1. In a *Joule-Thomson process*, a gas is allowed to expand adiabatically without doing any work. Initially, the gas is at temperature $T_1$ in a volume $V_1$. After the expansion, the same number of gas molecules occupy volume $V_2 > V_1$ at temperature $T_2$.

   (a) (5 points) In general, what is the change in the internal energy $E$ as a result of this process?

   (b) (5 points) If the gas is an ideal gas, what is the change in its entropy as a result of the expansion?

   (c) (10 points) Now consider a non-ideal (interacting) gas. Use the first law of thermodynamics to show that the change in temperature can be expressed as

   $$\Delta T = -\int_{V_1}^{V_2} \frac{T^2}{C_V} \frac{\partial (P/T)}{\partial T} \bigg|_V dV.$$

   It may be helpful to recall the surprising thermodynamic relationship

   $$\left. \frac{\partial E}{\partial V} \right|_T \left. \frac{\partial V}{\partial T} \right|_E \left. \frac{\partial T}{\partial E} \right|_V = -1.$$

   Note, in particular, the sign of the result.

   (d) (5 points) Some gases get cooler after free expansion, and others get hotter. What will happen to the temperature if the particles only experience short-ranged repulsive interactions?
Continuation to Problem 1:
2. A simple model for a polymer in two dimensions is a path on a square lattice. At every lattice point, the polymer can continue straight with no energy cost (white circles in diagram), or can bend at right angles with an energy cost of $\epsilon$ (filled circles). Each step presents two ways to bend, but only one way to continue straight. The total elastic energy cost to place a polymer in a given configuration is thus $\epsilon$ times the number of bends. Consider a polymer consisting of $N + 1$ segments, which therefore has $N$ possible points where it can bend. For simplicity, we will include all possible paths, including those that overlap and cross over themselves.

(a) (5 points) How many shapes have a total bending energy $E = n\epsilon$, where $0 \leq n \leq N$?

(b) (5 points) What is the entropy $S(E, N)$ of this system? Use Stirling’s formula to approximate any factorials.

(c) (5 points) Calculate the temperature of this system as a function of the total bending energy $E$ and the length $N$ of the polymer.

(d) (5 points) Calculate the energy of the polymer as a function of the temperature.

(e) (5 points) Calculate the heat capacity at constant length, $C_N(T, N)$. 
Continuation to Problem 2:
3. (25 points) A system of $N$ spin-$\frac{1}{2}$ fermions is in equilibrium at temperature $T$ in the three-dimensional potential $U(\vec{r}) = \frac{1}{2}m\omega^2 r^2$.

(a) Calculate the system’s Fermi energy, $\epsilon_F$, under the assumption that $N \gg 1$.

(b) Calculate the temperature dependence of the fermions’ chemical potential, $\mu(T)$, in the low-temperature limit. Express this in terms of $\epsilon_F$, keeping only the leading-order dependence on $T$.

(c) Calculate the system’s internal energy in the low-temperature limit. How is this related to the Fermi energy?

You may assume that the temperature is low enough that the Fermi-Dirac distribution function, $f(\epsilon) = [1 + \exp(\beta(\epsilon - \mu))]^{-1}$, may be approximated as a step function. In that case,

$$\int_0^\infty F(\epsilon)f(\epsilon) d\epsilon \approx \int_0^{\mu} F(\epsilon) d\epsilon + \frac{\pi^2}{6} F'(\mu) \left(\frac{k_B T}{\mu}\right)^2$$

for any function $F(\epsilon)$. 
Continuation to Problem 3: