Name: ______________________________

Question:	1	2	3	4	Total
Points:	20	20	35	25	100
Score:					

This is a closed-book exam. You may consult one sheet of notes. Please write your answers out completely and indicate if you have continued your work on the back of a page.

1. A substance has the following properties:

    1. The work done **by** the substance on the outside world as it undergoes an **isothermal** expansion at temperature T_0 from an initial volume V_0 to volume V is

    $$W = -Nk_B T_0 \ln \frac{V_0}{V}.$$

    2. The substance's entropy at temperature T and volume V is given by

    $$S(V,T) = Nk_B \frac{V_0}{V} \left(\frac{T}{T_0}\right)^{\alpha},$$

    where α is a constant.

    Using this information,

    (a) (10 points) Calculate the substance's Helmholtz free energy.

    (b) (5 points) Find its equation of state.

    (c) (5 points) Find the work done **by** the substance during an isothermal expansion at an arbitrary constant temperature T (not necessarily equal to T_0).

Continuation to Problem 1:

2. (20 points) *Adsorption* is the process by which objects, such as atoms and molecules, stick to surfaces. The sticky sites on a surface can be arranged randomly or in a regular pattern such as a lattice. Consider particles adsorbing to a d-dimensional square lattice from a reservoir of particles at fixed chemical potential μ. Each site on the lattice can bind one particle with binding energy, U. The occupation number for the site labeled j can take on the values $n_j = 0$ (empty) or $n_j = 1$ (occupied), and the microscopic state of this system is described by the set of occupation numbers, $\{n_j\}$. Adsorbed particles tend to attract each other and form islands on a surface. We model this interaction as a nearest-neighbor attraction that lowers the system's energy by an amount J. The Hamiltonian for this system is

$$\begin{aligned} H &= -U \sum_j n_j - \frac{1}{2} J \sum_{\{i,j\}_{\mathrm{nn}}} n_i n_j \\ &= -\sum_j n_j \left[U + \frac{1}{2} J \sum_{i \in \{j\}_{\mathrm{nn}}} n_i \right], \end{aligned}$$

where the sum in the second term of each expression runs over the $2d$ nearest neighbors of site j. The factor of $1/2$ avoids double-counting the nearest-neighbor attraction.

Develop a mean field theory for the density of particles bound to the lattice, and use this to compute $n = \langle n_j \rangle$, the mean fraction of occupied sites. It is fine to express your result as a self-consistency condition for the mean field theory.

Continuation to Problem 2:

3. According to Bekenstein and Hawking, the entropy of a black hole is proportional to its area A, and is given by

$$S = \frac{k_B c^3}{4\pi G\hbar}\, A.$$

   (a) (5 points) Calculate the escape velocity at a radius R from a mass M using classical mechanics. Find the relationship between the radius and mass of a black hole by setting this escape velocity ot the speed of light c. Interestingly, this also is the correct relativistic result.

   (b) (5 points) Does entropy increase or decrease when two black holes collapse into a single black hole? What is the entropy change for the Universe (in equivalent number of bits of information) when two solar mass black holes coalesce ($M = 2 \times 10^{30}\,\mathrm{kg}$)?

   (c) (5 points) The internal energy of the black hole is given by the Einstein relation, $E = Mc^2$. Find the temperature of the black hole in terms of its mass.

   (d) (5 points) A "black hole" actually emits thermal radiation due to pair creation processes on its event horizon. Find the rate of energy loss due to such radiation.

   (e) (5 points) Find the amount of time it takes an isolated black hole to evaporate. How long is this this time for a black hole of solar mass?

   (f) (5 points) What is the mass of a black hole that is in thermal equilibrium with the current cosmic background radiation at $T = 2.7\,\mathrm{K}$?

   (g) (5 points) Consider a spherical volume of space of radius R. According to the *Holographic Principle*, there is a maximum to the amount of entropy that this volume of space can have, independent of its contents. What is this maximal entropy?

Continuation to Problem 3:

4. (25 points) Hydrogen gas in the lab takes the form of diatomic molecules, H_2. Most of the hydrogen in space, by contrast, appears as individual atoms, H. This might seem surprising because the binding energy of H_2 is quite large, $\epsilon = 7.23 \times 10^{-19}\,\mathrm{J}$. To explain this phenomenon, **work out how the concentration of hydrogen molecules, n_2, depends on the concentration of hydrogen atoms, n_1, in equilibrium at temperature T.**

   In performing this calculation, treat the atoms and molecules as ideal gases, ignore the molecules' rotational and vibrational degrees of freedom, and remember that each molecule has twice the mass of an atom.

   Hint: The quantum of phase space volume for an object of mass m at temperature T is

$$V_Q(T) = \frac{h^3}{(2\pi m k_B T)^{3/2}}.$$

Continuation to Problem 4: