2012 Graduate Quantum Mechanics I
Preliminary Exam

All problems are of equal value.

Problem 1

For a free particle of mass m in one-dimensional quantum mechanics, $\psi(x,0) = Ne^{-\left(\frac{x}{2a}\right)^2}$. Determine N, and find the full solution $\psi(x,t)$.

Problem 2

A quantum system with two energy eigenstates ψ_0, ψ_1 with energies E_0, E_1 is weakly coupled to a heat bath at temperature T. Assuming a long time has passed since any external perturbation took place, write the state (pure or mixed) that best describes the two-state system. Explicitly compute the expectation values of energy E, free energy F, and entropy S, and check the thermodynamic relation $F = E - TS$.

Problem 3

(a) Consider a state of two non-identical spin 1/2 particles:

$$\psi_{12} = N\left(|0\rangle_1|1\rangle_2 + \alpha|1\rangle_1|0\rangle_2\right),$$

where $|0\rangle, |1\rangle$ indicates spin $+\hbar/2, -\hbar/2$ along the z-axis. Suppose an experimenter uses a Stern-Gerlach apparatus to measure the spin of particle 1 along the x-axis and obtains $-\hbar/2$. She then rotates her apparatus and measures the spin of particle 2 along the $z-$axis. What results may she obtain, with what probabilities?

(b) Consider three particles, one with spin 1, one with spin 3/2, and one with spin 3, in an arbitrary pure state. Suppose the experimenter has access only to the spin 1 and spin 3 particles. What is the maximum possible von Neumann entropy ($S_{vN} \equiv -\mathrm{Tr}\rho\ln\rho$) of the reduced density matrix (obtained by tracing over the states of the inaccessible particle) describing the spin state of the two particles she can access? *Hint:* you may use the fact that entanglement entropies are symmetric, in the sense that the entropy of a subsystem with the rest of the system is equal to the entanglement entropy of the rest of the system with the subsystem.

Problem 4

In one dimensional quantum mechanics, compute the transmission and reflection probabilities $T(E)$ and $R(E)$ for scattering of a particle of mass m and energy E off the potential $V(x) = \beta\delta(x)$.

Problem 5

Consider the matrix elements $\langle lm|x|l'm'\rangle$, where x is the first component of the Cartesian coordinate position operator $\vec{x}$ in 3D quantum mechanics, and $|lm\rangle$ is an eigenstate of total angular momentum $\vec{L}^2$ and the z-component L_z (i.e. l, m are orbital and magnetic angular momentum quantum numbers). What conditions does the Wigner-Eckart theorem impose on (l, m, l', m') for this matrix element to be non-zero? *Bonus:* What condition does parity impose?