Graduate Quantum Mechanics II (SPRING 2018): Prelims. PLEASE WRITE YOUR N-NUMBER AND NOT YOUR NAME. TOTAL TIME=3 HRS, TOTAL POINTS=50

- Problem 1 (20 points)

  Consider a three dimensional rotor $H = \frac{\vec{L}^2}{2I}$ where I is the moment of inertia, and $\vec{L}$ is the angular momentum.

  a). What are the energy eigenvalues and eigenfunctions? You can represent the latter in terms of the spherical harmonics Y_{lm}. What is the degeneracy of each level?

  b). If now an electric field E is applied such that the perturbing potential is $V = dE\cos\theta$, where d is the dipole moment of the rotor, discuss how all the energy eigenvalues (ground and excited states) are affected in first order in V.

  c). Discuss how all the energy eigenvalues are affected to second order in V. To what extent is the degeneracy of the levels lifted?

  d). Is the extent to which the degeneracy lifted affected at higher orders (> 2) in perturbation theory? Give your reasoning.

  Hint:

$$\begin{aligned}
&\cos\theta Y_{00} = -iY_{10}/\sqrt{3}, \\
&\cos\theta Y_{lm} = a_{lm}Y_{l+1,m} - a_{l-1,m}Y_{l-1,m}, l \geq 1 \\
&a_{lm} = -i\sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}
\end{aligned} \tag{1}$$

- Problem 2 (10 points)

  Consider a particle of mass m bound to a spherically symmetric potential $U(r)$. Give the WKB quantization condition for the allowed energy levels accounting for a non-zero angular momentum of the particle.

- Problem 3 (20 points)

  Consider particles of mass μ and momentum $\hbar k$, scattering off of a spherically symmetric potential. Recall that the wavefunction may be written as

$$\Psi(r,\theta) = e^{ikr\cos\theta} + f(\theta)\frac{e^{ikr}}{r} \tag{2}$$

  a). Show that for large distances from the scatterer, the radial component of the current density due to interference between the incident and scattered waves is

$$j_r^{\text{int}} \underset{r\to\infty}{\sim} \frac{\hbar k}{\mu}\frac{1}{r}\text{Im}\left[ie^{ikr(\cos\theta-1)}f^*(\theta)\cos\theta + ie^{ikr(1-\cos\theta)}f(\theta)\right] \tag{3}$$

  b). Argue that as long as $\theta \neq 0$, the average of j_r^{int} over any small solid angle is zero because $r \to \infty$.

  c). Integrate j_r^{int} over a small solid angle in the forward direction to derive an expression for:

$$\int_{\text{forwardcone}} j_r^{\text{int}} r^2 d\Omega \tag{4}$$

  d). Recall number conservation and steady-state implies that $\vec{\nabla}\cdot\vec{j} = 0$ where $\vec{j}$ is the total current density from the incident and scattered particles. Apply this at large distances from the scatterer, and use the results from above, to derive the optical theorem,

$$\sigma_{\text{scat}} = \frac{4\pi}{k}\text{Im}f(0) \tag{5}$$

  where σ_{scat} is the total scattering cross-section defined as $\sigma_{\text{scat}} = \int d\Omega|f|^2$ and $d\Omega = \sin\theta d\theta d\phi$. Hint: It is helpful to use Gauss's theorem $\int_{\text{surface}} \vec{j}\cdot\vec{da} = 0$.

  e). Give a physical interpretation for the optical theorem.