New York University
Physics Department
2011 Preliminary Examination in Statistical Physics

This is closed book, closed notes, no cell phones, and no internet exam.
No use of any computational equipment is allowed.

Full credit will be given for complete solution of any 4 of these 6 problems.

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1. Consider an Ising system (N spins s_i, each of them can adopt two states, $s_1 = +1$ or $s_i = -1$) in which every spin interacts with all other spins, with ferromagnetic interaction of strength J'.

   (a) Show that coupling constant J' in this system must be equal $J' = J/N$, where J is independent of N, in order for the system energy to be properly extensive.

   (b) Write down Hamiltonian of the system. Show that it can be expressed in terms of overall magnetization $S = \sum_i s_i$.

   (c) Write down the partition sum of the system. Show how it can be reduced to the single summation over S. Show that this summation can be approximated as integration.

   (d) Consider partial equilibrium of the system with fixed S. Find free energy of this state as a function of S, let us denote it $f(S)$. Sketch the plot of $f(S)$ at different values of J or T. Do these plots suggest phase transition in the system? At which temperature?

   (e) Estimate, up to numerical coefficients, the amount of fluctuations $\langle S^2 \rangle - \langle S \rangle^2$ above the transition temperature and at the transition point. Pay special attention to how these fluctuations depend on N.

   (f) Estimate the width of the transition range of temperature at large but finite N.

2. For non-relativistic gases of elementary particles with either Bose or Fermi statistics, there is a simple relation between pressure P and energy per unit volume E/V, namely, $P = (2/3)E/V$. This relation is exact as long as gas is ideal. Derive this relation.

   Hint: You may want to proceed as follows: start with thermodynamic potential Ω, which is natural thermodynamic potential in variables T, V, μ, write it down in terms of proper partition sum, for both Fermi and Bose statistics. Remember the relation between Ω and pressure P. Write down also the expressions for energy in terms. In both Ω and E remember, that energy of non-relativistic elementary particles is related to momentum via $\varepsilon_p = p^2/2m$. Performing integration by parts, establish the requisite relation.

3. In early Universe, temperature T was very high, there was light at this temperature, there were virtually no atoms, and electron-positron pairs were born or annihilated through the "reaction"

   $$e^- + e^+ \leftrightharpoons \gamma \ . \tag{1}$$

   Consider this "chemical reaction" and find equilibrium concentration of electrons at temperature T. Assume that all gases in question are ideal. Give an estimate of a typical distance between electrons as it depends on temperature and world constants.

   Hint: You may follow these steps: Write down the general condition of chemical equilibrium. You should know what is chemical potential of photons; this should allow you to find chemical potential of electrons. Knowing the chemical potential of electrons, and their statistics, find their number. Do not forget that thermal speed of electrons at high temperature T might be comparable to the speed of light, such that energy of an electron moving with momentum p is equal to $\varepsilon_p = \sqrt{(mc^2)^2 + (pc)^2}$.

4. Consider a gas with pressure P and temperature T in contact with a surface on which there are some K distinct adsorption sites. At most only one molecule can be adsorbed on each site at any given moment, and the adsorption free energy of one molecule is δ.

   (a) What physical condition determines the equilibrium fraction of occupied sites on the surface, ϕ, as a function of gas pressure P? Explain why the function $\phi(P)$ is called adsorption isotherm?

   (b) Show that fluctuations in the number of adsorbed particles satisfy

   $$\left\langle \left(N - \langle N \rangle\right)^2 \right\rangle = T \left(\frac{\partial \langle N \rangle}{\partial \mu} \right)_T ,$$

   where μ is chemical potential of the gas in the bulk.

(c) Find adsorption isotherm, that is, the average number of occupied adsorption sites on the surface as it depends on the gas pressure in the bulk at constant temperature, assuming gas in the bulk is ideal (this is called Langmuir isotherm).

(d) Show that the Langmuir isotherm can be presented in the form $\langle N \rangle = K \frac{P}{P+P_0}$, where $\langle N \rangle$ is the average number of adsorbed particles, and P_0 describes interaction between particles and the surface.

(e) Show that in this ideal gas case

$$\left\langle (N - \langle N \rangle)^2 \right\rangle = K\phi(1-\phi) \ ,$$

where $\phi = \langle N \rangle / K$ is the average fraction of occupied sites. Explain why the fluctuation vanishes in both $\phi \to 0$ and $\phi \to 1$ limits.

5. Find zero field susceptibility and mean squared fluctuations of order parameter for the system in 3D described by Landau theory with single scalar order parameter. Consider both above and below the transition cases.

6. Consider a system of non-interacting spins in a magnetic field B pointing in the z-direction. The work done by the field is given by $B\Delta M_z$, with a magnetization $M_z = \mu \sum_{i=1}^{N} m_i$. For each spin, m_i takes only two values, $-1/2$ and $+1/2$.

   (a) Calculate the Gibbs partition function $Z(T, B, N)$ (note that the ensemble corresponding to the macrostate (T, B, N) includes the work by magnetic field).

   (b) Calculate the Gibbs free energy $G(T, B, N)$ and find its asymptotics at small values of magnetic field B.

   (c) Calculate the zero field susceptibility $\chi = \partial M_z / \partial B|_{B=0}$ and show that it satisfies the Curie law (i.e., inversely proportional to temperature).