QMI PRELIM 2013

All problems have the same point value. If a problem is divided in parts, each part has equal value. Show all your work.

Problem 1

$\vec{L} = \vec{r} \times \vec{p}, \quad \vec{p} = -i\hbar\vec{\nabla}$

(a) Show that $L_z = i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$; (cyclic for other components)

(b) Show that the commutator $[L_x, L_y] = i\hbar L_z$ (and cyclic), and consequently that we can write $\vec{L}\times\vec{L} = i\hbar\vec{L}$. Also, show $[L_z, r_\pm] = \pm\, r_\pm\hbar$, where $r_\pm = x \pm iy$

(c) Show that $[L^2, L_z] = 0$, where $L^2 = L_x^2 + L_y^2 + L_z^2$.

(d) Show that $L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$, where $L_\pm = L_x \pm iL_y$.

Problem 2

Let ψ be an eigenfunction of L^2: $L^2\psi = l(l+1)\hbar^2\,\psi$. Since $[L^2, L_z] = 0$, we can make ψ an eigenfunction of L_z as well: $L_z\psi_m = m\hbar\,\psi_m$. L^2 and L_z are defined as in Problem 1.

(a) With *raising operator* $L_+ = L_x + iL_y$, and results from Problem 1, show that $L_+\psi_m$ generates an eigenfunction of same L^2 but with L_z eigenvalue of $(m+1)\hbar$.

(b) Compute the non-zero matrix elements of L_+.

Problem 3

Consider a spin vector $\sigma_n = \sigma_x \cos\alpha \sin\beta + \sigma_y \sin\alpha \sin\beta + \sigma_z \cos\beta$, and:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(a) Determine the eigenvalues of σ_n.
(Hint: compute σ_n^2).

(b) Pure rotation by an angle γ of the z-axis in the $x-z$ plane leads to the rotation matrix

$$\begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \\ -\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}$$

Interpret this result by comparing to rotation of ordinary vectors, *e.g.* $\vec{\ell}$, $\vec{r}$.

Problem 4

At fixed principal quantum number n and orbital quantum number l, the Hamiltonian of an alkali atom can be modelled as

$$H = A + B\ \vec{L}\cdot\vec{S}, \qquad A, B = \text{constants.}$$

(a) Diagonalize this Hamiltonian for $l = 1$ and $l = 2$.

(b) Which other operator(s) built in terms of $\vec{L}$ and $\vec{S}$ commutes with the Hamiltonian?

(c) The Hilbert space of states for $l = 1$ is the product of two spaces, $V = V_L \otimes V_S$. V_L is the orbital angular momentum space (dimension 3) and V_S is the spin space (dimension 2). By definition, given a pure state $|\psi\rangle \in V$, the reduced density matrix in V_L is $\rho_L = \mathrm{tr}_S |\psi\rangle\langle\psi|$. The entanglement entropy is defined as $S = -\mathrm{tr}\,\rho_L \log\rho_L$.

Compute the entanglement entropy of the $j = 1/2$ state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left|+\frac{1}{2}\right\rangle_S \left|0\right\rangle_L - \sqrt{\frac{2}{3}} \left|-\frac{1}{2}\right\rangle_S \left|1\right\rangle_L .$$

Problem 5

Consider the scattering of a particle of energy E in a square well potential $V = 0$ for $x < 0$ and $x > b$, $V = V_0 > 0$ for $0 < x < b$.

Assuming that the transmission coefficient $T \ll 1$ (thick barrier approximation), obtain its value in terms of the above parameters. You may use the definitions

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad k' = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

With this obtain the value of the transmission probability

$$T = \left| \frac{transmitted\ amplitude}{incident\ amplitude} \right|^2 .$$

Problem 6

Consider the Hamiltonian of a one-dimensional particle with mass m and potential $V = A/x^2$. A can be either positive or negative.

(a) Prove that the spectrum is unbounded below for $2mA < -1/4$ (Hint: study how the potential energy and the kinetic energy scale when the parameter $b \to 0$ in the test wave packet $\psi = b^{-1/2-a} x^a \exp(-x^2/2b^2)$, with $a > 1/2$ but arbitrarily close to 1/2: $a = 1/2 + \epsilon^+$.)

(b) When $2mA > -1/4$, it can be proven that the spectrum is bounded below (do *not* try to prove it!). In this case, is the spectrum continuous or discrete? What is the lowest value of the energy? (Hint: under the scaling $x \to y = \lambda x$, the Hamiltonian transforms as $H \to \lambda^{-2} H$.)