

New York University

Department of Physics

**PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE**

**DYNAMICS**

Fall, 2015

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. On the front cover of each booklet write your identification number.
4. Show ALL your work.

## Part I: Classical Dynamics

### Problem 1 (33 pts)

A particle of unit mass is attracted to two fixed point masses at points  $(\pm 1, 0)$  in the  $x, y$  plane. Using confocal elliptical coordinates  $\xi, \eta$  defined by

$$x = \cosh \xi \cos \eta, \quad y = \sinh \xi \sin \eta, \quad -\infty < \xi < \infty, \quad 0 \leq \eta < 2\pi,$$

the potential energy function takes the form

$$V(\xi, \eta) = -\frac{\kappa}{\cosh \xi - \cos \eta} - \frac{\lambda}{\cosh \xi + \cos \eta}.$$

- (a) (6 pts) Calculate the Lagrangian of the system.
- (b) (6 pts) Calculate the momenta  $p_\xi$  and  $p_\eta$  conjugate to  $\xi$  and  $\eta$ , respectively.
- (c) (8 pts) Show that the Hamiltonian of the system takes the form

$$H(\xi, \eta, p_\xi, p_\eta) = \frac{A(\xi, p_\xi) + B(\eta, p_\eta)}{C(\xi) + D(\eta)}$$

for appropriate functions  $A, B, C, D$ .

- (d) (13 pts) Show that for *any* pair of canonical coordinates  $\xi, \eta$  and *any* Hamiltonian of the form given in (c), the quantity

$$K(\xi, \eta, p_\xi, p_\eta) = A(\xi, p_\xi) - C(\xi)H(\xi, \eta, p_\xi, p_\eta) = -B(\eta, p_\eta) + D(\eta)H(\xi, \eta, p_\xi, p_\eta)$$

is a constant of the motion. What constraint must be satisfied in order that  $H$  and  $K$  be independent functions? Show that the system is both integrable and separable.

### Problem 2 (33 pts)

The motion of a particle of unit mass along the  $q$  axis is governed by the Hamiltonian

$$H(q, p) = \frac{1}{2}(p^2 + q^2) + qp^2 + \frac{3}{8}q^2p^2$$

- (a) (5 pts) Write down the Hamilton equations of motion for the system.
- (b) (8 pts) Determine all of the equilibrium points in phase space. Test all of these for linear stability. For any stable case, calculate the frequencies of small oscillations about equilibrium.
- (c) (8 pts) Make a canonical transformation (proving that it really is canonical) from coordinates  $(q, p)$  to new coordinates  $(\theta, J)$  such that

$$q = \sqrt{2J} \cos \theta$$

and the transformed Hamiltonian takes the form

$$K(\theta, J) = J + J^{3/2}f(\theta) + J^2g(\theta)$$

(d) (12 pts) We consider the terms in  $K(\theta, J)$  involving powers of  $J$  greater than one to be a small perturbation on the unperturbed harmonic oscillator Hamiltonian  $J$ . For convenience, you will want to introduce a perturbation parameter  $\epsilon$  multiplying them. With the aid of a suitable generating function, make a canonical transformation  $(\theta, J) \rightarrow (\bar{\theta}, \bar{J})$  such that the transformed Hamiltonian takes the form

$$\bar{K}(\bar{\theta}, \bar{J}) = \bar{J} + \epsilon\bar{K}_1(\bar{J}) + O(\epsilon^2).$$

Give an explicit expression for the function  $\bar{K}_1(\bar{J})$ .

## Part II: Continuum Mechanics

In dealing with the following problems, you may find useful some of the formulas in the Appendix.

### Problem 3 (20 pts)

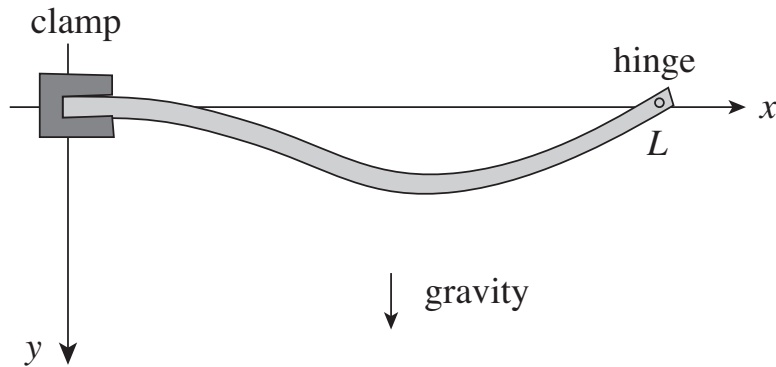
The space between two coaxial cylinders of radii  $R_1$  and  $R_2 > R_1$  is filled with a viscous fluid with viscosity  $\mu$  and mass density  $\rho$ . The cylinders are rotating about the common axis with angular velocities  $\Omega_1, \Omega_2$ , respectively. The pressure in the fluid at the inner boundary is  $p_0$ .

(a) (14 pts) Assuming non-slip boundary conditions, calculate the velocity and pressure fields in the fluid. Calculate the torque per unit length on the inner cylinder.

(b) (6 pts) Repeat the calculations of (a) assuming that the outer cylindrical surface is frictionless, with the non-slip constraint remaining on the inner boundary.

### Problem 4 (14 pts)

A long, thin elastic rod (Young's modulus  $E$ ) of length  $L$  and circular cross-section (moment of inertia  $I$ ) is placed on the  $x$  axis with one end clamped at  $x = 0$  and the other end attached at  $x = L$  with a frictionless hinge. Gravity pulls downward (positive  $y$  direction) with a uniform force per unit length  $K$ , distorting its shape as shown in the figure. Calculate the vertical displacement  $Y(x)$ , assuming that the radius of curvature is large compared to  $L$  everywhere along the rod (small deflection approximation). Calculate the force and torque exerted by the rod on the clamp at  $x = 0$ .



# Appendix A

## Miscellaneous elasticity formulas for bending of rods

$$\begin{aligned}\mathbf{t} &= \frac{d\mathbf{r}}{dl} \\ \frac{d\mathbf{F}}{dl} &= -\mathbf{K} \\ \frac{d\mathbf{M}}{dl} &= \mathbf{F} \times \mathbf{t} \\ \mathbf{M} &= EI\mathbf{t} \times \frac{d\mathbf{t}}{dl}\end{aligned}$$

## Vector calculus in orthogonal coordinate systems

Consider orthogonal coordinates  $u_1, u_2, u_3$  with arc length along the coordinate directions given by

$$ds_i = h_i(u_1, u_2, u_3) du_i.$$

Examples of such systems are *spherical coordinates* with

$$\begin{aligned}u_1 &= r, & u_2 &= \theta, & u_3 &= \phi \\ h_1 &= 1, & h_2 &= r, & h_3 &= r \sin \theta,\end{aligned}$$

and *cylindrical coordinates*, with

$$\begin{aligned}u_1 &= \rho & u_2 &= \phi, & u_3 &= z \\ h_1 &= 1, & h_2 &= \rho, & h_3 &= 1,\end{aligned}$$

In these orthogonal coordinate systems, the gradient, divergence, curl and Laplacian are given, respectively, by

$$\begin{aligned}(\nabla V)_i &= \frac{\partial V}{\partial s_i} = \frac{1}{h_i} \frac{\partial V}{\partial u_i}, \\ \nabla \cdot \mathbf{A} &= \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \text{cyclic permutations} \right) \\ (\nabla \times \mathbf{A})_3 &= \frac{1}{h_1 h_2} \left( \frac{\partial (h_2 A_2)}{\partial u_1} - \frac{\partial (h_1 A_1)}{\partial u_2} \right), \quad (\text{cyclic permutations}), \\ \nabla \cdot (\nabla V) &= \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial V}{\partial u_1} \right) + \text{cyclic permutations} \right).\end{aligned}$$

## Strain tensor in cylindrical coordinates

$$\begin{aligned}u_{rr} &= \frac{\partial u_r}{\partial r}, & u_{\phi\phi} &= \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, & u_{zz} &= \frac{\partial u_z}{\partial z}, \\u_{\phi z} &= \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} \right), & u_{rz} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \\u_{r\phi} &= \frac{1}{2} \left( \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right)\end{aligned}$$

## Appendix B: Stress-strain relations in a Newtonian fluid

### 0.1 Cylindrical coordinates

$$\begin{aligned}\sigma_{rr} &= -p + 2\mu \frac{\partial v_r}{\partial r}, \\ \sigma_{\phi\phi} &= -p + 2\mu \left( \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} \right), \\ \sigma_{zz} &= -p + 2\mu \frac{\partial v_z}{\partial z} \\ \sigma_{r\phi} &= \mu \left( \frac{1}{r} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right) \\ \sigma_{\phi z} &= \mu \left( \frac{\partial v_\phi}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \phi} \right) \\ \sigma_{zr} &= \mu \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)\end{aligned}$$

## Appendix C: Equations of Motion in Rectangular, Cylindrical and Spherical Coordinates <sup>[1]</sup>

### 1. Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5.23)$$

$$\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + F_x \quad (5.24)$$

together with similar equations for  $v$  and  $w$ .

2. Cylindrical polar coordinates ( $r$  = distance from axis,  $\phi$  = azimuthal angle about axis,  $z$  = distance along axis):

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0 \quad (5.25)$$

$$\begin{aligned} \rho \left[ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} \right] &= -\frac{\partial p}{\partial r} \\ + \mu \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \right] &+ F_r \end{aligned} \quad (5.26)$$

$$\begin{aligned} \rho \left[ \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z} \right] &= -\frac{1}{r} \frac{\partial p}{\partial \phi} \\ + \mu \left[ \frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial r} \right] &+ F_\phi \end{aligned} \quad (5.27)$$

$$\begin{aligned} \rho \left[ \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} \right] &= -\frac{\partial p}{\partial z} \\ + \mu \left[ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2} \right] &+ F_z. \end{aligned} \quad (5.28)$$

3. Spherical polar coordinates ( $r$  = distance from origin,  $\theta$  = angular displacement from reference direction,  $\phi$  = azimuthal angle about line