QMI PRELIM 2013

All problems have the same point value. If a problem is divided in parts, each part has equal value. Show all your work.

Problem 1

\[ \vec{L} = \vec{r} \times \vec{p}, \quad \vec{p} = -i\hbar \vec{\nabla} \]

(a) Show that \( L_z = i\hbar \left( \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \); (cyclic for other components)

Solution

\[ L_i \equiv \epsilon_{ijk} r_j p_k = -i\hbar \epsilon_{ijk} r_j \partial_k. \] So \( L_z = -i\hbar (x \partial_y - y \partial_x) = i\hbar (y \partial_x - x \partial_y) \).

(b) Show that the commutator \([L_x, L_y] = i\hbar L_z\) (and cyclic), and consequently that we can write \( \vec{L} \times \vec{L} = i\hbar \vec{L} \). Also, show \([L_z, r_{\pm}] = \pm r_{\pm}\hbar\), where \( r_{\pm} = x \pm iy \)

Solution

We know \([r_i, p_j] = i\hbar \delta_{ij}\). So:

\[
[L_i, L_j] = \epsilon_{ikl} \epsilon_{jmn} [r_k p_l, r_m p_n]
= \epsilon_{ikl} \epsilon_{jmn} (r_m [r_k, p_l] p_n + r_k [p_l, r_m] p_n)
= i\hbar \epsilon_{ikl} \epsilon_{jmn} (r_m p_l \delta_{kn} - r_k p_n \delta_{lm})
= i\hbar ((-\delta_{ij} \delta_{lm} + \delta_{in} \delta_{lj}) r_m p_l - (-\delta_{ij} \delta_{kn} + \delta_{in} \delta_{kj}) r_k p_n)
= i\hbar (r_i p_j - r_j p_i)
= i\hbar \epsilon_{ijk} L_k.
\]
Hence \([L_x, L_y] = i\hbar L_z\) (et cyc). Then:

\[
(\vec{L} \times \vec{L})_k = \epsilon_{ijk} L_i L_j
\]

\[
= \frac{1}{2} \epsilon_{ijk} (L_i L_j - L_j L_i)
\]

\[
= \frac{1}{2} \epsilon_{ijk} [L_i, L_j]
\]

\[
= \frac{1}{2} i\hbar \epsilon_{ijk} \epsilon_{ijl} L_l
\]

\[
= \frac{1}{2} i\hbar (2\delta_{kl}) L_l
\]

\[
= \frac{1}{2} i\hbar \epsilon_{ijk} \epsilon_{ijl} L_l
\]

\[
= \frac{1}{2} i\hbar L_k
\]

So \(\vec{L} \times \vec{L} = i\hbar \vec{L}\).

(c) Show that \([L^2, L_z] = 0\), where \(L^2 = L_x^2 + L_y^2 + L_z^2\).

Solution

\([L^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] = i\hbar (-L_x L_y - L_y L_x + L_y L_x + L_x L_y) = 0\).

(d) Show that \(L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z\), where \(L_\pm = L_x \pm iL_y\).

Solution

\(L_\pm L_\mp = L_x^2 + L_y^2 \pm iL_y L_x \mp iL_x L_y = L^2 - L_z^2 \pm i[L_y, L_x] = L^2 - L_z^2 \pm \hbar L_z\).

Problem 2

Let \(\psi\) be an eigenfunction of \(L^2\): \(L^2 \psi = l(l+1)\hbar^2 \psi\). Since \([L^2, L_z] = 0\), we can make \(\psi\) an eigenfunction of \(L_z\) as well: \(L_z \psi_m = m\hbar \psi_m\). \(L^2\) and \(L_z\) are defined as in Problem 1.

(a) With raising operator \(L_+ = L_x + iL_y\), and results from Problem 1, show that \(L_+ \psi_m\) generates an eigenfunction of same \(L^2\) but with \(L_z\) eigenvalue of \((m+1)\hbar\).
Solution

\[ L_z(L_+\psi_m) = (L_x L_z + i L_y L_z)\psi_m = (L_x L_z + i\hbar L_y + i(\hbar L_z - i\hbar L_x))\psi_m = (L_+ L_z + \hbar L_+)\psi_m = (m+1)\hbar (L_+\psi_m). \]

(b) Compute the non-zero matrix elements of \( L_+ \).

Solution

From part (a) we know \( L_+|l\ m\rangle = c_+|l\ m+1\rangle \), so \( \langle l\ m+1|L_+|l\ m\rangle = c_+ \), all other matrix elements 0.

\[ |c_+|^2 = \langle l\ m|L_- L_+|l\ m\rangle = \langle l\ m|(L^2 - L_z^2 - \hbar L_z)|l\ m\rangle = \hbar^2(l(l+1) - m^2 - \hbar^2 m) \]

So \( c_+ = \hbar \sqrt{l(l+1) - m(m+1)} \).
Problem 3

Consider a spin vector \( \sigma_n = \sigma_x \cos \alpha \sin \beta + \sigma_y \sin \alpha \sin \beta + \sigma_z \cos \beta \), and:

\[
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

(a) Determine the eigenvalues of \( \sigma_n \).
(Hint: compute \( \sigma_n^2 \)).

Solution

\[
\begin{align*}
\sigma_n &= \begin{pmatrix} \cos \beta & e^{-i\alpha} \sin \beta \\ e^{i\alpha} \sin \beta & -\cos \beta \end{pmatrix} \quad \Rightarrow \quad \sigma_n^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

\( \sigma_n^2 \) has an eigenvalue +1, so \( \sigma_n \) could have eigenvalues \( \pm 1 \). Need to check each.

\[
\begin{align*}
\det(\sigma_n - 1) &= (\cos \beta - 1)(-\cos \beta - 1) - \sin^2 \beta \\
&= 1 - \cos^2 \beta - \sin^2 \beta \\
&= 0 \\
\det(\sigma_n + 1) &= (\cos \beta + 1)(-\cos \beta + 1) - \sin^2 \beta \\
&= 1 - \cos^2 \beta - \sin^2 \beta \\
&= 0
\end{align*}
\]

The eigenvalues are \( \pm 1 \).

(b) Pure rotation by an angle \( \gamma \) of the z-axis in the \( x - z \) plane leads to the rotation matrix

\[
\begin{pmatrix} \cos \frac{\gamma}{2} & \sin \frac{\gamma}{2} \\ -\sin \frac{\gamma}{2} & \cos \frac{\gamma}{2} \end{pmatrix}
\]

Interpret this result by comparing to rotation of ordinary vectors, e.g. \( \vec{\ell}, \vec{r} \).

Solution

Pure spin rotation by an angle \( \gamma \) of the z-axis in \( x - z \) plane (\( \beta = 0 \)), leads to a rotation matrix as given. Comparing to rotation of ordinary vectors, this is seen to correspond to spin rotations by half angles.

Problem 4
At fixed principal quantum number \( n \) and orbital quantum number \( l \), the Hamiltonian of an alkali atom can be modelled as

\[
H = A + B \vec{L} \cdot \vec{S}, \quad A, B = \text{constants}.
\]

(a) Diagonalize this Hamiltonian for \( l = 1 \) and \( l = 2 \).

**Solution**

Use: \( 2\vec{L} \cdot \vec{S} = j(j+1) - l(l+1) - 3/4 \). Hence for \( j = l + 1/2 \) we find \( \vec{L} \cdot \vec{S} = l/2 \), while for \( j = l - 1/2 \) we find \( \vec{L} \cdot \vec{S} = -l/2 - 1 \).

(b) Which other operator(s) built in terms of \( \vec{L} \) and \( \vec{S} \) commutes with the Hamiltonian?

**Solution**

\( \vec{J} = \vec{L} + \vec{S} \) (and functions thereof).

(c) The Hilbert space of states for \( l = 1 \) is the product of two spaces, \( V = V_L \otimes V_S \). \( V_L \) is the orbital angular momentum space (dimension 3) and \( V_S \) is the spin space (dimension 2). By definition, given a pure state \( |\psi\rangle \in V \), the reduced density matrix in \( V_L \) is \( \rho_L = \text{tr}_S|\psi\rangle \langle \psi| \). The entanglement entropy is defined as \( S = -\text{tr} \rho_L \log \rho_L \).

Compute the entanglement entropy of the \( j = 1/2 \) state

\[
|\psi\rangle = \frac{1}{\sqrt{3}} \left[ +\frac{1}{2} \right]_S \langle 0 \rangle_L - \sqrt{\frac{2}{3}} \left[ -\frac{1}{2} \right]_S \langle 1 \rangle_L.
\]

**Solution**

\( \rho_L = \text{diag} \left( 2/3, 1/3, 0 \right) \); \( S = (1/3) \log 3 + (2/3) \log(3/2) = \log 3 - (2/3) \log 2 \).

The maximal entanglement entropy is \( \log 3 \).

**Problem 5**

Consider the scattering of a particle of energy \( E \) in a square well potential \( V = 0 \) for \( x < 0 \) and \( x > b \), \( V = V_0 > 0 \) for \( 0 < x < b \).

Assuming that the transmission coefficient \( T \ll 1 \) (thick barrier approximation), obtain its value in terms of the above parameters. You may use the
definitions

\[ k = \sqrt{\frac{2mE}{\hbar^2}} \quad k' = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \]

With this obtain the value of the transmission probability

\[ T = \left| \frac{\text{transmitted amplitude}}{\text{incident amplitude}} \right|^2. \]

Solution

\[ H\psi = E\psi = \left(\frac{p^2}{2m} + V\right)\psi \]

\[ p = \sqrt{2m(E - V)}, \psi = \psi(x)e^{-iEt/\hbar} \]

\[ \nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0 \]

In barrier region, from \( r = 0 \) to \( r = b \), \( V = V_0 \)

\[ k' \equiv \frac{1}{\hbar} \sqrt{2m(E - V)} \]

Outside of barrier region, for \( r \leq 0, r \geq b \),

\[ k \equiv \frac{1}{\hbar} \sqrt{2mE} \]

Say alpha particle incident from Region I:

Region I: \( \psi_I = De^{ikx} + Ae^{-ikx} \)

Region II: \( \psi_{II} = Be^{k'x} + Ce^{-k'x} \)

Region III: \( \psi_{III} = Ae^{ikx} \)

At \( x = 0 \), \( \psi_I = \psi_{II} \) or \( D + E = B + C \) and:

\[ \frac{\partial\psi_I}{\partial x} = \frac{\partial\psi_{II}}{\partial x} = Di ke^{ikx} - Eie^{-ikx} = Bk' e^{k'x} - Ck' e^{-k'x} \]

so:

\[ ik(D - E) = k'(B - C), D - E = \frac{k'}{ik}(B - C) = -i \frac{k'}{k}(B - C) \]

At \( x = b \), \( \psi_{II} = \psi_{III} \), \( Ae^{ikb} = Be^{k'b} + Ce^{-k'b} \), and \( ikAe^{ikb} = k'Be^{k'b} - k'Ce^{-k'b} \), which we solve:

\[ Be^{k'b} = Ae^{ikb} - Ce^{-k'b} \]

\[ ikAe^{ikb} = k'Be^{k'b} - k'Ce^{-k'b} - k'C e^{-k'b} \]

\[ Ae^{ikb}(ik - k') = -2k'C e^{-k'b} \]

\[ \implies C = \frac{1}{2} Ae^{ikb}(\frac{ik - k'}{-k'}) e^{k'b} \]

\[ = \frac{A}{2} e^{ikb + k'b}(\frac{-ik}{k'} + 1) \]
Solve for $B$:

$$Be^{kb} = Ae^{ikb} - \left[\frac{A}{2}e^{ikb+kk'}(1 - \frac{i}{k'})\right]e^{-k'b}$$

$$= Ae^{ikb} - \frac{A}{2}e^{ikb}(1 - \frac{i}{k'})$$

$$= Ae^{ikb}[1 - \frac{i}{2}(1 - \frac{i}{k'})]$$

$$\Rightarrow B = \frac{A}{2}e^{(ik-k')b}(1 + \frac{i}{k'})$$

From $D + E = B + C$ and $D - E = -\frac{ik'}{k}(B - C)$ or $2D = B(1 - \frac{ik'}{k}) + C(1 + \frac{ik'}{k})$ we obtain for a thick barrier, $B \to 0$:

$$D = \frac{1}{2}(1 + \frac{ik'}{k})[\frac{A}{2}(1 - \frac{i}{k'})e^{ikb+kk'}]$$

$$A = \frac{4e^{-ikb-k'}b}{(1 + \frac{ik'}{k})(1 - \frac{i}{k'})}$$

and finally:

$$\frac{|A|^2}{|D|^2} = \frac{16e^{-2k'b}}{[1 + (\frac{k}{k'})^2][1 + (\frac{k}{k'})^2]}.$$ 

**Problem 6**

Consider the Hamiltonian of a one-dimensional particle with mass $m$ and potential $V = A/x^2$. $A$ can be either positive or negative.

(a) Prove that the spectrum is unbounded below for $2mA < -1/4$ (Hint: study how the potential energy and the kinetic energy scale when the parameter $b \to 0$ in the test wave packet $\psi = b^{-1/2-a}x^a \exp(-x^2/2b^2)$, with $a > 1/2$ but arbitrarily close to 1/2: $a = 1/2 + \epsilon^+$.)

**Solution**

$$\int_0^\infty dx |\frac{d\psi}{dx}|^2 \sim a^2b^{-2}, \quad \int_0^\infty dx^2mA |\psi|^2 \sim 2mA b^{-2}.$$ For any $2mA < -1/4$ one can choose an $a$ such that $a^2 < |2mA|$, so the potential drives the energy to $-\infty$.

(b) When $2mA > -1/4$, it can be proven that the spectrum is bounded below (do not try to prove it!). In this case, is the spectrum continuous or discrete? What is the lowest value of the energy? (Hint: under the scaling
$x \rightarrow y = \lambda x$, the Hamiltonian transforms as $H \rightarrow \lambda^{-2}H$.)

Solution
Because of the scaling, if $\psi(x)$ is an eigenstate of $H$ with eigenvalue $E$, $\psi(\lambda x)$ has eigenvalue $E/\lambda^2$. So the spectrum is continuous and the lowest eigenvalue is $E = 0$. 