The Julius Silver, Roslyn S. Silver, and Enid Silver Winslow

DIALOGUES IN ARTS AND SCIENCE

UNIVERSALITY FOR MATHEMATICAL AND PHYSICAL SYSTEMS

Percy Deift

Silver Professor of Mathematics

NEW YORK UNIVERSITY

Universality For Mathematical And Physical Systems[1]

All physical systems in equilibrium obey the laws of thermodynamics. The first law asserts the conservation of energy. The second law has a variety of formulations, one of which is the following: Suppose that in a work cycle a heat engine, such as a locomotive, extracts Q_1 units of heat from a heat reservoir, such as a boiler, at temperature T_1, performs W units of work, moving a load, for example, from to bottom to the top of a hill, and then exhausts the remaining $Q_2 = Q_1 - W$ units of heat to a heat sink, such as a lake, at a lower temperature T_2, $T_2 < T_1$. Let $E = W/Q_1$ denote the efficiency of the conversion of heat into work. Then the second law tells us there is a maximal efficiency

$$E_{\max} = \frac{T_1 - T_2}{T_1}$$

depending only on T_1 and T_2, so that for all heat engines, and all work cycles, E is never greater than $E_{\max}$,

$$E \leq E_{\max} .$$

Nature is so set up that we just cannot do any better.

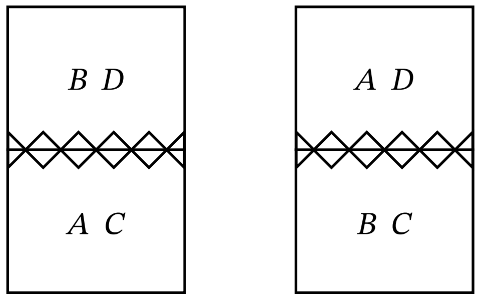
On the other hand, it is a very old thought going back at least to

Democritus and the Greeks, that matter, all matter, is built out of tiny constituents — atoms — obeying their own laws of interaction. The juxtaposition of these points of view, the macroscopic world of tangible objects such as boilers, heat engines and lakes, and the microscopic world of atoms, presents a fundamental, difficult and long-standing challenge to scientists: Namely, how does one derive the macroscopic laws of thermodynamics from the microscopic laws of atoms? The special, salient feature of this challenge is that the *same* laws of thermodynamics should emerge no matter what the details of the atomic interaction. In other words, on the macroscopic scale on which we live, physical systems, be they composed of benzene molecules, for example, or gold atoms in a bar, should exhibit universal behavior. Indeed, it is the very emergence of universal behavior for macroscopic systems that makes possible the existence of physical laws.

This kind of thinking, however is not common in the world of mathematics. Mathematicians tend to think of their problems as sui generis, each with its own special distinguishing features. Two problems are regarded as "the same" *only* if some "isomorphism," explicit or otherwise, can be constructed between them. For example, consider

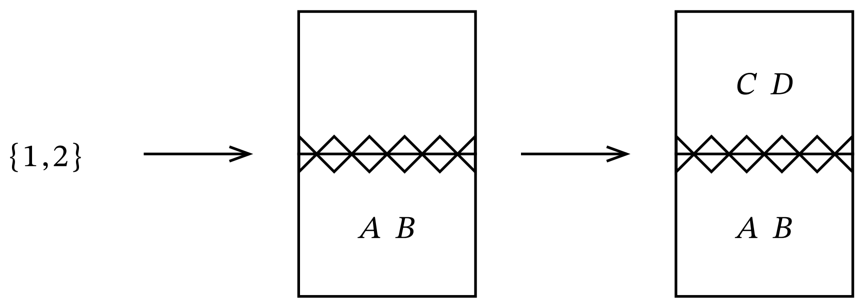
Problem 1. How many distinct pairs of numbers can be formed from the numbers 1, 2, 3, 4? For example, $\{1,\ 2\}$, $\{2,\ 4\}$, ... etc.

Problem 2. Let $A = \mathit{Abby}$, $B = \mathit{Brenda}$, $C = \mathit{Charles}$, and $D = \mathit{Daisy}$, be four tennis players. How many different double games can they play? For example:

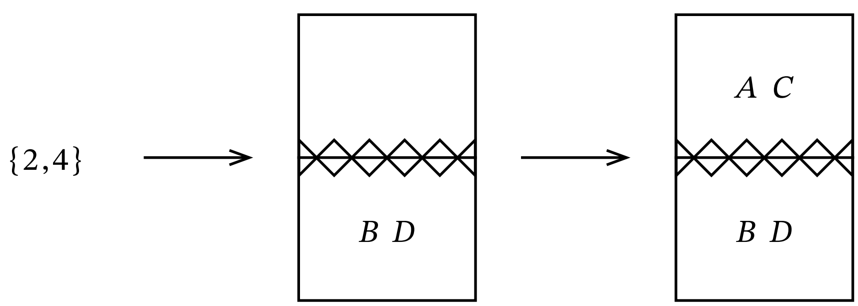


, and so on.

Let 1 correspond to *A*, 2 to *B*, 3 to *C* and 4 to *D*. We can construct an isomorphism between Problems 1 and 2 in the following way: To any pair of numbers, say $\{1,2\}$, we associate the corresponding players, {*A*, *B*} in this case, and place *Abby* and *Brenda* in the lower court. The game is then set up by placing the remaining players *Charles* and *Daisy* in the upper court, as follows:

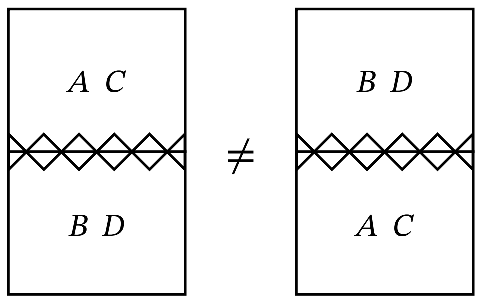


Similarly:



and so on.

Now as every game can be achieved in this way, and as different pairs of numbers give rise to different games, we see that there are precisely as many games (there are six! Exercise!) as there are distinct pairs of numbers from 1 to 4. With this isomorphism established, a mathematician would conclude that Problems 1 and 2 are "the same." Remark: Here we are distinguishing the game where *AC* play in the upper court, from the game in which they play in the lower court:



In general it is very difficult to determine whether two problems are "the same" in the above strict mathematical sense. In recent years, however, universality in the sense of macroscopic physics, when no isomorphism is known or apparent, has started to emerge in a wide variety of mathematical problems, and the goal of this article is to illustrate some of these developments. As we will see, there are problems from diverse areas, often with no discernible, mechanistic connections, and with no known isomorphisms as in Problem 1/Problem 2 above, all of which behave, on some appropriate scale, in precisely the same way. The list of such problems is varied, long and growing, and points to the emergence of what one might call "macroscopic mathematics."

A precedent for the kind of result that we are going to describe is given by the celebrated central limit theorem of probability theory, where one considers independent, identically distributed variables, x_1, x_2, … , with average 0. For example x_1 could be the outcome of flipping a (fair) coin, +1 for heads and −1 for tails: Then the

$$\text{average of } x_1 = (+1) \times (\text{probability of a heads}) + (-1) \times (\text{probability of a tail})$$
$$= 1 \times ½ + (-1) \times ½ = 0.$$

Following on, x_2 is the outcome of flipping the coin again, and so on. Alternatively x_1 could denote the outcome of choosing a number uniformly between −1 and 1, and x_2, x_3, … , denote subsequent choices. The central limit theorem tells us that whatever the distribution of x_1, x_2, … , the probability distribution for the sum $x^1 + x^2 + \cdots + x_n$, suitably scaled, converges, as n becomes large, to the famous bell curve. In other words, the bell curve is universal, independent of the distribution of x_1, x_2, …

In order to describe the emergence of "macroscopic mathematics," we must consider the theory of random matrices. Recall that a $k \times l$ matrix M is a rectangular array of numbers with k rows and l columns. For example:

$$M = \begin{pmatrix} 1 & 10 & 9 \\ 7 & 6 & 4 \end{pmatrix}$$

is a 2×3 matrix with $M_{11} = 1$ in the first row and first column, $\mathrm{M}_{23} = 4$ in the second row and third column, and so on. If $k = l$, the matrix is square, for example:

$$M = \begin{pmatrix} 7 & 17 \\ 8 & 64 \end{pmatrix}$$

is a 2×2 matrix with $M_{12} = 17$, etc. Now associated to any square $k \times k$ matrix there are k distinguished numbers $y_1, y_2, \dots, y_n$ called *eigenvalues*, or *characteristic values*. For general matrices, the eigenvalues $y_1, y_2, \dots, y_k$ could be complex numbers, but if M is a *symmetric* matrix, the eigenvalues $y_1, \dots, y_k$ are all real numbers. A matrix is symmetric if the entry in row i and column j is the same as the entry in the j^{th} row and i^{th} column. Thus:

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is symmetric as $M_{12} = -1 = M_{21}$. For this 2×2 matrix, the eigenvalues are obtained by solving the quadratic equation $(y - 1|^2 - 1 = 0$, i.e. $y - 1 = +1$ or -1, so that $y = 0$ or 2. Thus $y_1 = 0$ and $y_2 = 2$.

In order to understand what an eigenvalue is from a physical point of view, consider, for example, a swing in a park with constituent parts consisting of a seat, two ropes, a crossbar and two side stands. It turns out that the dynamics of the swing can be described by a matrix M where M_{12}, say, describes the connection between the seat and one of the ropes, M_{23} describes the connection between the rope and the crossbar, and so on. Now when you push your daughter in the swing, you discover that there is a special frequency, the so-called *resonant* frequency, with the property that if you push the swing precisely at that frequency, the swing resonates with your pushing and moves back and forth, high and easily, with little effort on your part. That resonance frequency corresponds to an eigenvalue of the matrix M for the swing system.

A random matrix M is a matrix in which the entries M_{11} , M_{12} , … of the matrix are chosen randomly. For example a random 2 × 2 Bernoulli matrix M is one in which the four entries M_{11}, M_{12}, M_{21}, M_{22} take on the values 1 or −1 depending on the flip of a coin, +1 for heads and −1 for tails. If M is random, then the eigenvalues y_1, y_2, … of M are clearly also random. Now the remarkable and serendipitous fact is that *many systems, physical and mathematical, behave like the eigenvalues of a (large) random matrix.*

Let me illustrate how this works with two example, the first from a card game, and the second from a bus scheduling problem in Cuernavaca, Mexico.

Example 1. Consider the solitaire card game known as *patience sorting.* The game is played with N cards, numbered 1, 2, … , N for convenience. The deck is shuffled and the first card is placed face up on the table in front of the dealer. If the next card is smaller than the card on the table, it is placed face up on the top of the card; if it is bigger, the card is placed to the right of the first card, making a new pile. If the third card in the deck is smaller than one of the cards on the table, it is placed on top of that card; if it is smaller than both cards, it is placed as far to the left as possible. If it is bigger than both cards, it is placed face up to the right of the pile(s), making a new pile. One continues in this fashion until all the cards are dealt out. Let q_N denote the number of piles obtained. Clearly q_N depends on the particular shuffle which we denote by S, and we write $q_N = q_N(S)$.

For example, if $N = 6$ and $S = 341562$, where 3 is the top card, 4 is the next card and so on, then patience sorting proceeds as follows:

		1	1	1	1
3	3 4	3 4	3 4 5	3 4 5 6	3 4 5 6

and so $q_6(S) = 4$.

Question 1. Suppose all the shuffles S are equally likely. If each card is of unit size, how big a table does one typically need to play patience sorting with N cards? Or more precisely, how does the probability distribution for $q_N = q_N(S)$ behaves as $N \to \infty$?

Example 2. The city of Cuernavaca in Mexico (population about 500,000) has an extensive bus system, but there is no municipal transit authority to control the city transport. In particular, there is no time table, which gives rise to so-called Poisson-like phenomena, with bunching and long waits between buses. Typically the buses are owned by drivers as individual entrepreneurs, and all too often a bus arrives at a stop just as another bus is loading up. The driver then has to move on to the next stop to find some fares. In order to remedy the situation the drivers in Cuernavaca came up with a novel solution: They introduced "recorders" at specific locations along the bus routes in the city. The recorders kept tract of when buses passed their locations, and then transmitted this information, via various pre-arranged hand signals, to the next driver, who could then speed up or slow down in order to optimize the distance to the preceding bus. The upshot of this ingenious scheme is that the drivers do not lose out on fares and the citizens of Cuernavaca now have a reliable and regular bus service. In the late 1990s two Czech physicists with interest in transportation problems, M. Krbálek and P. Sěba, learned about the buses in Cuernavaca and decided to investigate. For about a month they studied the statistics of the bus arrivals on a particular line close to the city center.

Question 2. What did Krbálek and Sěba learn about the statistics of the bus system in Cuernavaca?

Quite remarkably, the answer to both questions is given by random matrix theory. In particular, in Example 1, as N goes to infinity, the probability distribution for q_N, suitably centered and scaled, converges to the distribution function for the largest eigenvalue of a (large) random Hermitian[2] matrix. (It turns out that on average the table should be wide enough to accommodate 11 piles side by side.) In Example 2, at a fixed location (bus stop) Krbálek and Sěba found that the statistics of waiting times, that is, the times between the arrival of one bus and the next, was precisely described by the separation between the eigenvalues of a (large) random Hermitian matrix.

This is universality in the sense of macroscopic physics. There is no isomorphism/correspondence as in Problems 1/2, connecting patience sorting or the buses in Cuernavaca, to the eigenvalues of a random matrix. Nevertheless, on the appropriate scale, all these systems behave statistically in the same way.

Problems 1 and 2 are just two examples of the burgeoning field of macroscopic mathematics.

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1 Part of this article is taken from the author's address to the International Congress of Mathematics, Madrid, Spain, 2006. See, Intl. Congress of Mathematics, Vol. I, 125–152, Eur. Math. Soc. Zurich, 2007

2 Hermitian is the analog of symmetric when the entries M_{11}, M_{12}, . . . of a matrix M are complex numbers.