\lambda\Delta}) + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda}\right)$$

By this hypothesis, the distribution of Z_T computed at time $t + \Delta \in \mathcal{T}$ given $z_{t+\Delta}$ is

$$\mathcal{N}\left(p\left(\frac{X_t - x_t}{Y_t - y_t}\right)(1 - e^{-\lambda(T-t-\Delta)}) + z_{t+\Delta} e^{-\lambda(T-t-\Delta)}, \frac{\sigma^2(1 - e^{-2\lambda(T-t-\Delta)})}{2\lambda}\right)$$

The compound of normal distributions is also a normal distribution. Therefore, the distribution of Z_T at time t , given (x_t, y_t, z_t) is normal with mean and variance:

$$\begin{aligned} \mu_{Z_T|t} &= p\left(\frac{X_t - x_t}{Y_t - y_t}\right)(1 - e^{-\lambda(T-t-\Delta)}) + p\left(\frac{x_t}{y_t}\right)(e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}) + z_t e^{-\lambda(T-t)} \\ \sigma_{Z_T|t}^2 &= \frac{\sigma^2(1 - e^{-2\lambda(T-t)})}{2\lambda}. \end{aligned}$$

These expressions follow from the law of iterated expectation, $\mu_{Z_T|t} = E_t[E_{t+1}[Z_T]]$, and the law of iterated variance, $\sigma_{Z_T|t}^2 = E_t[Var_{t+1}[Z_T]] + Var_t[E_{t+1}[Z_T]]$.

Now, define the standardized random variable

$$\tilde{Z}_T = \frac{Z_T - \mu_{Z_T|t}}{\sigma_{Z_T|t}}.$$

Candidate 1 wins if $Z_T > 0$ or, equivalently, if

$$\tilde{Z}_T > -\frac{\mu_{Z_T|t}}{\sigma_{Z_T|t}} =: \tilde{z}_T^*$$

Therefore, taking y_t as given, candidate 1's objective is to maximize his probability of winning, which is given by

$$\pi_t(x_t, y_t | X_t, Y_t, z_t) := \int_{\tilde{z}_T^*}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

Factoring common constants, the first order condition for this optimization problem is satisfied if and only if $0 = \partial\mu_{Z_T|t}/\partial x_t$, i.e.,

$$0 = p' \left(\frac{x_t}{y_t} \right) \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{y_t} - p' \left(\frac{X_t - x_t}{Y_t - y_t} \right) \cdot \frac{1 - e^{-\lambda(T-t-\Delta)}}{Y_t - y_t} \quad (13)$$

Moreover, substituting the first order condition in the second order condition and rearranging terms, we get that the second order expression is given by a positive constant that multiplies

$$\frac{\partial^2 \mu_{Z_T|t}}{\partial (x_t)^2} = p'' \left(\frac{x_t}{y_t} \right) \frac{(e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)})}{(y_t)^2} + p'' \left(\frac{X_t - x_t}{Y_t - y_t} \right) \cdot \frac{1 - e^{-\lambda(T-t-\Delta)}}{(Y_t - y_t)^2}$$

Because the function q is strictly concave, p is strictly concave as well. Hence, the second order condition is always satisfied and the objective function is strictly quasi-concave in x_t . By an analogous argument, we can show that candidate 2's problem is strictly quasi-concave in y_t .

Therefore, the first order approach in the main text of Section 3.1 is valid, and we have $x_t/y_t = X_t/Y_t$ for all $\tau \geq t$. This implies $(X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Therefore, we can conclude that the distribution of Z_T computed at time t is given by a normal distribution with mean and variance:

$$\begin{aligned} \mu_{Z_T|t} &= p \left(\frac{X_t}{Y_t} \right) (1 - e^{-\lambda(T-t)}) + z_t e^{-\lambda(T-t)}, \\ \sigma_{Z_T|t}^2 &= \frac{\sigma^2 (1 - e^{-2\lambda(T-t)})}{2\lambda}. \end{aligned}$$

This concludes the inductive step. The statement of the proposition follows by induction.

A.2 Proof of Proposition 2

Suppose that $\lambda > 0$. Then, the first order condition for x_t from (13) is

$$p' \left(\frac{x_t}{y_t} \right) \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{y_t} = p' \left(\frac{X_t - x_t}{Y_t - y_t} \right) \cdot \frac{1 - e^{-\lambda(T-t-\Delta)}}{Y_t - y_t}$$

This equation together with the fact that from Proposition 1 we know that $x_t/y_t = (X_t - x_t)/(Y_t - y_t)$

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}}.$$

Now consider the $\lambda = 0$ case. The first order conditions for x_t and y_t are, respectively,

$$\begin{aligned} p' \left(\frac{x_t}{y_t} \right) \frac{\Delta}{y_t} &= p' \left(\frac{X_t - x_t}{Y_t - y_t} \right) \cdot \frac{T - t}{Y_t - y_t}, \\ p' \left(\frac{x_t}{y_t} \right) \frac{x_t \Delta}{(y_t)^2} &= p' \left(\frac{X_t - x_t}{Y_t - y_t} \right) \frac{(X_t - x_t)(T - t)}{(Y_t - y_t)^2}. \end{aligned}$$

Therefore, we have $x_t/X_t = y_t/Y_t = \Delta/(T - t)$.

A.3 Proof of Proposition 3

Since spending ratios are equal for the two candidates, we can focus without loss of generality on candidate 1. From Proposition 2, we have

$$\frac{x_{n\Delta}}{X_{n\Delta}} = \frac{e^{-\lambda(T-(n+1)\Delta)} - e^{-\lambda(T-n\Delta)}}{1 - e^{-\lambda(T-n\Delta)}} = \frac{e^{\lambda\Delta} - 1}{e^{\lambda(T-n\Delta)} - 1}$$

Then since

$$\frac{e^{\lambda(T-(n+1)\Delta)} - 1}{e^{\lambda(T-n\Delta)} - 1} = \frac{x_{n\Delta}/X_{n\Delta}}{x_{(n+1)\Delta}/X_{(n+1)\Delta}} = \frac{x_{n\Delta}}{x_{(n+1)\Delta}} \frac{X_{(n+1)\Delta}}{X_{n\Delta}} = \frac{x_{n\Delta}}{x_{(n+1)\Delta}} \frac{X_{n\Delta} - x_{n\Delta}}{X_{n\Delta}}$$

we have

$$\begin{aligned} r_n(\lambda) = \frac{x_{(n+1)\Delta}}{x_{n\Delta}} &= \left(1 - \frac{x_{n\Delta}}{X_{n\Delta}} \right) \frac{e^{\lambda(T-n\Delta)} - 1}{e^{\lambda(T-(n+1)\Delta)} - 1} \\ &= \left(1 - \frac{e^{\lambda\Delta} - 1}{e^{\lambda(T-n\Delta)} - 1} \right) \frac{e^{\lambda(T-n\Delta)} - 1}{e^{\lambda(T-(n+1)\Delta)} - 1} = e^{\lambda\Delta} \end{aligned}$$

This gives us

$$x_{n\Delta} = e^{\lambda\Delta n} x_0 \quad \text{and} \quad X_0 = \sum_{n=0}^{N-1} x_{n\Delta} = \sum_{n=0}^{N-1} e^{\lambda\Delta n} x_0 = \frac{e^{\lambda\Delta N} - 1}{e^{\lambda\Delta} - 1} x_0$$

Therefore, we have

$$\gamma_\lambda(n) = \frac{x_{n\Delta}}{X_0} = \frac{e^{\lambda\Delta} - 1}{e^{\lambda\Delta N} - 1} e^{\lambda\Delta n}.$$

A.4 Proof of Proposition 4

Existence of an interior equilibrium under the conditions posited in the proposition, and independence of spending ratios from the history of relative popularity, both follow from the argument laid out in the main text above the proposition.

To prove that Assumption A implies the equal spending ratio result, write Z_T as in equation (9) in the main text, and note that at any time $t \in \mathcal{T}$ candidate 1 maximizes $\Pr[Z_T \geq 0 \mid z_t, X_t, Y_t]$ under the constraint $\sum_{n=t/\Delta}^{N-1} x_{n\Delta} \leq X_t$, while candidate 2 minimizes this probability under the constraint $\sum_{n=t/\Delta}^{N-1} y_{n\Delta} \leq Y_t$.

Consider the final period. Because money-left over has no value, candidates will spend all of their remaining budget in the last period so that the equal spending ratio result holds trivially in the last period.

Now consider any period m that is not the final period. Reasoning as in the proof of Proposition 1, candidate 1 will maximize the mean of Z_T while candidate 2 minimizes it. By the budget constraint, this implies that equilibrium spending $x_{n\Delta}$ and $y_{n\Delta}$ for any period $n \in \{0, 1, \dots, N-2\}$ solve the following pair of first order conditions

$$\begin{aligned} e^{-\lambda\Delta(N-1-n)} p_x(x_{n\Delta}, y_{n\Delta}) &= p_x \left(X_0 - \sum_{m=0}^{N-2} x_{m\Delta}, Y_0 - \sum_{m=0}^{N-2} y_{m\Delta} \right) \\ e^{-\lambda\Delta(N-1-n)} p_y(x_{n\Delta}, y_{n\Delta}) &= p_y \left(X_0 - \sum_{m=0}^{N-2} x_{m\Delta}, Y_0 - \sum_{m=0}^{N-2} y_{m\Delta} \right) \end{aligned}$$

Taking the ratio of these first order conditions, applying Assumption A and inverting function ψ , we get that $\forall n < N-2$

$$\frac{x_{n\Delta}}{X_0 - \sum_{m=0}^{N-2} x_{m\Delta}} = \frac{y_{n\Delta}}{Y_0 - \sum_{m=0}^{N-2} y_{m\Delta}}.$$

or equivalently

$$x_{n\Delta} = \frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}} y_{n\Delta}$$

Thus for every $n < N-2$, we have

$$\frac{x_{n\Delta}}{X_{n\Delta}} = \frac{x_{n\Delta}}{\sum_{m=n}^{N-1} x_{m\Delta}} = \frac{\frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}} y_{n\Delta}}{\sum_{m=n}^{N-2} \left(\frac{x_{(N-1)\Delta}}{y_{(N-1)\Delta}} y_{m\Delta} \right) + x_{(N-1)\Delta}} = \frac{y_{n\Delta}}{\sum_{m=n}^{N-1} y_{m\Delta}} = \frac{y_{n\Delta}}{Y_{n\Delta}}.$$

Therefore, the equal spending result holds for all periods.

A.5 Proof of Proposition 5

We will in fact prove a more general result than Proposition 5 under which we also characterize the stochastic path of spending over time for this extension.

Applying Itô's lemma, we can write the process governing the evolution of this ratio for this model as:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_{XY}(z_t)dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y, \quad (14)$$

where

$$\mu_{XY}(z_t) = (a - b)z_t + \sigma_Y^2 - \rho\sigma_X\sigma_Y.$$

Hence, the instantaneous volatility of this process is simply $\sigma_{XY} = \sqrt{\sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y}$. Therefore, if at time $t \in \mathcal{T}$ the candidates have an amount of available resources equal to X_t and Y_t and spend x_t and y_t , then $Z_{t+\Delta}$ conditional on all information, \mathcal{I}_t , available at time t is a normal random variable:

$$Z_{t+\Delta} | \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{x_t}{y_t}\right) \frac{1 - e^{-\lambda\Delta}}{\lambda} + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda}\right),$$

and Itô's lemma implies that

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) | \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + \mu_{XY}(z_t)\Delta, \sigma_{XY}^2\Delta\right).$$

Last, let $g_1(0) = 1$ and $g_2(0) = 0$, and define recursively for every $m \in \{1, \dots, N - 1\}$,

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} & a - b \\ \frac{1 - e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} g_1(m - 1) \\ g_2(m - 1) \end{pmatrix} \quad (15)$$

Then we have the following result, which implies Proposition 5 in the main text.

Proposition 5'. *Let $t = (N - m)\Delta \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in the essentially unique equilibrium, spending ratios are equal to*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{g_1(m - 1)}{g_1(m - 1) + g_2(m - 1)\frac{\lambda}{1 - e^{-\lambda\Delta}}}. \quad (16)$$

Moreover, in equilibrium, $(\log(x_{t+n\Delta}/y_{t+n\Delta}), z_{t+n\Delta}) \mid \mathcal{I}_t$ follows a bivariate normal distribution with mean

$$\begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \log\left(\frac{X_t}{Y_t}\right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} - \begin{pmatrix} \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{a-b} \end{pmatrix}$$

and variance

$$\begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \sigma_{XY}^2\Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-e^{-\lambda\Delta}}{\lambda} \\ (a-b)\Delta & e^{-\lambda\Delta} \end{pmatrix}^n.$$

Proof. Consider time $t = n\Delta \in \mathcal{T}$ and suppose that at time t both candidates have still a positive budget, $X_t, Y_t > 0$. We will prove the proposition by induction on the times at which candidates take actions, $t = (N-m)\Delta \in \mathcal{T}$, $m = 1, 2, \dots, N$.

To simplify notation, let $g_1(0) = 1, g_2(0) = 0, g_3(0) = 0$ and $g_4(0) = 0$. Furthermore, using (15), recursively write for every $m \in \{1, 2, \dots, N\}$,

$$\begin{aligned} g_3(m) &= g_2(m-1)\Delta + g_3(m-1) \\ g_4(m) &= (g_1(m-1))^2 \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} + (g_2(m-1))^2 \sigma_{XY}^2\Delta + g_4(m-1) \end{aligned}$$

Diagonalizing the matrix in (15) and solving for $(g_1(m), g_2(m))'$ with initial conditions $g_0(1) = 1$ and $g_2(0) = 0$, we can conclude that, for each $N \in \mathbb{N}$ and $\lambda, \Delta > 0$, there exists $-\eta < 0$ such that, if $a-b \geq -\eta$, both $g_1(m)$ and $g_2(m)$ are non-negative for each m . In the proof, we will thus assume that $g_1(m) \geq 0$ and $g_2(m) \geq 0$ for every $m = 1, \dots, N$.

The inductive hypothesis is the following: for every $\tau = (N-m)\Delta \in \mathcal{T}$, $m \in \{1, \dots, N\}$, if $X_\tau, Y_\tau > 0$, then

- (i) the continuation payoff of each candidate is a function of current popularity z_τ , current budget ratio X_τ/Y_τ and calendar time τ ;
- (ii) the distribution of Z_T given z_τ and X_τ/Y_τ is $\mathcal{N}\left(\hat{\mu}_{(N-m)\Delta}(z_\tau), \hat{\sigma}_{(N-m)\Delta}^2\right)$, where

$$\begin{aligned} \hat{\mu}_{(N-m)\Delta}(z_{(N-m)\Delta}) &= g_1(m)z_{(N-m)\Delta} + g_2(m) \log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m)(\sigma_Y^2 - \rho\sigma_X\sigma_Y), \\ \hat{\sigma}_{(N-m)\Delta}^2 &= g_4(m). \end{aligned}$$

Base Step Consider $m = 1$, the subgame reached in the final period $t = (N - 1)\Delta$ and suppose both candidates still have a positive amount of resources, $X_{(N-1)\Delta}, Y_{(N-1)\Delta} > 0$. Both candidates will spend their remaining resources: $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Hence, $x_{(N-1)\Delta}/y_{(N-1)\Delta} = X_{(N-1)\Delta}/Y_{(N-1)\Delta}$ and

$$Z_T | \mathcal{I}_{(N-1)\Delta} \sim \mathcal{N} \left(\log \left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}} \right) \frac{1 - e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta} e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda} \right).$$

Because Z_T fully determines the candidates' payoffs, the continuation payoff of the candidates is a function of current popularity $z_{(N-1)\Delta}$, the ratio $X_{(N-1)\Delta}/Y_{(N-1)\Delta}$, and calendar time. Furthermore, given the recursive definition of g_1, g_2, g_3 and g_4 , we can conclude that the second part of the inductive hypothesis also holds at $t = (N - 1)\Delta$. This concludes the base step.

Inductive Step Suppose the inductive hypothesis holds true at any time $(N - m)\Delta \in \mathcal{T}$ with $m \in \{1, 2, \dots, m^* - 1\}$, $m^* \leq N$. We want to show that at time $(N - m^*)\Delta \in \mathcal{T}$, if $X_t, Y_t > 0$, then (i) an equilibrium exists, (ii) in all equilibria, $x_t/y_t = X_t/Y_t$ and the continuation payoffs of both candidates are functions of relative popularity z_t , the ratio X_t/Y_t , and calendar time t , and (iii) Z_T given period t information is distributed according to $\mathcal{N} \left(\hat{\mu}_{(N-m^*)\Delta}(z_t), \hat{\sigma}_{(N-m^*)\Delta}^2 \right)$.

Consider period $t = N - m^*$ and let $x, y > 0$ be the candidates' spending in this period. Exploiting the inductive hypothesis, the distribution of $Z_{t+\Delta} | \mathcal{I}_t$ and the one of $\log \left(\frac{X_{t+\Delta}}{Y_{t+\Delta}} \right) | \mathcal{I}_t$, we can compound normal distributions and conclude that $Z_T | \mathcal{I}_t \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$, where

$$\begin{aligned} \tilde{\mu} &= \hat{\mu}_t(x, y) := G_1 \log \left(\frac{x}{y} \right) + G_2 \log \left(\frac{X_{(N-m^*)\Delta} - x}{Y_{(N-m^*)\Delta} - y} \right) + G_3 \\ \tilde{\sigma}^2 &= G_4 \end{aligned}$$

with G_1, G_2, G_3 and G_4 defined as follows:

$$G_1 = g_1(m^* - 1) \frac{1 - e^{-\lambda\Delta}}{\lambda} \tag{17}$$

$$G_2 = g_2(m^* - 1) \tag{18}$$

$$G_3 = g_1(m^* - 1) z_t e^{-\lambda\Delta} + g_2(m^* - 1) \mu_{XY}(z_t) \Delta + g_3(m^* - 1) (\sigma_Y^2 - \rho \sigma_X \sigma_Y) \tag{19}$$

$$G_4 = (g_1(m^* - 1))^2 \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda} + (g_2(m^* - 1))^2 \sigma_{XY}^2 \Delta + g_4(m^* - 1) \tag{20}$$

Note that $\tilde{\sigma}^2$ is independent of x and y .

Candidate 1 wins the election if $Z_T > 0$. Thus, in equilibrium he chooses x to maximize his winning probability

$$\int_{\frac{-\hat{\mu}_t(x,y)}{\tilde{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

The first order necessary condition for x is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}_t}} \frac{\hat{\mu}'_t(x,y)}{\tilde{\sigma}} = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x) - G_2x}{x(X_t - x)} \right].$$

Furthermore, when the first order necessary condition holds, the second order condition is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}_t}} \frac{\hat{\mu}''_t(x,y)}{\tilde{\sigma}} = \frac{-1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x)^2 + G_2x^2}{x^2(X_t - x)^2} \right] < 0.$$

Hence, the problem is strictly quasi-concave for candidate 1 for each y . A symmetric argument shows that the corresponding problem for candidate 2 is strictly quasi-concave for each x . Hence an equilibrium exists and the optimal investment of the two candidates is pinned down by the first order necessary conditions, which yields

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{G_1}{G_1 + G_2}. \quad (21)$$

Thus, in equilibrium, $x_t/y_t = X_t/Y_t$ and $(X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Because the continuation payoffs of candidates is fully determined by Z_T , these expected payoffs from the perspective of time t depend only on calendar time, the level of current popularity and the ratio of budget at time t . Furthermore, recalling the definition of $\mu_{XY}(z_t)$, we conclude that the second part of the inductive hypothesis is also true.

Next, we know that

$$Z_T \mid \mathcal{I}_{(N-m^*)\Delta} \sim \mathcal{N}(\hat{\mu}_{(N-m^*)\Delta}, \hat{\sigma}_{(N-m^*)\Delta}^2)$$

where

$$\hat{\mu}_{(N-m^*)\Delta}(z_{(N-m^*)\Delta}) = g_1(m^*)z_{(N-m^*)\Delta} + g_2(m^*) \log \left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}} \right) + g_3(m^*)(\sigma_Y^2 - \rho\sigma_X\sigma_Y),$$

$$\hat{\sigma}_{(N-m^*)\Delta}^2 = g_4(m^*).$$

The expression for x_t/X_t and y_t/Y_t in the proposition thus follows from (15), (17), (18) and (21).

To derive the distribution of $(x_t/y_t, z_t)$, we first use the proof of Proposition 5 to derive the distribution of $x_{t+j\Delta}/y_{t+j\Delta}$ and $z_{t+j\Delta}$ given x_t/y_t and z_t . Let

$$\Sigma = \begin{pmatrix} \sigma_{XY}^2 \Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix}.$$

Because $X_t/Y_t = x_t/y_t$ for each t , we can write

$$\begin{pmatrix} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) \\ z_{t+n\Delta} \end{pmatrix} \middle| \begin{pmatrix} \frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \\ z_{t+(n-1)\Delta} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \mu_{XY}(z_{t+(n-1)\Delta})\Delta \\ \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) \frac{1-e^{-\lambda\Delta}}{\lambda} + z_{t+(n-1)\Delta}e^{-\lambda\Delta} \end{pmatrix}, \Sigma \right)$$

Define

$$A = \begin{pmatrix} 1 & (a-b)\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}.$$

and notice that the previous distribution implies

$$\begin{pmatrix} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} \middle| \begin{pmatrix} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix}$$

follows a multivariate normal distribution

$$\mathcal{N} \left(A \begin{pmatrix} \log \left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+(n-1)\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix}, \Sigma \right)$$

Therefore, we conclude that

$$\begin{pmatrix} \log \left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_{t+n\Delta} + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} \middle| \begin{pmatrix} \log \left(\frac{x_t}{y_t} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix}$$

follows the multivariate normal distribution

$$\mathcal{N} \left(A^n \begin{pmatrix} \log \left(\frac{X_t}{Y_t} \right) + \frac{\lambda(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \\ z_t + \frac{(\sigma_Y^2 - \rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix}, A^n \Sigma (A^T)^n \right).$$

■

A.6 Proof of Proposition 6

Fix λ and Δ . and let $n = N - m$. We must show that for all $n \in \{0, \dots, N - 1\}$,

$$\hat{r}_n(a - b) = \frac{x_{n\Delta}}{X_{n\Delta}} / \frac{x_{(n+1)\Delta}}{X_{(n+1)\Delta}}$$

is decreasing in $\alpha := a - b$ around $\alpha = 0$. Note that \hat{r}_n is the same as \tilde{r}_{N-m} .

Proposition 5' and (15) imply

$$\hat{r}_m(\alpha) = \frac{g_1(m-1) \left(g_1(m) + g_2(m) \frac{\lambda}{1-e^{-\lambda\Delta}} \right)}{\left(g_1(m-1) + g_2(m-1) \frac{\lambda}{1-e^{-\lambda\Delta}} \right) g_1(m)} = \frac{g_1(m-1)}{g_1(m)} \frac{g_2(m+1)}{g_2(m)}.$$

Furthermore, (15) also implies

$$g_1(m) = \frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} g_1(m-1) + \alpha g_2(m), \quad (22)$$

$$g_2(m+1) = \frac{(1 - e^{-\lambda\Delta}) ((\lambda + \alpha) e^{-\lambda\Delta} - \alpha)}{\lambda^2} g_1(m-1) + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} g_2(m). \quad (23)$$

Substituting in the expression for $\hat{r}_m(\alpha)$ and simplifying, we get

$$\hat{r}_m(\alpha) = \frac{1}{\frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} + \alpha g_m} \left(\frac{(1 - e^{-\lambda\Delta}) ((\lambda + \alpha) e^{-\lambda\Delta} - \alpha)}{\lambda^2} \frac{1}{g_m} + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} \right) \quad (24)$$

where $g_m := g_2(m) / g_1(m-1)$. We can thus identify two values of g_m for which (24) holds. However, if α is sufficiently low, namely if $\alpha < \lambda / (1 + e^{\lambda\Delta})$, one of these two values is negative and thus not feasible. Thus, if α is sufficiently small, (24) enables us to express g_m as a function of $\hat{r}_m(\alpha)$. Moreover, from (22) and (23), we further have

$$g_{m+1} = \frac{\frac{1 - e^{-\lambda\Delta}}{\lambda} \frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} + \frac{\alpha + \lambda - \alpha e^{-\lambda\Delta}}{\lambda} g_m}{\frac{(\lambda + \alpha) e^{-\lambda\Delta} - \alpha}{\lambda} + \alpha g_m}. \quad (25)$$

Computing (24) one step forward and substituting for g_{m+1} as obtained from (25) and, subsequently, for g_m as obtained from (24), we get \hat{r}_{m+1} as a function of α and \hat{r}_m , $\hat{r}_{m+1}(\alpha, \hat{r}_m)$.

Given the expression for \hat{r}_{m+1} , we can show by induction that $\hat{r}_m > e^{\lambda\Delta} > 1$ for each m around $\alpha = 0$. When $m = 1$, we have $x_{(N-1)\Delta}/X_{(N-1)\Delta} = 1$ and $x_{(N-2)\Delta}/X_{(N-2)\Delta} = g_1(1) / \left(g_1(1) + g_2(1) \frac{\lambda}{1-e^{-\lambda\Delta}} \right)$. Substituting for $g_1(1)$ and $g_2(1)$, we get $\hat{r}_1 - e^{\lambda\Delta} = 1$. Thus, $\hat{r}_1 > e^{\lambda\Delta} > 1$. Suppose $\hat{r}_m > e^{\lambda\Delta} > 1$. Then, subtracting $e^{\lambda\Delta}$ from the right hand side of the expression of \hat{r}_{m+1} and setting $\alpha = 0$, we get $\hat{r}_{m+1} - e^{\lambda\Delta} = 1 - e^{\lambda\Delta}/\hat{r}_m > 0$. We conclude that, if $\hat{r}_m > e^{\lambda\Delta}$, then $\hat{r}_{m+1} > e^{\lambda\Delta}$ in a neighborhood of $\alpha = 0$. Therefore, $\hat{r}_m > e^{\lambda\Delta}$ for each m in a neighborhood of $\alpha = 0$.

Furthermore, $\hat{r}_{m+1}(\alpha, \hat{r}_m)$ is decreasing in α and increasing in \hat{r}_m at $\alpha = 0$:

$$\begin{aligned} \left. \frac{\partial \hat{r}_{m+1}(\alpha, \hat{r}_m)}{\partial \alpha} \right|_{\alpha=0} &= -\frac{(\hat{r}_m - 1) e^{\lambda\Delta} (e^{2\lambda\Delta} - 1)}{\hat{r}_m (\hat{r}_m - e^{\lambda\Delta})} < 0; \\ \left. \frac{\partial \hat{r}_{m+1}(\alpha, \hat{r}_m)}{\partial \hat{r}_m} \right|_{\alpha=0} &= \frac{e^{\lambda\Delta}}{(\hat{r}_m)^2} > 0. \end{aligned}$$

Hence, a simple induction argument implies that $\hat{r}_m(\alpha)$ is decreasing in α for each m in a neighborhood of $\alpha = 0$.

Finally, \hat{r}_m is increasing in λ as well:

$$\left. \frac{\partial \hat{r}_{m+1}(\alpha, \hat{r}_m, \lambda)}{\partial \lambda} \right|_{\alpha=0} = \frac{e^{\lambda\Delta} (\hat{r}_m - 1) \Delta}{\hat{r}_m} > 0 \text{ for each } \lambda > 0.$$

Thus, a symmetric inductive argument shows that \hat{r}_m is increasing in λ for every m in a neighborhood of $\alpha = 0$.

A.7 Proof of Proposition 7

Note that the game ends in a defeat for any candidate that spends 0 in any district in any period. Therefore, in equilibrium spending must be interior (i.e., satisfy the first order conditions) for any district and any period.

Given this, we will prove the proposition by induction. Consider the final period as the basis case. Fix $(z_{T-\Delta}^s)_{s=1}^S$ arbitrarily. Suppose candidates 1 and 2 have budgets X and Y , respectively in the last period. Fix an equilibrium strategy profile $(x_{T-\Delta}^{s,*}, y_{T-\Delta}^{s,*})_{s=1}^S$. We show that, if they have budgets θX and θY , then $(\theta x_{T-\Delta}^{s,*}, \theta y_{T-\Delta}^{s,*})_{s=1}^S$

is an equilibrium. This implies that the equilibrium payoff in the last period is determined by $(z_{T-\Delta}^s)_{s=1}^S$ and $X_{t-\Delta}/Y_{t-\Delta}$.

Suppose otherwise. Without loss, assume that there is $(\tilde{x}_{T-\Delta}^{s,*})_{s=1}^S$ such that it gives a higher probability of winning to candidate 1 given $(z_{T-\Delta}^s)_{s=1}^S$ and $\theta y_{T-\Delta}^{s,*}$, satisfying $\sum_{s=1}^S \tilde{x}_{T-\Delta}^{s,*} \leq \theta X$. Since the distribution of $(Z_T^s)_{s=1}^S$ is determined by $(z_{T-\Delta}^s)_{s=1}^S$ and $(x_{t-\Delta}^s/y_{t-\Delta}^s)_{s=1}^S$, this means that the distribution of $(Z_T^s)_{s=1}^S$ given $(z_{T-\Delta}^s)_{s=1}^S$ and $(\tilde{x}_{t-\Delta}^s/\theta y_{t-\Delta}^{s,*})_{s=1}^S$ is more favorable to candidate 1 than that given $(z_{T-\Delta}^s)_{s=1}^S$ and

$$(\theta x_{t-\Delta}^{*,s}/\theta y_{t-\Delta}^{*,s})_{s=1}^S = (x_{t-\Delta}^{*,s}/y_{t-\Delta}^{*,s})_{s=1}^S.$$

On the other hand, candidate 1 could spend $(\frac{1}{\theta} \tilde{x}_{T-\Delta}^{s,*})_{s=1}^S$ when the budgets are (X, Y) . Since $(x_{T-\Delta}^{s,*}, y_{T-\Delta}^{s,*})_{s=1}^S$ is an equilibrium, the distribution of $(Z_T^s)_{s=1}^S$ given $(z_{T-\Delta}^s)_{s=1}^S$ and $(\frac{1}{\theta} \tilde{x}_{t-\Delta}^s/y_{t-\Delta}^{*,s})_{s=1}^S = (\tilde{x}_{t-\Delta}^s/\theta y_{t-\Delta}^{*,s})_{s=1}^S$ is no more favorable to candidate 1 than that given $(z_{T-\Delta}^s)_{s=1}^S$ and $(x_{t-\Delta}^{*,s}/y_{t-\Delta}^{*,s})_{s=1}^S$. This is a contradiction.

Now consider the inductive step. Take the inductive hypothesis to be that the continuation payoff for either candidate in period $t \in \mathcal{T}$ can be written as a function of only the budget ratio X_{t+1}/Y_{t+1} and vector $(z_{t+1}^s)_{s=1}^S$ and candidates spend a positive amount in each district and in each following period. We have to show that $x_t^s/X_t = y_t^s/Y_t$.

Denote the continuation payoff of candidate 1 in period t with $W_{t+1}(\frac{X_{t+1}}{Y_{t+1}}, (z_{t+1}^s)_{s=1}^S)$. Candidate 1's objective is

$$\max_{x_t} \int W_{t+1} \left(\frac{X_t - \sum_{s=1}^S x_t^s}{Y_t - \sum_{s=1}^S y_t^s}, (z_{t+1}^s)_{s=1}^S \right) f_t \left((z_{t+1}^s)_{s=1}^S \mid \left(\frac{x_t^s}{y_t^s} \right)_{s=1}^S, (z_{t+1}^s)_{s=1}^S \right) dz_{t+1}.$$

The first order condition for an interior optimum is

$$\begin{aligned} & \frac{1}{Y_t - \sum_{s=1}^S y_t^s} \int \frac{\partial W_{t+1}((X_t - x_t)/(Y_t - y_t), z_{t+1})}{\partial (x_t^s/y_t^s)} f_t \left(z_{t+1} \mid \frac{x_t}{y_t}, z_t \right) dz_{t+1} = \\ & = \frac{1}{y_t^s} \int W_{t+1} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1} \right) \frac{\partial f_t(z_{t+1} \mid (x_t^s/y_t^s)_{s=1}^S, z_t)}{\partial (x_t^s/y_t^s)} dz_{t+1}. \end{aligned}$$

Similarly, the objective for candidate 2 is

$$\min_{y_t} \int W_{t+1} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1} \right) f_t \left(z_{t+1} \mid \frac{x_t}{y_t}, z_t \right) dz_{t+1}.$$

and the corresponding first order condition is

$$\begin{aligned} & \frac{X_t - \sum_{s=1}^S x_t^s}{\left(Y_t - \sum_{s=1}^S y_t^s\right)^2} \int \frac{\partial W_{t+1} \left((X_t - x_t)/(Y_t - y_t), z_{t+1} \right)}{\partial (x_t^s/y_t^s)} f_t \left(z_{t+1} \mid \frac{x_t}{y_t}, z_t \right) dz_{t+1} \\ &= \frac{x_t^s}{(y_t^s)^2} \int W_{t+1} \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+1} \right) \frac{\partial f_t \left((z_{t+1}^s)_{s=1}^S \mid (x_t^s/y_t^s)_{s=1}^S, (z_t^s)_{s=1}^S \right)}{\partial (x_t^s/y_t^s)} dz_{t+1}. \end{aligned}$$

Dividing the candidate 1's first order condition by candidate 2's, we have

$$\frac{X_t - \sum_{s=1}^S x_t^s}{Y_t - \sum_{s=1}^S y_t^s} = \frac{x_t^s}{y_t^s}.$$

Hence there exists θ such that $x_t^s = \theta y_t^s$ for each s , and so

$$\theta = \frac{X_t - \theta \sum_{s=1}^S y_t^s}{Y_t - \sum_{s=1}^S y_t^s},$$

which implies $\theta = X_t/Y_t$. Therefore, $x_t^s/y_t^s = X_t/Y_t$ for each s .

A.8 Proof of Proposition 8

We prove the statement of the proposition only for candidate 1 because the analysis for candidate 2 is identical. Let $\lambda > 0$ and $\gamma_\lambda^\Delta(n) := x_{n\Delta}^\Delta/X_0$ and $\gamma_\lambda^{\bar{\Delta}}(n) := x_{n\bar{\Delta}}^{\bar{\Delta}}/X_0$. Let $\lambda > 0$. If $K = 1$, the result holds trivially. Thus, suppose $K > 1$ and fix any $t = n\Delta \in \mathcal{T}$ with $n \leq N - 1$. Then,

$$\begin{aligned} \sum_{k=0}^{K-1} x_{t+k\bar{\Delta}}^{\bar{\Delta}} &= X_0 \sum_{k=0}^{K-1} \gamma_\lambda^{\Delta/K}(nK+k) = X_0 \frac{e^{\lambda\Delta/K} - 1}{e^{\lambda NK\Delta/K} - 1} \sum_{k=0}^{K-1} e^{\lambda\Delta(nK+k)/K} = \\ &= X_0 \frac{e^{\lambda\Delta/K} - 1}{e^{\lambda N\Delta} - 1} e^{\lambda\Delta n} \frac{e^{\lambda\Delta} - 1}{e^{\lambda\Delta/K} - 1} = X_0 \frac{e^{\lambda\Delta} - 1}{e^{\lambda N\Delta} - 1} e^{\lambda\Delta n} = X_0 \gamma_\lambda^\Delta(n) = x_t^\Delta \end{aligned}$$

The proof for the case in which $\lambda = 0$ is similar and omitted.

Table B.1: Time trend of $x_t/X_t - y_t/Y_t$

	<i>Dependent variable: $x_t/X_t - y_t/Y_t$</i>				
	(All)	(All)	(House)	(Senate)	(Governor)
Constant	0.0005 (0.0009)	0.0005 (0.001)	0.001 (0.001)	-0.0008 (0.002)	0.0002 (0.0002)
Time	-0.0001 (0.0001)	-0.0001 (0.0001)	0.0001 (0.0001)	-0.0002 (0.0002)	-0.0007** (0.0002)
Time \times Dem Victory		0.0001 (0.0002)	-0.0003 (0.0002)	0.0003 (0.0003)	0.0010** (0.0003)
Observations	11,780	11,780	6,137	2,888	2,755
R ²	0.0001	0.0002	0.001	0.0005	0.006

*p<0.1; **p<0.05; ***p<0.01

Note: Standard errors are robust to heteroskedasticity and autocorrelation. The sample includes 620 elections (out of a total of 928) for which we can identify the winner from our data.

B Additional Empirical Analyses

B.1 Further Results on the Equal Spending Ratio

Although we have evidence that the equal spending ratio result holds across many weeks of the many elections we look at, one possibility is that its failures are driven by the candidate that eventually wins the election spending higher ratios than the one that eventually loses, especially as the race nears its end. This may be the case, for example, if donors adjust their contributions to how candidates are doing in comparison to expectations, and this might be more likely in state-wide elections than House races due to the greater importance of state-wide offices.²⁸

We investigate this possibility in Table B.1. To start, column (1) of the table presents the result of fitting a linear regression with a time trend to the data, showing that on average there is no overall time trend across elections. To investigate the possibility that the difference $x_t/X_t - y_t/Y_t$ tilts towards the winning candidate, column (2) of the table adds an interaction of the time-trend with Democratic victory, and columns (3) - (5) disaggregate the data to House, Senate and gubernatorial elections. Although the

²⁸Recall that the extension in Section 4.2 in which the candidates' budgets evolve over time in response to the realizations of their relative popularity is motivated in part by the possibility that the amount of money the candidates raise could be sensitive to popularity path.

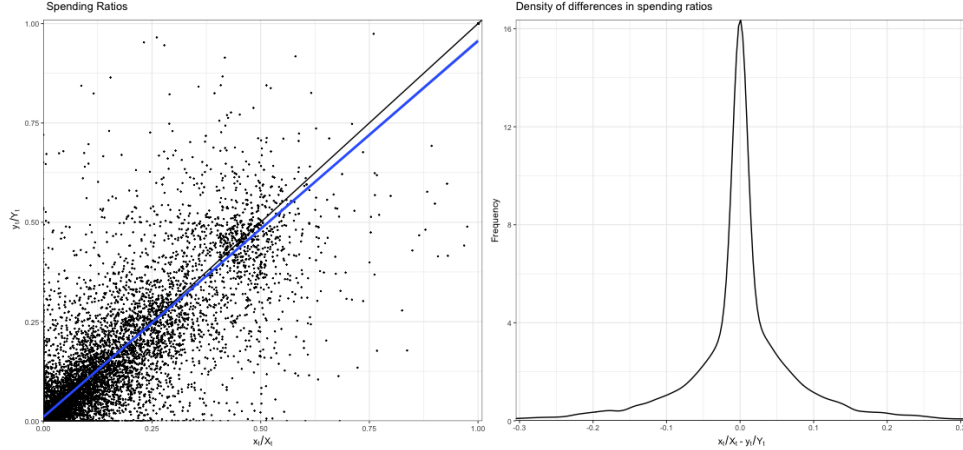


Figure B.1: Replication of Figure 3 in the main text using only the last 12 weeks of spending data for each election.

time trend coefficients are statistically significant in the gubernatorial sample at the 5% level, the substantive magnitudes are very small. Thus, we find very limited evidence that the spending ratios tilt toward the candidate that eventually wins the election.

B.2 Results Using Only 12 Weeks of Spending Data

We replicate the analysis of Section 5.2 using only the last 12 weeks of spending data, and discuss the differences with the results reported in the main text.

Figure B.1 plots x_t/X_t , y_t/Y_t , and the density of the differences in the spending ratios using the last 12 weeks of spending. The distribution is similar to the one reported in the main text, and the differences in spending ratios are very small. In 68% of the data, the absolute difference between the spending ratios is less than 0.01, and in 82% of the data, the difference is less than 0.05.

Figure B.2 plots the estimated decay rates and the densities of the decay rates for Senate, House, and gubernatorial elections. We observe similar patterns where the decay rates for statewide races are lower than the decay rates for house races. Restricting attention to the last 12 weeks leads to lower estimates of decay rates, however. This is not surprising since for some of the races, the general election campaign has begun prior to 12 weeks from election day, but our cutoff is treating the data as if the campaign starts at 12 weeks from election day. The average difference between these estimates and the estimates reported in the main text is -12.5 percentage points.

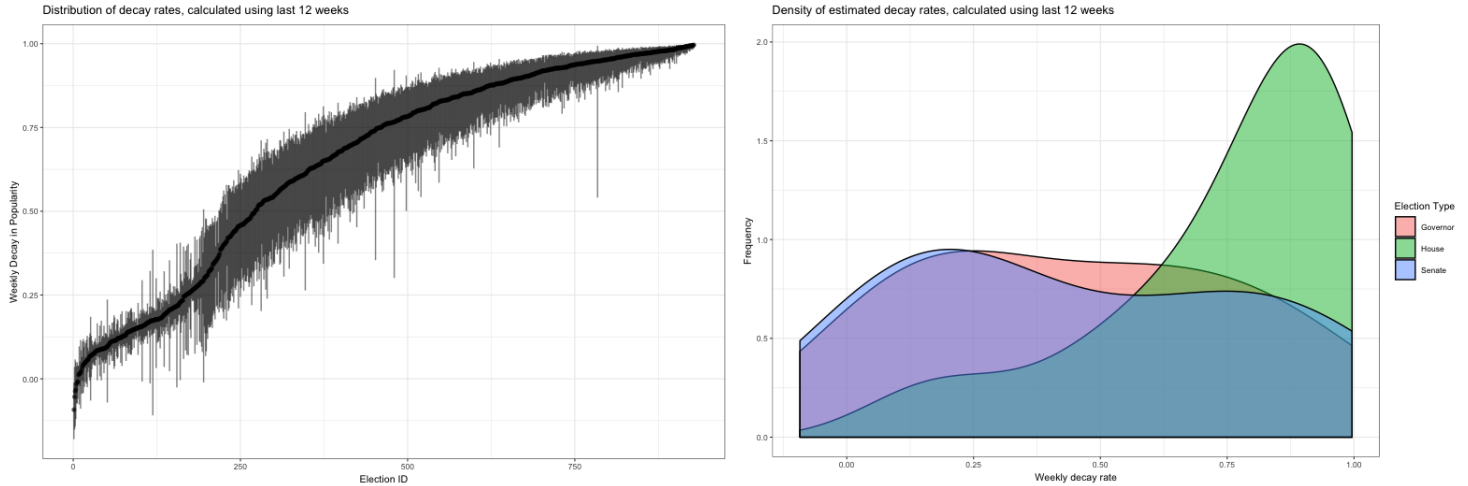


Figure B.2: Replication of the figures in Figure 4 in the main text using only the last 12 weeks of spending data for each election.

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