New York University
Physics Department
2012 Preliminary Examination in Statistical Physics

This is closed book, closed notes, no cell phones, and no internet exam.
No use of any computational equipment is allowed.

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1. Consider an ideal Bose gas of spin s particles with dispersion relation $\epsilon = p^2/2m$ (independent of spin). Derive the expression and sketch a plot of pressure against volume at constant temperature and constant number of particles. For comparison, draw in the same figure the plot of pressure against volume for a classical ideal gas of the same number of particles at the same temperature.

   Feel free to use the following mathematical identity:

$$\int_0^\infty \frac{x^\xi dx}{e^x - 1} = \Gamma(1+\xi)\,\zeta(1+\xi) \quad \text{for any positive } \xi > 0 \ . \tag{1}$$

2. Consider a system of non-interacting spins in a magnetic field B pointing in the z-direction. The work done by the field is given by $B\Delta M_z$, with a magnetization $M_z = \mu \sum_{i=1}^{N} m_i$. For each spin, m_i takes only two values, $-1/2$ and $+1/2$.

   (a) Calculate the Gibbs partition function $Z(T, B, N)$ (note that the ensemble corresponding to the macrostate (T, B, N) includes the work by magnetic field).

   (b) Calculate the Gibbs free energy $G(T, B, N)$ and find its asymptotics at small values of magnetic field B. (Note: what you have to find is not the limit when $B \to 0$, but asymptotics when B is small – and you have to identify the characteristic scale for which B is small).

   (c) Calculate the zero field susceptibility $\chi = \partial M_z/\partial B|_{B=0}$ and show that it satisfies the Curie law (i.e., inversely proportional to temperature).

3. Consider some extensive variable X, and look at how its equilibrium average value changes depending on the applied force (or field) f conjugate to X. To achieve this, consider the quantity

$$\chi = \frac{\partial \langle X \rangle}{\partial f} \ . \tag{2}$$

   - Show how this quantity χ is related to the amount of fluctuations of X away from its average value when the system is in equilibrium at a given f.
   - The relation you just obtained is called fluctuation-response theorem. Can you explain this name? Identify fluctuation and response.

4. Van der Waals gas has equation of state

$$p = \frac{NT}{V - Nb} - a\frac{N^2}{V^2} \ . \tag{3}$$

   Derive, on purely thermodynamic basis, expressions for free energy, entropy, and energy of this gas.

   Hint: To find pressure from free energy, you differentiate. To find free energy from pressure – you will have to do an integration, leading to an integration constant. You can establish the integration constant by "correspondence principle" (remember, in quantum mechanics, this principle says that in classical limit we should obtain classical results). In this context, you can rely on the idea that in the limit of very low density ($N/V \ll$???) you should be obtaining an ideal gas (you will have to establish what is the characteristic density ???).