MATHEMATICAL EXISTENCE

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Anyone who has thought about the nature of mathematics has probably been puzzled over the status of its objects. Are the objects with which mathematics deals — numbers, sets, functions and the like — created or are they discovered? Should we think of them in the manner of the stars and the planets, whose character and existence is entirely independent of our investigations and activities? Or should we think of them in the manner of the objects of fiction, whose existence and character is entirely dependent upon what their authors make of them?

This question has divided philosophical opinion from the very beginning. As is well known, Plato thought of the mathematical objects and the "forms" as existing independently of the ordinary world and, so great was his influence on this question, that this general position is now called "Platonism." Aristotle, on the other hand, thought of mathematical objects as the result of abstraction on the objects of the ordinary world. They do not exist "up there" in a Platonic Heaven but "down here," on earth, though in a highly rarefied form. And in more recent times the debate has continued. Thus we find Goedel and Church — two great philosopher-logicians of the last century — each espousing an extreme Platonic position, on the one side; and we find Hilbert and Brouwer

— two great mathematicians of the last century — each espousing a strongly anti-Platonic position, on the other.

It is not hard to see why the debate might have been so difficult to settle. For we need only give the matter the most superficial thought to find ourselves torn in both directions. On the one hand, mathematics presents itself as dealing with an "objective reality." Abstract as the subject might be, it is hard not to have the sense, when one is doing mathematics, that one is up against a "firm reality" that is equally impervious to our activities and investigations as are the stars and planets of astronomy. On the other hand, it is hard not to be impressed by the great freedom that mathematicians seem to enjoy in coming up with new objects. The equation $3 + n = 0$ does not have a natural number solution. Not to worry, just take a negative number -3 to be its solution! The equation $x^2 = -1$ does not have a real number solution. Again, not to worry, just take the imaginary number i to be its solution. Mathematicians appear to enjoy a freedom in coming up with new objects and 'narrative structures' that should be the envy of any creative writer and it is hard not to avoid the impression that their reality is not altogether firm and might to some extent be of our making.

Further reflection only reinforces one's sense of puzzlement. If mathematical objects are independent of us or the world in which they live, then "where" are they? Presumably not in space and time? But then what kind of object is it that can enjoy an existence outside of space and time? How do we refer or identify such objects? Clearly, not by pointing, since we can only point to objects in

space or time. By description? Might the number 1 for example be identified as the "successor" of the number 0, the number 2 as the successor of the number 1, and so on? But that is to presuppose that one has already fixed a meaning for "successor" (and the initial number 0). But how is this to be done? Easy one might think; successor is an operation taking 0 to 1, 1 to 2, and so on. But this explanation already presupposes that we can identify the numbers! We are caught in a circle: for we cannot identify the numbers independently of the arithmetical operations, such as successor, or the operations independently of the numbers.

Nor does the alternative anti-Platonist position seem any more acceptable. If numbers are created, for example, there was presumably a time when they first came into existence. Perhaps it was when man first began to count. But don't we want to say that there was once a time when the number of dinosaurs roaming the earth was over 1,000? And yet how can that be? There were plenty of dinosaurs alright, but not a single number, and so how could *their number* have then been over 1,000? The position also makes a mockery of how we wish to apply mathematics. Consider the following piece of reasoning: at some time in the past the number of male dinosaurs was over 500; at the same time, the number of female dinosaurs was also over 500; and so, at that time, the total number of dinosaurs was over 1,000. Here we appeal to the arithmetical fact that $500 + 500 = 1,000$. But since were then no numbers, this simple piece of arithmetical reasoning cannot even get going.

These problems with the positions, which I have merely sketched, are

only the beginning of our difficulties. There are many other problems — both on the "input" side, concerning our access to the realm of mathematics, and on the "output" side, concerning its application to the world.

This suggests that we might go for an intermediate position. I have so far drawn a stark contrast between "discovery" and "invention." The question has been whether mathematical objects are discovered or invented. But why not both? Might not the objects of mathematics have some of the characteristic features of discovered objects and some of the characteristic features of invented objects? Such an intermediate position would then hold out the hope that we might be able to take on what is good in the Platonist and anti-Platonist positions and yet leave out what is bad.

In exploring such an intermediate position, we naturally turn to Kant. He espoused a doctrine of "transcendental idealism" that is intermediate in more or less the sense we have in mind. His concern here was not with the objects of mathematics but with the objects of the external world; and just as we considered the contrast between Platonism and anti-Platonism in mathematics, so he considered the contrast between realism and idealism about the external world. According to the realist, the external world and its objects are independent of our minds; according to the idealist, they are constituted or created by our minds. But he suggested that even though the external world might not be dependent upon our minds in any crude, empirical way (we do not make the world in the way we make cookies), still there might be a subtle

non-empirical way in which the external is dependent upon our minds. Hence the transcendental in "transcendental idealism"; the dependence of the external world upon our minds is not itself to be found within the external world.

The problem with this position is to understand what it is. What is this transcendental form of dependence? And how can it be transcendental and not empirical? Kant sometimes explains the notion by appeal to the distinction between the transcendental and empirical self. The transcendental self, unlike the empirical self, is not in the world; and the transcendental dependence of the world on our minds relates to the "activities" of the transcendental rather than the empirical self. But you would not be alone if you thought that this distinction between two kinds of self was just as problematic as the original distinction between the two kinds of dependence.

So given the obscurity in Kant's original doctrine of transcendental idealism, the prospects of making out a similar doctrine in the case of mathematics do not look good.

This is more or less how matters stood in my mind a few years ago. I recognized the difficulties in the Platonist and anti-Platonist positions and regarded the doctrine of transcendental idealism as hopelessly obscure. I was inclined to adopt a Platonist position, but more from a general inclination towards realism rather than from a reasoned response to the arguments.

I am now much more optimistic about the viability of an intermediate position. This change of heart was the result of reflecting about certain developments

in the foundations of mathematics at the beginning of the last century. These center on Russell's famous antimony; and so let me say a little about the antimony before discussing its relevance to the doctrine of transcendental idealism.

Any number of objects, it would seem, can be collected together into a set. Thus we can form the set of all philosophy professors or the set of all natural numbers or the set of all sets. Now normally a set is not a member of itself. A philosophy professor, for example, is not a set (though he may be set in his ways) and so the set of all philosophy professors will be normal. Similarly, the set of all natural numbers is not a natural number and so again will be normal. The set of all sets, on the other hand, *is* a set and so this set will *not* be normal.

Consider now the set of all normal sets. Is or is it not normal? If it is a normal, then it is not a member of itself and so, since it contains all normal sets, it is not normal. On the other hand, if it is not normal, then it is a member of itself and so, since it only contains normal sets, it is normal. Either way we have a contradiction!

It is easy to dismiss the antimony as frivolous and not worthy of serious thought. It strikes, however, at the very foundations of mathematics — and for two rather different reasons. In the first place, many important branches of mathematics make extensive and essential use of sets. The prime example is Cantor's theory of transfinite numbers (for which set theory was originally developed), but there is hardly a branch of mathematics today that does not

involve some appeal to sets. Second, one might think of set theory as not merely one branch of mathematics among others but as providing the foundation for the rest of mathematics: all mathematical notions are to be defined from the basic notions of set theory, and all mathematical theorems are to be derived from its basic assumptions. However, the antimony threatens the very coherence of set theory and hence the very coherence of its role either as an essential adjunct to or as a foundation for the rest of mathematics.

There have been many different responses to the antimony. But one line of thought goes as follows. Suppose we have a domain of sets. Then the antimony shows that it cannot, on pain of inconsistency, be taken to include the set of all normal sets from the domain. But surely we are in a position to *add* this set to the domain, thereby forming a larger domain. Of course, this larger domain cannot be taken to include the set of all normal sets from this larger domain. But not to worry, since this larger domain can itself be further extended. Thus whatever domain is our starting point, it can always be enlarged with the Russell-set (of all normal sets from the domain).

Suppose, however, that our starting point is the domain of all sets whatever. Then how can it be enlarged? For if there is a new set to put in the domain it must already have been there! So what should we say: that we were wrong in thinking that any domain whatever can be enlarged with the Russell-set; or that we were wrong in thinking that we could take as our starting point the domain of all sets whatever?

The line of thought I wish to consider is one which accepts the second of these alternatives. It maintains that we can form no intelligible conception of an *inextendible* domain. Whatever it is that one takes to be the domain of discourse, even if it as the domain of all sets, there is *always* the possibility of extending the domain.

If this is correct, then it raises the possibility of vindicating a form of transcendental idealism. For suppose I take my domain to include everything there is. I now postulate a Russell set for the domain; and the domain is thereby enlarged. The new set would then appear to depend for its existence upon an act of postulation. And is it not possible that this dependence is of a transcendental rather than of an ordinary empirical sort?

There are certainly ways of understanding what one is doing that would make the dependence straightforwardly empirical. One might think, for example, that in postulating a set one was bringing it into existence in perhaps something like the manner in which an author brings his characters into existence. But there is no need to think of what one is doing in this way. For we might suppose that, once the set was postulated, it would be correct to say that it always existed — or, better still, that it was not even the kind of thing to exist in space or time. Postulation enables one to refer to the set but it need not commit one to the strange view that the existence of the set is dependent upon one's ability to refer to it.

But if that is so, then can I be so sure that the existence of the set actually depends upon the act of postulation. Suppose I say "everyone is coming to the party" meaning "all of our friends." Someone then mentions Bert, who is an acquaintance rather than a friend, and I generously respond "OK, everyone including Bert." Here I extend the domain of discourse to include Bert. But no one would suppose that Bert depends for his existence upon my decision to include him in the discourse. So why is it not the same for sets? Is not the Russell-set like Bert — dependent upon us for its admission into our discourse, though not for its very existence?

This response, however, ignores the fact that our initial domain was meant to include everything that there is. When I said "everyone is coming to the party," the quantifier "everyone" was implicitly restricted; I meant "all my friends." It is for this reason that the admission of Bert simply consists in relaxing the restriction on the quantifier. But when I postulate the Russell-set, I began with a domain, or a quantifier, that was already unrestricted. Thus there is no possibility of understanding what I do as a way of relaxing an implicit restriction on the domain.

But how, you may ask, can the domain be both unrestricted and extendible? For does not the mere fact that the domain can be extended prove that it must originally have been restricted? It is absolutely critical at this point to appreciate that there is no contradiction in supposing that the domain might be both unrestricted and extendible. In saying that it is unrestricted, I am saying

that it is not understood as the restriction of a possibly broader domain. In saying that it is extendible, I am saying that it can be enlarged through acts of postulation. Of course, once it is enlarged I can then "revisit" the previous domain and understand it as the restriction of the enlarged domain. But that is not to say that it was *previously* understood as the restriction of a larger domain. Indeed, from our own point of view, the objector gets things backwards. For it is not that the initial smaller domain is understood, through restriction, in terms of the larger domain; rather, it is the larger domain that is understood, through postulation or "extension," in terms of the smaller domain.
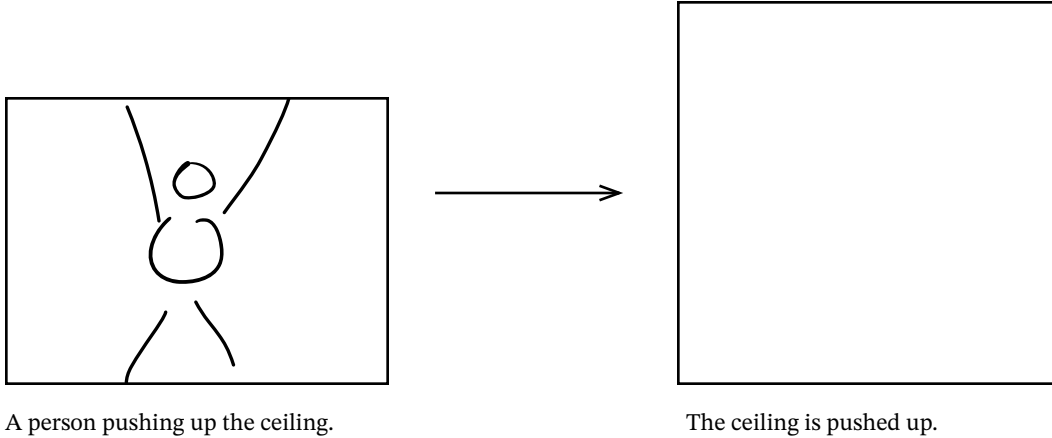
Here are a couple of pictures that may help illustrate the present point of view:
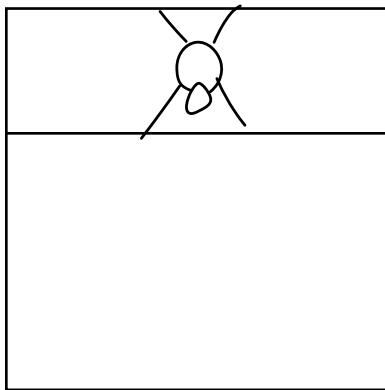
A person pulling up the ceiling.
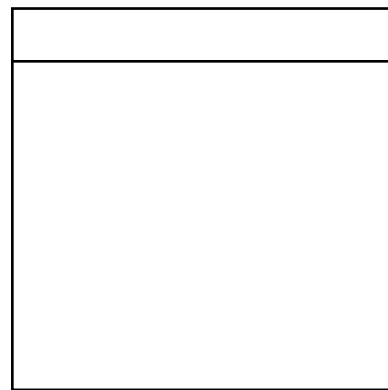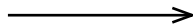
The box is of the same size and the ceiling is pulled up.

A person pushing up the ceiling.   The ceiling is pushed up.

In the first picture we have someone whose domain of discourse is understood as the restriction of a larger domain; and it is by pulling "upwards," i.e. by relaxing the restriction on the larger domain, that he is able to enlarge the domain of discourse. In the second picture we have someone whose domain of discourse is understood as unrestricted; and it is by pushing "outwards," i.e. by postulating new objects, that he is able to enlarge the domain. There is in this case no more extensive domain by reference to which we can understand what we are doing. It is as if we were to extend the boundaries of the universe without there being anything on the other side.

If I am right, then postulation has the following two features. First, the objects of postulation are — or need not be — the kind of thing that exist in space or time. There is therefore no possibility of there being *created* through acts of postulation. Second, postulation enlarges the domain of what there is not merely of what we choose to talk about. It is therefore the existence of these objects and not merely possibility of referring to them that is dependent

upon the acts of postulation. These two features therefore make it plausible that postulation does indeed provide us with a form of transcendental idealism.

These remarks merely constitute the beginning of an attempt to develop an intermediate position, one that does justice to the status of mathematical objects as both "discovered" and "invented." Two major tasks remain. The first is to work out a detailed theory of postulation. A central question here is to spell out the conditions under which we are justified in postulating objects. We cannot postulate anything we like. It is acceptable to postulate negative numbers, for example, but not guardian angels; and the theory of postulation should explain what we can postulate and why. It would be desirable if we could show that the theory of postulation was forced upon us; and this suggests that the principles of such a theory should emerge form a consideration of the conditions for the possibility of postulation (just as, for Kant, a priori principles of metaphysics are meant to emerge from a consideration of the conditions for the possibility of experience). The other task is to show how mathematics can be seen to be the product of postulation. Ideally, we would like to see all mathematical objects and all mathematical axioms as consequential upon acts of postulation. In this way, the whole of mathematics could be seen as an instance (though an especially interesting instance) of the general theory of postulation.

In working on these two tasks, I have found it helpful to adopt a procedural view. The postulates are not to be regarded as statements of what one wants to be true but as instructions for the construction of the new domain.

The postulates are literally programs, written in a programming language especially designed for that purpose. But of course there is no computer on which these programs are run. If you like, the computer is our own mind or, as I sometimes like to think, a fictitious mathematical genie, who automatically carries out our postulational wishes. This genie might be compared to Kant's transcendental self; and so we have another way in which the two approaches appear to overlap.