New York University
Physics Department
Winter 2016 Preliminary Examination in Statistical Physics

This is closed book, closed notes, no cell phones, and no internet exam.
No use of any computational equipment is allowed.

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1. Consider an Ising system (N spins s_i, each of them can adopt two states, $s_1 = +1$ or $s_i = -1$) in which every spin interacts with all other spins, with ferromagnetic interaction of strength J'.
   (a) Show that coupling constant J' in this system must be equal $J' = J/N$, where J is independent of N, in order for the system energy to be properly extensive.
   (b) Write down Hamiltonian of the system. Show that it can be expressed in terms of overall magnetization $S = \sum_i s_i$.
   (c) Write down the partition sum of the system. Show how it can be reduced to the single summation over S. Show that this summation can be approximated as integration.
   (d) Consider partial equilibrium of the system with fixed S. Find free energy of this state as a function of S, let us denote it $f(S)$. Sketch the plot of $f(S)$ at different values of J or T. Do these plots suggest phase transition in the system? At which temperature?
   (e) Estimate, up to numerical coefficients, the amount of fluctuations $\left\langle S^2 \right\rangle - \left\langle S \right\rangle^2$ above the transition temperature and at the transition point. Pay special attention to how these fluctuations depend on N.
   (f) Estimate the width of the transition range of temperature at large but finite N.

2. For non-relativistic gases of elementary particles with either Bose or Fermi statistics, there is a simple relation between pressure P and energy per unit volume E/V, namely, $P = (2/3)E/V$. This relation is exact as long as gas is ideal. Derive this relation.

   Hint: You may want to proceed as follows: start with thermodynamic potential Ω, which is natural thermodynamic potential in variables T, V, μ, write it down in terms of proper partition sum, separately for Fermi and Bose statistics. Remember the relation between Ω and pressure P. Write down also the expressions for energy in terms of Fermi- or Bose- distribution. In both Ω and E remember, that energy of non-relativistic elementary particles is related to momentum via $\varepsilon_p = p^2/2m$. Performing integration by parts in the expression for Ω, establish the requisite relation.

3. Consider some extensive variable X, and look at how its equilibrium average value changes depending on the applied force (or field) f conjugate to X. To achieve this, consider the quantity

$$\chi = \frac{\partial \langle X \rangle}{\partial f} \ . \tag{1}$$

   - Show how this quantity χ is related to the amount of fluctuations of X away from its average value when the system is in equilibrium at a given f.
   - The relation you just obtained is called fluctuation-response theorem. Can you explain this name? Identify fluctuation and response.

4. Consider a system of non-interacting spins in a magnetic field B pointing in the z-direction. The work done by the field is given by $B\Delta M_z$, with a magnetization $M_z = \mu \sum_{i=1}^{N} m_i$. For each spin, m_i takes only two values, $-1/2$ and $+1/2$.
   (a) Calculate the Gibbs partition function $Z(T, B, N)$ (note that the ensemble corresponding to the macrostate (T, B, N) includes the work by magnetic field).
   (b) Calculate the Gibbs free energy $G(T, B, N)$ and find its asymptotics at small values of magnetic field B.
   (c) Calculate the zero field susceptibility $\chi = \partial M_z/\partial B|_{B=0}$ and show that it satisfies the Curie law (i.e., inversely proportional to temperature).

5. Air conditioner usually works by consuming electric energy. Is it possible to design an air conditioner receiving energy from lukewarm water, which is colder than outside air? To address this question, suppose the air temperature outside is T_0, while you want to maintain a temperature $T_r < T_0$ in the room, and you have a source of water at some temperature T_w intermediate between T_0 and T_r: $T_0 > T_w > T_r$. How would you characterize the efficiency of such a device? What does fundamental physics have to say about this efficiency? Given the three temperatures T_0, T_r, and T_w and nothing else, is it possible to find how much water per minute will be needed?