1. [30 points]

a) Suppose the Hamiltonian $\mathcal{H}$ is such that there are functions $f$ and $g$ in phase-space that do not depend explicitly on time and that they satisfy

$$[f, \mathcal{H}] = f, \quad [g, \mathcal{H}] = -g,$$

Show that $fg$ and $[f, g]$ are constants of motion.

b) Consider phase-space with canonical coordinates $(q, p)$. Do there exist new canonical coordinates $(Q, P)$ such that the new momentum is $P = qp$? If so, find $Q$. If not, give a proof or argument that such new canonical coordinates do not exist.

c) A particle moves in 1D under the influence of the potential shown in the Figure below. Draw qualitatively the phase-space portrait, indicating regions of libration (bounded motion), rotation (unbounded motion) and the separatrix curve.

![Potential energy](image)

FIG. 1: Potential energy for problem 1c
2. [20 points] A mass \( m \) moves constrained on an inclined plane of angle \( \alpha \) with respect to the horizontal (there is gravity acting in the vertical direction). When the mass reaches the end of the inclined plane, it collides elastically with a wall and bounces back up the plane. Using as a generalized coordinate the distance \( d \) along the inclined plane to the bouncing point,

a) Write down the Hamiltonian, and construct the phase-space diagram

b) Show that the frequency of oscillation \( \nu \) can be written in terms of the action \( J \) as

\[
\nu = \left( \frac{\pi^2}{3J} mg^2 \sin^2 \alpha \right)^{1/3}
\]

(2)

c) Find the angle variable as a function of \( d \) and \( J \).

3. Two ideal vortices with ideal circulation \( \Gamma \) are located at \( x_1 = (-b, b) \) and \( x_2 = (b, b) \) \((b > 0)\) at the time of interest in a semi-infinite \((y > 0)\) two-dimensional inviscid fluid. A flat impenetrable surface lies at \( y = 0 \). Determine the velocities of the point vortices presuming they are free to move. [20 points]

4. Consider the behavior of an incompressible viscous fluid in two dimensions. The fluid is originally at rest and occupies the region \( y > 0 \). At \( y = 0 \) there is a flat plate that at \( t = 0 \) starts moving parallel to itself (along the \( x \) direction). Assuming the flow is laminar, calculate the flow at \( t > 0 \). Hint: show that the problem is self-similar and look for a similarity solution. [30 points]