Graduate Quantum Mechanics I (FALL 2015): Prelims. PLEASE WRITE YOUR N-NUMBER AND NOT YOUR NAME. TOTAL TIME=3 HRS

- Problem 1 (20 points)
  Suppose A and B are two two-level systems represented by the Pauli-matrices $\sigma_x^{A,B}, \sigma_y^{A,B}, \sigma_z^{A,B}$. Recall

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

  Suppose you are given the state vector that lives in the $A \otimes B$ space,

$$|\Phi\rangle_{AB} = \alpha| + A, -B\rangle + \beta| - A, +B\rangle \tag{2}$$

  where $|\pm, A\rangle, |\pm, B\rangle$ represent the eigenvectors of σ_z^A, σ_z^B respectively with eigenvalues ± 1 and $| \pm A, \pm B\rangle$ is a product state. For simplicity take α, β to be real, and the wave-function to be normalized $\alpha^2 + \beta^2 = 1$.
  a). Write the density matrix ρ_{AB} for the above state. What is Tr $\left[\rho_{AB}^2\right]$?
  b). Construct the reduced density matrix $\rho_A = \mathrm{Tr_B}\left[\rho_{\mathrm{AB}}\right]$. Write ρ_A in the form $(1 + \vec{P} \cdot \vec{\sigma}^A)/2$ and give $\vec{P}$ explicitly in terms of α, β.

  c). What is the range of Tr $\left[\rho_A^2\right]$? .

  d). Compute the entanglement entropy $S_A = -\mathrm{Tr}\left[\rho_A \ln \rho_A\right]$. For what values of α, β are the spins maximally entangled?

- Problem 2 (15 points)
  For the wave-function in Eq. 2, if now an observer in sub-system A uses a Stern-Gerlach set-up to measure the spin A along a direction $\hat{n}$,
  a) What is the average outcome of the measurement?
  b). What is the probability of finding the spin to be aligned along the direction $\hat{n}$? What is the probability of finding the spin anti-aligned to the direction $\hat{n}$?
  c). For what values of α, β, will the outcome become independent of $\hat{n}$?

- Problem 3 (10 points)
  Use the Wigner-Eckart theorem to determine the following ratio

$$\frac{\langle l = 5, m = 3|x|l = 5, m' = 2\rangle}{\langle l = 5, m = 3|y|l = 5, m'' = 4\rangle} \tag{3}$$

  where $\vec{r} = (x, y, z)$ is the position operator and $|l, m\rangle$ is the eigenstate of the orbital angular-momentum operators L^2, L_z. (Useful formulae: $J_x = (J_+ + J_-)/2, J_y = (J_+ - J_-)/(2i)$, $J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$).

- Problem 4 (10 points)
  The wave-function of a particle subjected to a spherically symmetric potential $V(r)$ is given by

$$\psi(x, y, z) = (x + y + 3z)\, f(r) \tag{4}$$

  a). Is ψ an eigenfunction of L^2? If so what is the l value? If not what are the possible values of l that we may obtain when L^2 is measured?
  b). What are the probabilities for the particle to be found in various m_l states?
  Useful relations $Y_0^0(\theta, \phi) = \frac{1}{2\sqrt{\pi}}; Y_1^{-1}(\theta, \phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{-i\phi}; Y_1^0(\theta, \phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta; Y_1^1(\theta, \phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{i\phi}$

- Problem 5 (15 points)
  Consider a particle of mass m, in a one-dimensional potential $U(x) = -\alpha\delta(x) - \alpha\delta(x - a)$. Solve for **all** the bound state solutions, and determine the condition for the number of bound states in terms of m, α, a.