2011 Graduate Quantum Mechanics I
Preliminary Exam

All problems are of equal value. Closed book, no notes allowed.

Problem 1

Consider a system of 3 spin-1/2 particles. Two possible states for the system

$$|\psi_A\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \tag{1}$$

and

$$|\psi_B\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}, \tag{2}$$

in the standard notation where “1” and “0” represent spin $+\hbar/2$ and $-\hbar/2$ respectively along the $z-$axis, and *e.g.* $|010\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3$.

1. What is the von Neumann entropy $S \equiv -\mathrm{Tr}\,(\rho \ln \rho)$ for the states ψ_A and ψ_B?
2. Compute the reduced density matrices $\rho_{A,12}$ and $\rho_{B,12}$ for each state by partial-tracing out the state of particle 3.
3. What is S for each reduced density matrix? Which is larger?

Problem 2

Consider the states of three spin 1/2 particles ψ_A and ψ_B from Problem 1.

1. Which of the two states is an eigenstate of the z-component of the total spin angular momentum $\vec{S}_{\mathrm{tot}}$? What is its eigenvalue?

   Now suppose a (projective) measurement is made of the spin along the z-axis of particle 3. Answer the following questions for each of the two states:

2. What are the possible outcomes of the measurement, and what are the probabilities for each result?

3. For each possible measurement outcome, compute the reduced density matrix ρ_1 by tracing over the states of particles 2 and 3 on the state (after the measurement). What is the von Neumann entropy S in each case?

Problem 3

Consider a 1D potential that has two wells: $V(x) = V_0$ for $-2a < x < -a$, $V(x) = V_0$ for $a < x < 2a$, and $V(x) = \infty$ for all other x.

1. Find the energy levels and normalized eigenstates. What is the degeneracy of the energy eigenstates?

2. Consider the parity operator $P\psi(x) = \psi(-x)$. Find a basis for the energy eigenstates that simultaneously diagonalizes P and H.

3. Now consider the same potential, but with $V(x) = V_1 \gg E_0$ for $-a < x < a$, where E_0 is the ground state energy. Sketch a graph of the ground state and first excited state for this potential. What are their degeneracies? Estimate the gap in energy between the ground state and first excited state.

Problem 4

In 3D quantum mechanics, consider the spherically symmetric potential $V(r) = A/r$. Argue convincingly that the energy spectrum is bounded from below for $A < 0$. You may do this either by solving the Schrodinger equation directly, or by another technique.

Again in 3D quantum mechanics, consider the potential $V(r) = B/r^2$. Is the spectrum bounded from below? Why or why not? If your answer depends on B, explain how.

Problem 5

Compute the transmission and reflection probabilities $T(E)$ and $R(E)$ as functions of the energy E for 1D scattering with potential $V = A\theta(x)$, where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 1$.

Problem 6

Consider the state of two spin 1/2 particles. A basis for the set of all spin states is

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \tag{3}$$

where the notation is the same as Problem 1 ("1" and "0" represent spin $+\hbar/2$ and $-\hbar/2$ respectively along the $z-$axis).

1. Which of these states are eigenstates of the total spin operator $\vec{S}^2_{\text{tot}}$?

2. Construct a complete basis of states that simultaneously diagonalize $\vec{S}^2_{\text{tot}}$ and $S_{\text{tot},z}$, the z-component of the total spin angular momentum.

3. Does a basis exists that simultaneously diagonalizes $S_{1,z}, S_{2,z}$, and $\vec{S}^2_{\text{tot}}$, where $S_{i,z}$ is the z-component of the spin operator for the ith particle? If not, prove it. If yes, construct the basis.