

QMII, FALL 2012, PRELIM EXAM

Natural units $\hbar = c = 1$ are used throughout.

Total Point Value: 20 Points

INSTRUCTIONS:

- This is a no book exam. You have 3 hours to complete the exam.
- Do all problems, using separate booklets for each problem.
- Please show your work: credit will not be given otherwise.

Problem 1 (4 points)

N identical spin one-half particles of mass m are confined in the 3D harmonic potential well

$$V(x, y, z) = \frac{1}{2}m\Omega^2(x^2 + y^2 + z^2)$$

The particle self-interaction is negligible.

I (2 points)

Find the maximum energy of an electron in the ground state of the system –i.e. the Fermi Energy–, the total ground state energy of the system, and the ground state degeneracy for $N = 21$.

I (2 points)

Find the Fermi Energy for $N = 10^{10}$.

Problem 2 (4 points)

An atom undergoes a sudden change of charge $Z \rightarrow Z - 2$ (by α decay e.g.). Compute the probability of exciting the K-shell electron (i.e. the lowest-energy electron state, which you can assume to be in an S-wave).

Useful Formulae

The wave function of the 1S state normalized to $\int d^3x |\psi(x)|^2 = 1$ is $\psi_Z(r) = \frac{1}{\sqrt{\pi a_Z^3}} \exp(-r/a_Z)$.
The Bohr radius of a particle of mass m for a nucleus of atomic number Z is $a_Z = 1/mZe^2$.

Problem 3 (7 points)

A non relativistic particle of momentum k scatters off a "black hole": a sphere of radius R that absorbs all particles falling on it.

For total absorption boundary condition on the radial wave function is

$$\left. \frac{d\chi(r)}{dr} \right|_{r=R} = ik\chi(R).$$

I (1 point)

Explain briefly why this corresponds to total absorption.

II (3 points)

Compute the elastic cross section and the total cross section in the high-frequency limit $kR \gg 1$.

Hint

Draw the effective potential for a particle of angular momentum l and notice that in the limit $kR \gg 1$ the semiclassical approximation, in which the wave function is either reflected by the barrier or passes above the barrier, is valid.

III (2 points)

Assume that the scattering phases δ_l behave as $\delta_l \sim (kR)^{2l+1}$ for nonzero l and $kR \rightarrow 0$. Show that instead the scattering phase δ_0 is nonzero in the limit $kR \rightarrow 0$. Find $\exp(2i\delta_0)$ in that limit and use your result to compute the total cross section.

Notice that δ_0 can be complex!

Useful formulae

Scattering amplitude:

$$f(\theta) = \sum_l (2l+1) P_l(\cos\theta) f_l, \quad f_l = \frac{e^{2i\delta_l} - 1}{2ik}.$$

Elastic cross section:

$$\sigma_{el} = \sum_l 4\pi(2l+1) |f_l|^2.$$

Optical theorem:

$$\sigma_{tot} = \frac{4\pi}{k} \text{Im} f(0).$$

IV (1 point)

The elastic cross section is nonzero even in the semiclassical, high-momentum limit $kR \gg 1$. Can you explain why?

Problem 4 (5 points)

Consider a relativistic scalar defined by the annihilation operator

$$\phi(x, t) = \int \frac{d^3p}{(2\pi)^3 2E} e^{ikx - iEt} a_p, \quad E = \sqrt{p^2 + m^2},$$

where $[a_p, a_q^\dagger] = (2\pi)^3 2E \delta^3(p - q)$.

I (2 points)

Show that the commutator $[\phi(x, t), \phi^\dagger(y, s)]$ does not vanish when the separation between the events (x, t) and (y, s) is space-like. In other words, $\phi(x, t)$ and $\phi^\dagger(y, s)$ are not *mutually local* operators.

II (3 points)

Find an operator that is mutually local with itself and its space and time derivatives. Show that it is local.