Problem 1 (20 points)

Consider a three dimensional rotor \( H = \frac{\vec{L}^2}{2I} \) where \( I \) is the moment of inertia, and \( \vec{L} \) is the angular momentum.

a). What are the energy eigenvalues and eigenfunctions? You can represent the latter in terms of the spherical harmonics \( Y_{lm} \). What is the degeneracy of each level?

b). If now an electric field \( E \) is applied such that the perturbing potential is \( V = dE \cos \theta \), where \( d \) is the dipole moment of the rotor, discuss how all the energy eigenvalues (ground and excited states) are affected in first order in \( V \).

c). Discuss how all the energy eigenvalues are affected to second order in \( V \). To what extent is the degeneracy of the levels lifted?

d). Is the extent to which the degeneracy lifted affected at higher orders (> 2) in perturbation theory? Give your reasoning.

Hint:

\[
\begin{align*}
\cos \theta Y_{00} &= -iY_{10}/\sqrt{3}, \\
\cos \theta Y_{lm} &= a_{lm}Y_{l+1,m} - a_{l-1,m}Y_{l-1,m}, l \geq 1 \\
a_{lm} &= -i\sqrt{(l-m+1)(l+m+1)}/(2l+1)(2l+3)
\end{align*}
\]  

(1)

Solution:

a). Eigenfunctions are \( Y_{lm} \), and eigenvalues are \( E_{lm}^{(0)} = \hbar^2 l(l+1)/(2I) \). Each energy level has a degeneracy of \( 2l+1 \) corresponding to \( m = -l, -l+1, \ldots, l \).

b). To first order, the energy levels are unaffected.

c). At second order the energy shift is the following for the ground state,

\[
E_{00}^{(2)} = -\frac{d^2 I}{3\hbar^2} E^2
\]  

(2)

For the excited states it is,

\[
E_{lm}^{(2)} = \frac{|a_{l,m}|^2}{E_{l,m}^{(0)}} + \frac{|a_{l-1,m}|^2}{E_{l-1,m}^{(0)}} - \frac{d^2 I}{\hbar^2} E^2 \left[ \frac{l(l+1) - 3m^2}{l(l+1)(2l-1)(2l+3)} \right]
\]  

(3)

Thus the \( 2l+1 \) degenerate levels split into \( l+1 \) distinct levels. Except the \( m = 0 \) level, the rest are doubly degenerate. This is because the electric field does not differentiate between states with eigenvalues \( \pm |m| \).

d). \( L_z \) commutes with the total Hamiltonian even with the electric field. Thus the electric field cannot mix states of different \( m \). Moreover the electric field preserves time reversal and therefore does not differentiate between \( m \) and \( -m \). Thus higher orders will not further lift the double degeneracy of the \( m \neq 0 \) states.
Problem 2 (10 points)
Consider a particle of mass $m$ bound to a spherically symmetric potential $U(r)$. Give the WKB quantization condition for the allowed energy levels accounting for a non-zero angular momentum of the particle.

Solution:
The radial part resembles the one-dimensional Schrödinger equation in an effective potential $U_{\text{eff}} = U(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$.

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + U_{\text{eff}}(r)u = Eu$$

(4)

Treating the origin as an infinite wall, the quantization condition becomes

$$\int_0^a p_r dr = (n - 1/4)\pi \hbar, \quad p_r = \sqrt{2m(E - U_{\text{eff}})}$$

(5)

where $a$ is the turning point.
Problem 3 (20 points)
Consider particles of mass \( \mu \) and momentum \( \hbar k \), scattering off of a spherically symmetric potential. Recall that the wavefunction may be written as
\[
\Psi(r, \theta) = e^{ikr \cos \theta} + f(\theta) e^{ikr} / r
\]  
(6)
a). Show that for large distances from the scatterer, the radial component of the current density due to interference between the incident and scattered waves is
\[
j_r^{\text{int}} \sim r \to \infty \frac{\hbar k}{\mu} \frac{1}{r} \text{Im} \left[ ie^{ikr(\cos \theta - 1)} f^*(\theta) \cos \theta + ie^{ikr(1 - \cos \theta)} f(\theta) \right]
\]  
(7)
b). Argue that as long as \( \theta \neq 0 \), the average of \( j_r^{\text{int}} \) over any small solid angle is zero because \( r \to \infty \).
c). Integrate \( j_r^{\text{int}} \) over a small solid angle in the forward direction to derive an expression for:
\[
\int_{\text{forward cone}} j_r^{\text{int}} r^2 d\Omega
\]  
(8)
d). Recall number conservation and steady-state implies that \( \vec{\nabla} \cdot \vec{j} = 0 \) where \( \vec{j} \) is the total current density from the incident and scattered particles. Apply this at large distances from the scatterer, and use the results from above, to derive the optical theorem,
\[
\sigma_{\text{scat}} = \frac{4\pi}{k} \frac{\hbar k}{\mu} \frac{1}{r} \text{Im} \left[ f(0) \right]
\]  
(9)
where \( \sigma_{\text{scat}} \) is the total scattering cross-section defined as \( \sigma_{\text{scat}} = \int d\Omega |f|^2 \) and \( d\Omega = \sin \theta d\theta d\phi \). Hint: It is helpful to use Gauss’s theorem \( \int_{\text{surface}} \vec{j} \cdot d\vec{a} = 0 \).
e). Give a physical interpretation for the optical theorem.

Solution:
a). The radial current density is \( j_r = \frac{k}{2im} (\Psi^* \partial_r \Psi - \Psi \partial_r \Psi^*) \). Then,
\[
\partial_r \Psi(r) = ik \cos \theta e^{ikr \cos \theta} + \frac{ik}{r} f e^{ikr} - \frac{f}{r^2} e^{ikr}
\]  
(10)
After some straight-forward algebra, one finds that the current has a part coming entirely from the incident beam, a part coming entirely from the scattered beam, and a part coming from the interference between the two.
b). The interfering terms go as, \( e^{ikr(\cos \theta - 1)} \). In the limit \( r \to \infty \), this is a rapidly oscillating function unless \( \theta = 0 \). Thus the integral over the solid angle is non-zero only around \( \theta \sim 0 \).
c). Here we notice that due to b), \( \int d(\cos \theta) e^{ikr(\cos \theta - 1)} f(\theta) \sim f(0) / (ikr) \). Then one finds,
\[
\int_{\text{forward cone}} j_r^{\text{int}} r^2 d\Omega = \left( \frac{\hbar k}{\mu} \right) \frac{4\pi}{k} \text{Im} \left[ f(0) \right]
\]  
(11)
d). Use Gauss’s theorem to write
\[
\int_{\text{surface}} \vec{j} \cdot d\vec{a} = 0.
\]  
(12)
For a surface of radius \( r \), this reduces to
\[
r^2 \int d\Omega (\Psi^* \partial_r \Psi - \Psi \partial_r \Psi^*) = 0
\]  
(13)
After some algebra, the optical theorem can be derived.
e). The optical theorem is a consequence of number conservation. Particles depleted from the forward direction by the scatterer must appear in other directions.