Graduate Quantum Mechanics I: Prelims and Solutions (Fall 2017)

- Problem 1 (10 points)
  A particle in the harmonic oscillator potential has the initial state

$$\psi(x,0) = A\left[1 - 3\sqrt{\frac{m\omega}{\hbar}}x + 2\frac{m\omega}{\hbar}x^2\right]e^{-m\omega x^2/2\hbar} \tag{1}$$

  a). Calculate the expectation value of the energy.
  b). At a later time T, the wavefunction is

$$\psi(x,T) = B\left[3 - 3i\sqrt{\frac{m\omega}{\hbar}}x - 2\frac{m\omega}{\hbar}x^2\right]e^{-m\omega x^2/2\hbar} \tag{2}$$

  for some constant B. What is the smallest possible value of T?

  HINT: Eigenstates of the harmonic oscillator are

$$\phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}e^{-\xi^2/2}H_n(\xi),\ \ \xi = x\sqrt{\frac{m\omega}{\hbar}}$$
$$H_0(\xi) = 1,\ H_1(\xi) = 2\xi,\ H_2(\xi) = 4\xi^2 - 2$$

- Problem 2 (30 points)
  Suppose the initial wavefunction of a free particle of mass m in one dimension is a Gaussian wavefunction in position space $\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}}e^{-x^2/2\Delta^2}e^{ip_0x/\hbar}$.
  a). Give results for the time-evolution of the wavefunction in position space.
  b). Give results for the time-evolution of the wavefunction in momentum space.
  c). Discuss how Ehrenfest's theorem is obeyed for this example.

- Problem 3 (20 points)
  Consider a system whose angular momentum consists of two parts, $\vec{J}_1$ and $\vec{J}_2$, and whose magnetic moment is $\vec{\mu} = \gamma_1\vec{J}_1 + \gamma_2\vec{J}_2$. In the state $|j, m, j_1, j_2\rangle$, determine the following expectation values,
  a). $\langle\mu_x\rangle$
  b). $\langle\mu_y\rangle$
  c). $\langle\mu_z\rangle$.

  (Useful formulae: $J_x = (J_+ + J_-)/2, J_y = (J_+ - J_-)/(2i)$, $J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j, m\pm1\rangle$)

  Hint: The Projection theorem says:

$$\langle\alpha' jm'|A^q|\alpha jm\rangle = \frac{\langle\alpha' jm|\vec{J}\cdot\vec{A}|\alpha jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J^q|jm\rangle$$

- Problem 4 (10 points)
  A particle on a sphere is in the state

$$\psi(\theta,\phi) = \sqrt{\frac{15}{16\pi}}\sin(2\theta)\cos(\phi)$$

  What are the probabilities of angular momentum L^2 and L^z measurements?