Quantum Mechanics I Prelim Exam 2018

Instructions: The exam will begin at 2:05 and run until 5:00. In order to pass, most of the credit (∼75%) must be achieved on each of at least 3 questions. The Pauli matrices are:

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Problem 1: States that maximize angular momentum along the z-axis but do not have an in-plane angular momentum component are often called “cat” states. Consider states defined as:

\[ |C(s)\rangle = \frac{|s,m_s\rangle + |s,-m_s\rangle}{\sqrt{2}}, \]

where \( s \) is the total spin eigenvalue and \( m_s \) is the z-axis spin moment. For what values of \( s \) will \( |C(s)\rangle \) fit the above definition of a cat state? Support your answer.

Problem 2: A spin-1/2 particle exists in two-dimensional space, with the following kinetic Hamiltonian:

\[ H_K = v_D(p_x\sigma_y - p_y\sigma_x), \]

where \( v_D \) is a constant, \( p_x \) (\( p_y \)) is the x- (y-) axis momentum operator, and \( \sigma_x \) and \( \sigma_y \) are Pauli matrices. (Note: this may be familiar as the massless 2D Dirac Hamiltonian)

(a) Please identify an eigenstate of \( H_K \) with non-zero momentum along the x-axis.

(b) The \( R_x \) operator reflects all kets across the y-axis by flipping the sign of the \( x \) coordinate, and is defined as follows:

\[ R_x = \sum_s \int dy \int dx |x,y,s\rangle \langle -x,y,s|, \]

where the sum is over spin orientations in the ±z directions. Consider the action of \( R_x \) on the eigenstate you provided in part (a). Can you identify why \( R_x \) does not commute with \( H_K \)? Would \( R_x \) commute with the classical Hamiltonian of a massive particle, \( H_C = p^2/2m \)?

Problem 3: A spinless particle in a harmonic oscillator potential has eigenstates \( |n\rangle \) with eigenenergies \( E_n = \hbar\omega(n + 1/2) \), and integer eigenvalues \( n \geq 0 \).

(a) Define the density matrix at finite temperature, and provide an equation for the ensemble expectation value \( \langle n \rangle_T \).

(b) Now consider a scenario in which there are many weakly interacting Harmonic oscillators with the same Hamiltonian, each containing one particle. Suppose weak interactions between the oscillators suppress the oscillator frequencies, giving \( \omega(T) = \omega_0(1 - c|n|) \).

For what values of the constant \( c \) is there a local energy minimum with \( |n|=0 \), and for what values is this a global energy minimum? Draw a rough diagram of how \( |n| \) will depend on temperature for \( c > 0 \) and \( c < 0 \) with \( |c| << 1 \), if the system is initialized in the \( |n|=0 \) state and heated rapidly with the constraint that \( |n| \) must evolve continuously.

Problem 4: Consider a massive particle in a 1D space with width \( 'b' \) and repeating boundary conditions.

Use a superposition of two plane waves to construct a normalized state \( |\Psi\rangle \) for which the expectation value of position \( \langle \Psi | x | \Psi \rangle \) initially moves in the positive direction, and for which quantum amplitude vanishes at the \( x = 0 \).
and \( b \). Evaluate the time dependence of \( \langle \Psi | x(t) | \Psi \rangle \) in terms of the mass \( m \) for \( t \sim 0 \).

**Problem 5:** For a massive particle in a 1D harmonic oscillator potential, the spatial translation operator 
\( T(d) = \exp(-idp/\hbar) \) can be written in terms of ladder operators \( a \) and \( a^\dagger \) that create transitions within the eigenbasis. These operators behave as bosonic annihilation and creation operators. The raising/creation operator is defined as:

\[
a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{i}{m\omega} p \right)
\]  

Please write the translation operator in terms of \( a \) and \( a^\dagger \), and show that when acting on the harmonic oscillator ground state \( |0\rangle \), the lowering operator \( a \) is not needed. Represent \( T(d)|0\rangle \) as \( \alpha_0 \exp(\alpha_1 a^\dagger)|0\rangle \), and provide equations for the constants \( \alpha_0 \) and \( \alpha_1 \). You may wish to use the operator exponentiation identity \( \exp(A)\exp(B) = \exp(A + B + [A, B]/2) \), which comes from the Zassenhaus formula, and applies only if \( [A, [A, B]] = [B, [A, B]] = 0 \).