New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

QUANTUM MECHANICS I

Fall, 2014

READ INSTRUCTIONS CAREFULLY

1. ANSWER 4 PROBLEMS AT YOUR CHOICE.
2. All problems have the same point value. When a problem is divided into parts, their relative weights are given as percentages in parentheses. If not otherwise indicated, all parts are equally weighted.
3. Write your own identification number on your answer booklets.
4. Show ALL your work.

Problem 1

Let a and $a^\dagger$ be canonical annihilation and creation operators obeying

$$[a, a^\dagger] = 1.$$

(a)

Find the most general linear transformation $b = \alpha a + \beta a^\dagger$ such that b and $b^\dagger$ are also canonical annihilation and creation operators.

(b)

Using the result of section a), find the ground state energy of the following Hamiltonian

$$H = A(aa + a^\dagger a^\dagger) + Ba^\dagger a, \qquad A, B = \text{real, positive constants}, \qquad 2A < B.$$

Problem 2

The nuclear magnetic octupole moment is given by a tensor of rank $k = 3$.

(a)

Show that this interaction cannot be measured in the ground state of alkali atoms.

(b)

Considering the states with principal quantum number n to label *pure* states, determine the ratio of intensities R in the two yellow lines from the first excited states to the ground state in sodium (principal series) doublet.

(c)

If you allow for the operator $\mathbf{L} \cdot \mathbf{S}$ to mix different n states, give an argument why, for example as observed in cesium, the ratio of the principal series intensities (varying n) would deviate from the value R of part (b).

Problem 3

(a)

With use of the minimum uncertainty principle applied to the electron in hydrogen, show that the radius is a_0, where a_0 is the Bohr radius.

(b)

Show that the solution of the Schrödinger equation for the $1s$ atom in hydrogen gives the same result, *i.e.* the maximum probability for the position of the electron is one Bohr radius from the nucleus.

Problem 4

Given the spin matrices

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \qquad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

and the spin eigenstates

$$\psi_1 = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_2 = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \psi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

25% have spin as in ψ_1, 50% have $|S_x+\rangle$, 25% have $|S_x-\rangle$.

(a)

Calculate the density operator ρ

(b)

Calculate the trace $\mathcal{T}r\ \ \rho$ and interpret the result physically

(c)

If $\mathcal{T}r\ \ \rho^2$ were equal to 1, what would be the meaning?

Problem 5

There is a potential step of height V_0 at $x \geq 0$ and an incoming particle of mass m, energy E, and velocity v from a region $x \leq 0$ where $V_0 = 0$. Determine the reflection coefficient R and the transmission coefficient T (intensities) and $R + T$ in terms of the wavenumbers $k = 2\pi/\lambda$ for the cases of

(a)

$E > V_0$

(b)

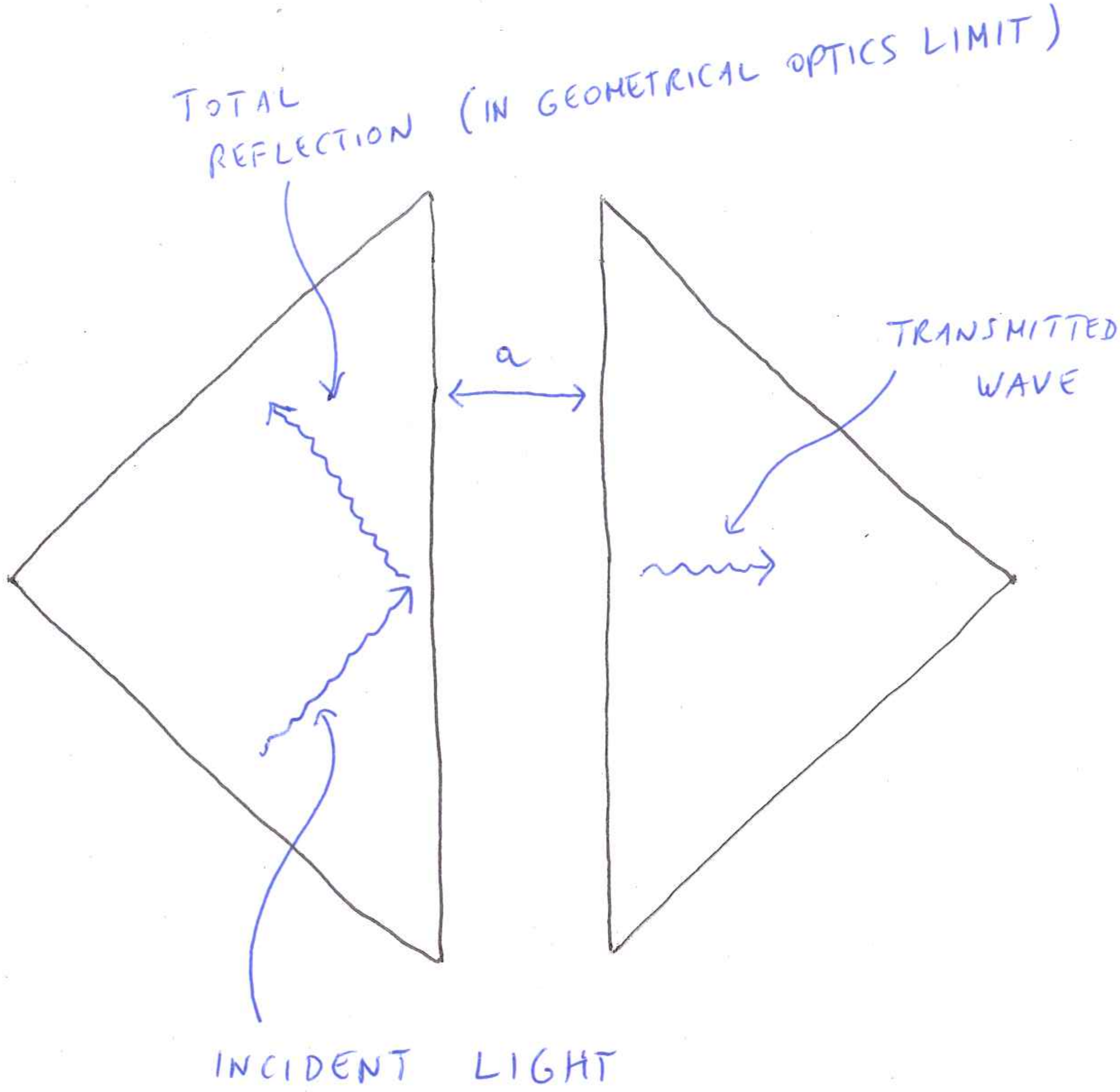
$E < V_0$

(c)

In part (b) there is a classical counterpart for a finite width a of the potential V_0: a wave incident from within a 45^0 prism on the large face of a right-angle prism, at an angle larger than the critical angle (total internal reflection). For example, the light incident on the first face of the prism could be in a direction which is perpendicular to the long prism face. If we place a second identical prism, at a separation distance a, with its large face parallel to the one of the first prism, we can couple some energy and transmit it through the second prism. In our case this would correspond to the potential step extending from $x = 0$ to $x = a$, for the air space between the two prisms. (This is used frequently in laser power couplers.)

Write, (**but do not solve**) equations that describe the quantum-mechanical waves at the boundaries $x = 0$ and $x = a$. **Sketch** (schematically) the wave behavior for $x < 0$, $0 \leq x \leq a$, $x > a$.

Problem 5c



TOTAL REFLECTION (IN GEOMETRICAL OPTICS LIMIT)
a
TRANSMITTED WAVE
INCIDENT LIGHT