Quantum Mechanics I  Final Exam  Fall 2007

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Read the problems carefully. Make your reasoning clear!

**Problem 1:** [5 points] Consider two non-orthogonal quantum states $|u\rangle$ and $|v\rangle$, with $(u|v) = \alpha \neq 0$. A measurement defined by the (projection) operator $P_u = 1 - |u\rangle \langle u|$ is made on each of the these two states. For each state $(|u\rangle$ and $|v\rangle$), what are the possible outcomes of the measurement $P_u$, and what are the probabilities of each of these outcomes? (Note, I'm *not* asking you to find the final states after the measurement.)

**Problem 2:** [5 points per part] Consider 2 (distinguishable) spin-1/2 particles, $\vec{s}_1$ and $\vec{s}_2$ in a state with total angular momentum $S = 1$, described by the following density matrix (defined in terms of the $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$ basis of states $|S, S_z\rangle$)

$$\rho = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (1)

a) Compute the probability that a measurement of total spin $S$ along the x-axis (that is $S_x$), will yield the value zero.

b) Find the expectation value $(s_{1z}, s_{2z})$, where $s_{iz}$ refers to the z-component of individual spin $i$ ($i = 1, 2$).

c) Find the probability that a measurement of $s_{1z}$ [on state (1)] will yield a value of 1/2.

d) If a measurement of $s_{1z}$ yielded a value of 1/2, what would the state $\rho$ (properly normalized) be after this measurement result? Note: you may express your answer either in terms of the basis $|S, S_z\rangle$ listed above, or in terms of the basis $|s_{1z}, s_{2z}\rangle$: $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}$.\}

**Problem 3:** [5 points per part] Consider the one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{\hbar \omega}{2} (p^2 + q^2)$$  \hspace{1cm} (2)

where $p$ and $q$ are the unitless momentum and position operators with $[p, q] = i$.

a) Find the Heisenberg equations of motion of the operators $p(t)$ and $q(t)$, and write the formal solutions (in terms of initial conditions).

b) If the initial state of the harmonic oscillator (in the Schrödinger picture) is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle),$$  \hspace{1cm} (3)

write down an expression for $|\psi(t)\rangle$. (States $|n\rangle$ are defined by $N|n\rangle = n|n\rangle$.)

c) What is the expectation value for the energy as a function of time?

d) For the state $|\psi(t)\rangle$ in part b), find the the expectation values of $q$ and $p$ as a function of time.
Problem 4: [10 points] A particle of mass $m$ moves in one dimension where the only potential $V(x) = C\delta(x)$ is at the origin with $C > 0$. A free particle of wave vector $k$ approaches the origin from the left. Derive an expression for the amplitude $T$ of the transmitted wave as a function of $k$, $C$, $m$, and $\hbar$.

Problem 5: [10 points] A spin-1/2 particle interacts with a magnetic field $\vec{B} = B_0 \hat{z}$ through the interaction $H = \mu \vec{\sigma} \cdot \vec{B},$ where $\mu$ is the magnetic moment and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. At $t = 0$, a measurement of $\sigma_x$ yields a value of +1. What is the probability that a measurement of $\sigma_y$ at a later time $t$ will yield a value -1?

Problem 6: [10 points] Consider harmonic oscillator with Hamiltonian

$$H_0 = \frac{\hbar \omega}{2}(q^2 + p^2)$$

for unitless position and momentum operators $q$ and $p$ in the presence of a small perturbation $H_1 = V_0 q$.

Starting with the unperturbed energies and eigenstates, use perturbation theory to compute to lowest non-zero order the perturbed energy and wavefunction of the (unperturbed) state $|n\rangle$.

Problem 7: [5 points per part] Two spin-1/2 particles are separated by a distance $\vec{a} = a \hat{z}$ and interact only through the magnetic dipole energy

$$H = \frac{\mu_1 \cdot \mu_2}{a^3} - 3 \frac{(\vec{\mu}_1 \cdot \vec{a})(\vec{\mu}_2 \cdot \vec{a})}{a^5}$$

where $\vec{\mu}_j$ is the magnetic moment of spin $j$. The system of two spins consists of eigenstates of the total spin $(S^2)$ and total $S_z$.

a) Write the Hamiltonian in terms of spin operators $\vec{s}_1$ and $\vec{s}_2$.

b) Write the Hamiltonian in terms of $S^2$ and $S_z$.

c) Give the energy eigenvalues of all states.

Problem 8: [5 points per part]

a) Consider two vector operators $\vec{V}$ and $\vec{W}$, which, in the spherical basis can be written $V_{m_1}$ and $W_{m_2}$. Suppose you want to combine these two operators to produce a rank-2 irreducible tensor operator $T^{(2)}_q$. In particular, how would you express the $q = 0$ component $T^{(2)}_0$ in terms of the spherical components of the operators $V_{m_1}$ and $W_{m_2}$? (You may use the table on the back of this exam to evaluate any Clebsch-Gordan coefficients.)

b) Suppose you want to evaluate the matrix elements of $T^{(2)}_0$ between two states $|j, m\rangle$ and $|j, m'\rangle$ of total angular moment $j = 1$. (That is you want to know $\langle 1, m | T^{(2)}_0 | 1, m' \rangle$.) Given that $\langle 1, 0 | T^{(2)}_0 | 1, 0 \rangle = A$, find the other nonzero matrix elements $\langle 1, m | T^{(2)}_0 | 1, m' \rangle$. 

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Problem 1:  \( P_\alpha = 1 - |u\rangle \langle u| \). This is a projection operator so possible outcomes are 0 and 1.

Measurement on state \( |u\rangle \): \( P_\alpha |u\rangle = 0 \), so \( \langle u|P_\alpha|u\rangle = \langle P_\alpha \rangle = 0 \). Thus, 
\( p_1 \equiv \) (Probability of outcome 1) = \( \langle P_\alpha \rangle = 0 \). \( p_0 \equiv \) (Probability of outcome 0) = 1.

Measurement on state \( |v\rangle \): \( \langle v|P_\alpha|v\rangle = p_1 = \langle v|v\rangle - \langle v|u\rangle \langle u|v\rangle = 1 - |\langle u|v\rangle|^2 = 1 - |\alpha|^2 \). Also \( p_0 = 1 - p_1 = |\alpha|^2 \).

Problem 2:  

a) \( p = \frac{1}{3} \), since the state is invariant under rotations so Prob(\( S_x = 0 \)) = Prob(\( S_z = 0 \)) = \( \frac{1}{3} \).

b) 
\[ s_{1z}s_{2z}|1,1\rangle = s_{1z}s_{2z}|++,\rangle = \frac{1}{4}|1,1\rangle \]
\[ s_{1z}s_{2z}|1,-1\rangle = s_{1z}s_{2z}|--,\rangle = \frac{1}{4}|1,-1\rangle \]
\[ s_{1z}s_{2z}|1,0\rangle = s_{1z}s_{2z}|++\rangle = \frac{1}{4}|1,0\rangle \]

so
\[ \langle s_{1z}s_{2z}\rangle = \frac{1}{3} \left( \frac{1}{4} + \frac{1}{4} - \frac{1}{4} \right) = \frac{1}{12} \]

or one can just use the fact that \( \langle s_{1z}s_{2z}\rangle = Tr(s_{1z}s_{2z}\rho) \). [Alternatively, you could show first that \( s_{1z}s_{2z} = S_z^2/2 - 1/4 \). See problem 7.]

c) If the density matrix is diagonal, the probability of finding a given measurement result can be written
\[ p = \sum_i \rho_{ii} p_i \]

where \( \rho_{ii} \) is a diagonal element of the density matrix, and \( p_i \) is the probability of that result given that the system is in the state \( |i\rangle \). In this case, the probability \( p_{1+} \) would therefore be
\[ p_{1+} = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{2} \]

where the 1/2 in the second term above is the probability of measuring \( S_z = 1/2 \) when a measurement is made on the state \( |1,0\rangle = (|++\rangle + |--\rangle)/\sqrt{2} \).

Alternatively, we can find the projection operator onto the subspace corresponding to eigenvalue +1/2 of the operator \( S_z \): \( P_{++} = |++\rangle_1 \langle ++| \otimes I_2 \), where the subscripts 1,2 refer to particle 1 or 2. We can therefore write
\[ p_{1+} = Tr(\rho P_{++}) = \frac{1}{3} \left[ |\langle 1,1|P_{++}|1,1\rangle + |\langle 1,0|P_{++}|1,0\rangle + |\langle 1,1|P_{++}|1,-1\rangle \right] = \frac{1}{2} \]
d) The (not necessarily normalized) final state is given by \( \rho_f = P_{1+} \rho P_{1+} \).

The projection operator \( P_{1+} \) expressed in the basis of individual spins is \( P_{1+} = |+\rangle_{11} \langle + | \otimes I_2 = |++\rangle \langle ++ | + |+-\rangle \langle -- | \). The density matrix \( \rho \) expressed in this same basis is

\[
\rho = \frac{1}{3} |++\rangle \langle ++ | + \frac{11}{3} |(+ -) + (- +) \rangle \langle (+ -) + (- +) | + \frac{1}{3} |--\rangle \langle -- | .
\]

Then

\[
P_{1+} \rho = \frac{1}{3} |++\rangle \langle ++ | + \frac{1}{6} |+-\rangle \langle +- | + \frac{1}{6} |+-\rangle \langle +- |
\]

and

\[
\rho_f = P_{1+} \rho P_{1+} = \frac{1}{3} |++\rangle \langle ++ | + \frac{1}{6} |+-\rangle \langle +- |
\]

Normalizing gives

\[
\rho_f = \frac{2}{3} |++\rangle \langle ++ | + \frac{1}{3} |+-\rangle \langle +- |
\]

One can also solve the problem in the \( |S, S_z \rangle \) basis, and the final state is

\[
\rho_f = \frac{2}{3} |11\rangle \langle 11 | + \frac{1}{6} (|1, 0 \rangle + |0, 0 \rangle) (\langle 1, 0 | + \langle 0, 0 |)
\]

\[
= \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6}
\end{pmatrix}
\]

**Problem 3:** a) \( H = \frac{\hbar \omega}{2} (p^2 + q^2), \) \( |q, p\rangle = i, \dot{A} = -\frac{i}{\hbar} [A, H], \) so

\[
\dot{q} = -\frac{i}{\hbar} [q, H] = -\frac{\hbar \omega}{2} [q, p^2].
\]

but \( [q, p^2] = qpp - pqp + qpq - pqp = [q, p]p + p[q, p] = 2ip, \) so

\[
\dot{q} = -\frac{i}{\hbar} \frac{\hbar \omega}{2} \cdot 2ip = \omega p.
\]

similarly,

\[
\dot{p} = -\omega q.
\]

Assume initial values \( p_0 \) and \( q_0 \). Solutions to these equations are

\[
p(t) = p_0 \cos \omega t - q_0 \sin \omega t
\]

\[
q(t) = q_0 \cos \omega t + p_0 \sin \omega t
\]

b) \( |\psi(t)\rangle = \frac{1}{\sqrt{2}} \left[ \exp \left( -\frac{i\omega t}{2} \right) |0\rangle - \exp \left( -\frac{i3\omega t}{2} \right) |1\rangle \right] \)

\[
\langle E \rangle = \hbar \omega (n + 1/2) = \hbar \omega (\frac{1}{2} + \frac{3}{2}) / 2 = \hbar \omega.
\]
d) $q = (a + a^d) / \sqrt{2}$, so $\langle q \rangle = \frac{1}{\sqrt{2}}(\langle a \rangle + \langle a^d \rangle)$. Now,

$$\langle a \rangle = \langle \psi | a | \psi \rangle$$

$$= \frac{1}{2} \left[ e^{i\omega t/2} \langle 0 | - e^{-3i\omega t/2} \langle 1 | a \right] \left[ e^{-i\omega t/2} \langle 0 | - e^{3i\omega t/2} \langle 1 | \right]$$

$$= -\frac{1}{2} e^{i\omega t/2} e^{-3i\omega t/2} \langle 0 | a | 1 \rangle$$

$$= -\frac{1}{2} e^{-i\omega t} \langle 0 | a | 1 \rangle = -\frac{1}{2} e^{-i\omega t}$$

Similarly $\langle a^d \rangle = \frac{1}{2} e^{i\omega t}$. Thus, $\langle q \rangle = -\frac{1}{2\sqrt{2}}(e^{i\omega t} + e^{-i\omega t}) = -\frac{1}{\sqrt{2}} \cos \omega t$.

**Problem 4:** $V(x) = C\delta(x)$.

Solution on the left side: $u_L(x) = e^{ikx} + Be^{-ikx}$.

Solution on the right side: $u_R(x) = Te^{ikx}$.

Boundary conditions:

$u_L(0) = u_R(0) \Rightarrow T = 1 + B$, and

$u''_L(0) - u''_R(0) - \frac{2m}{k^2} Cu(0) = 0$, where

$u''_L(x) = ike^{ikx} - ikBe^{-ikx}$

$u''_R(0) = ik - ikB$ and

$u''_R(x) = ikTe^{ikx}$

$u''_R(0) = ikT$

So $ikT - ik + ikB - \frac{2m}{k^2} CT = 0$. We now solve for $T$:

$ikT - ik + ik(T - 1) - \frac{2m}{k^2} CT = 0$

$2ikT - 2ik - \frac{2m}{k^2} CT = 0$

$T(2ik - \frac{2mC}{k^2}) = 2ik$, and

$$T = \frac{ik}{ik - \frac{mc}{k^2}} = \frac{1}{1 + \frac{imc}{k^2}}$$

**Problem 5:** Hamiltonian

$$H = \mu B_0 \sigma_z = \mu B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Also

$$e^{iHt/\hbar} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}$$

Just after measurement at $t = 0$, state is eigenstate of $\sigma_z$ with eigenvalue +1. That is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
At a later time
\[ |\psi(t)⟩ = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iωt} & 0 \\ 0 & e^{iωt} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iωt} \\ e^{iωt} \end{pmatrix} \]  
(29)

Eigenstate of \( σ_y \) with eigenvalue with eigenvalue -1 is
\[ |ψ_{-1,0}⟩ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \]  
(30)

Probability of getting -1 when \( σ_y \) is measured:
\[ \left| \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} e^{-iωt} \\ e^{iωt} \end{pmatrix} \right|^2 = \frac{1}{4} |e^{-iωt} + ie^{iωt}|^2 = \frac{1}{2} - \frac{1}{2} \sin(2ωt). \]  
(31)

**Problem 6:** \( H_1 = V_0q = \frac{V_0}{\sqrt{2}}(a + a^\dagger), \ E_n^{(0)} = (n + \frac{1}{2}) \hbarω, |ψ_n^{(0)}⟩ = |n⟩. \)

First order energy correction:
\[ E_n^{(1)} = ⟨n|H_1|n⟩ = \frac{V_0}{\sqrt{2}}(n|(a^\dagger + a)|n) = 0. \]  
(32)

Second order energy correction:
\[ E_n^{(2)} = \frac{V_0^2}{2} \sum_{n'\neq n} \frac{⟨n'|a^\dagger + a|n⟩}{\varepsilon_n - \varepsilon_{n'}} = \frac{V_0^2}{2} \left[ -\frac{n+1}{\hbarω} + \frac{n}{\hbarω} \right] = -\frac{V_0^2}{2\hbarω}. \]  
(33)

First order wavefunction correction:
\[ |ψ_n^{(1)}⟩ = |n⟩ + \frac{V_0}{\sqrt{2}} \sum_{n'\neq n} \frac{|n'|⟨n'|a^\dagger + a|n⟩}{\varepsilon_n - \varepsilon_{n'}} \]  
(34)

\[ = |n⟩ + \frac{V_0}{\sqrt{2}\hbarω} \left[ \sqrt{n}|n-1⟩ - \sqrt{n+1}|n+1⟩ \right] \]  
(35)

**Problem 7:** \( \bar{μ}_i = μ_B\bar{s}_i, \bar{a} = a^z, \bar{μ}_i · \bar{a} = μ_ia = μ_Ba, \bar{μ}_1 · \bar{μ}_2 = μ_B\bar{s}_1 · \bar{s}_2, s_1 = s_2 = 1/2. \)

a) \( H = \frac{μ_B^2}{a^2} \bar{s}_1 · \bar{s}_2 - \frac{3μ_B}{a^3} s_{1z}s_{2z} = \frac{μ_B}{a^2}[\bar{s}_1 · \bar{s}_2 - 3s_{1z}s_{2z}] \)  
(36)

b) \( S^2 = (\bar{s}_1 + \bar{s}_2)^2 = \bar{s}_1^2 + \bar{s}_2^2 + 2\bar{s}_1 · \bar{s}_2, \)  
(37)

so
\[ \bar{s}_1 · \bar{s}_2 = \frac{1}{2} \left[ S^2 - \bar{s}_1^2 - \bar{s}_2^2 \right] = \frac{1}{2} \left[ S(S+1) - \frac{3}{2} \right] = \frac{S(S+1)}{2} - \frac{3}{4} \]  
(38)

since \( \bar{s}_1^2 = s_1(s_1 + 1) = \frac{s_2^2}{2} = 3/4. \)

\[ S_2^2 = (s_{1z} + s_{2z})^2 = s_{1z}^2 + s_{2z}^2 + 2s_{1z}s_{2z}, \]  
(39)
\[ s_{1z}s_{2z} = \frac{1}{2} \left[ S^2_z - s^2_{1z} - s^2_{2z} \right] = \frac{S^2_z}{2} - \frac{1}{4} \]  

(40)

Thus,

\[ H = \frac{\mu_B^2}{a^3} \left[ \frac{S(S + 1)}{2} - \frac{3}{4} - \frac{3S^2_z}{2} + \frac{3}{4} \right] = \frac{\mu_B^2}{a^3} \left[ \frac{S(S + 1)}{2} - \frac{3S^2_z}{2} \right] \]  

(41)

c) Total spin \( S = 0, 1 \). Eigenstates \(|S, m\rangle\) are therefore \(|0, 0\rangle\) and \(|1, m\rangle\), where \( m = -1, 0, 1 \). Energies are:

\[ E_{00} = \frac{\mu_B^2}{a^3} \cdot 0 = 0 \]  

(42)

\[ E_{1m} = \frac{\mu_B^2}{a^3} \left[ 1 - \frac{3}{2} m^2 \right] \]  

(43)

so

\[ E_{10} = \frac{\mu_B^2}{a^3}, E_{11} = E_{1, -1} = -\frac{\mu_B^2}{2a^3}. \]  

(44)

**Problem 8:**

a)  

\[ T^{(2)}_0 = \sum_{m_1, m_2} V_{m_1} W_{m_2} \langle 11m_1m_2 | 20 \rangle \]  

(45)

From the C-G tables: \( \langle 111, -1 | 20 \rangle = \langle 11, -11 | 20 \rangle = 1/\sqrt{6}, \langle 1100 | 20 \rangle = \sqrt{2/3} = 2/\sqrt{6}, \) so

\[ T^{(2)}_0 = \frac{1}{\sqrt{6}} (V_1 W_{-1} + 2V_0 W_0 + V_{-1} W_1) \]  

(46)

b) Wigner-Eckart theorem states

\[ \langle 1, m | T^{(2)}_0 | 1, m' \rangle = \langle T | 12m0 | 1m' \rangle \]  

(47)

where \( \langle T \rangle \) is a reduced matrix element (independent of \( m, m' \)), and \( \langle 12m0 | 1m' \rangle \) is a C-G coefficient, which is non-zero only when \( m' = m \).

We are given \( \langle 1, 0 | T^{(2)}_0 | 1, 0 \rangle = A \), so

\[ \langle 1, 0 | T^{(2)}_0 | 1, 0 \rangle = \langle T | 1200 | 10 \rangle = A \]  

(48)

\[ = \langle T | (-\sqrt{5/2}) = A \]  

(49)

so \( \langle T \rangle = -\sqrt{5/2} A \). Thus

\[ \langle 1, 1 | T^{(2)}_0 | 1, 1 \rangle = -\sqrt{5/2} A (1210 | 11) \]  

(50)

\[ = -\sqrt{5/2} A \sqrt{1/10} = -A/2 \]  

(51)

\[ \langle 1, -1 | T^{(2)}_0 | 1, -1 \rangle = -\sqrt{5/2} A (12, -10 | 1, -1) \]  

(52)

\[ = -\sqrt{5/2} A \sqrt{1/10} = -A/2 \]  

(53)
Be quantitative where you can, but avoid getting bogged down if exact coefficients or overall irrelevant phases are not at your fingertips. All problems have the same weight, except that Problem 3 has 0.5 weight and Problem 4 has 1.5.

**Problem 1:** Two electrons are confined to a one-dimensional box of length $L$. The situation has been set up so that both electrons have the same spin state. Ignore the Coulomb interactions between electrons.

a) Write the ground state wavefunction $\psi(x_1, x_2)$ for the two-electron system.

b) What is the ground state energy of the system?

c) What is the probability that both electrons are found in the same half of the box?

**Problem 2:** The setup in problem 1 lends itself to analysis in terms of a 2nd quantized electron field.

a) Write the 2nd quantized electron field appropriate to this problem. Give a brief explanation of your reasoning. What commutation or anti-commutation relations do you impose on the creation and annihilation operators and why? Be explicit about the position dependence of the field. You need not determine the overall normalization factor.

b) Write the ground state of 2 electrons both with spin $+1/2$, in terms of creation and annihilation operators acting on the vacuum. Prove this is the lowest energy state of 2 electrons with identical spin.

c) Derive an expression for the energy of a state, in terms of creation and annihilation operators. Point out how this determines the correct choice of commutation or anti-commutation relations you adopted in a).

**Problem 3:** Estimate $\beta = |v|/c$ for an electron in the hydrogen atom. Is this sufficiently non-relativistic that the Dirac equation is irrelevant? If you believe it is not irrelevant, give an example of how the Dirac equation is relevant and give an estimate for the size of its effect.

**Problem 4:** Suppose that a new type of (neutral) vector boson $Z'$ is discovered at the LHC — much like a photon but massive: $m_{Z'} = 1$ TeV. Its coupling is found to exactly parallel the coupling of the photon to electrically charged particles, but with strength $e' \approx 3e$.

a) Write the interaction Hamiltonian between the $Z'$ field and the electron and muon fields.

b) Draw the lowest order Feynman diagrams for $e^+(p) e^-(q) \rightarrow \mu^+(p') \mu^-(q')$; write down the corresponding scattering amplitude.
c) This new interaction induces an effective 4-fermi interaction involving electrons and muons that is not present in the absence of the $Z'$. Write down the effective Hamiltonian for this interaction and give an expression for the coupling constant analogous to $G_F$ in terms of the fundamental parameters.

d) Estimate at what CM energy in a future $e^+e^-$ collider, the existence of the $Z'$-induced interaction would make a $\gtrsim 10\%$ change in the cross section for $e^+e^- \rightarrow \mu^+\mu^-$ compared to what it would be with only photons. (Ignore all other particles besides electrons, muons, photon and $Z'$.)

e) Suppose a bremsstrahlung photon is emitted during the scattering in b), so the process becomes $e^+(p)e^-(q) \rightarrow \mu^+(p')\mu^-(q')\gamma(k)$. Draw the complete set of Feynman diagrams for this process; give the Feynman amplitude corresponding to one of them, specifying which diagram you are doing.

Problem 5: A hypothetical new attractive "fifth" force might help explain some puzzling aspects of Dark Matter interactions. Suppose you want to model the new force on the Yukawa interaction, by postulating the existence of a new weakly coupled scalar boson, $X$, taking the DM to be a neutral fermion with mass $m_{DM}$. To have the desired impact on cosmology, the range of the interaction must be of order 10 million light years or greater.

a) What order-of-magnitude limit can you put on the mass $m_X$ of the $X$ boson?

b) What if the $X$ boson were a vector instead of a scalar but was otherwise identical in its mass and coupling strength – would that make a qualitative change in the force?

c) If you think of the gravitational interaction between two particles as being due to the exchange of a massless graviton, which you can take as a scalar boson for simplicity, how big (order of magnitude) does the coupling of the Dark Matter particle to the $X$ boson have to be, for the strength of the fifth force between DM particles to be comparable to their gravitational interaction, for distances small compared to the range of the fifth force?

Problem 6:

a) Outline the EPR argument leading to the conclusion in their 1935 paper that quantum mechanics is not complete.

b) Briefly state the content and significance of John Bell’s work that lead to the “Bell Inequality”.

c) In the discussion of the teleportation scheme in lecture, it was stated that if Alice’s particle were initially entangled with some fourth particle, then after teleportation, the state of Bob’s particle would be entangled with this fourth particle in exactly the same way as Alice’s original particle. Provide a proof or convincing argument for this statement.
Problem 1: Two electrons are confined to a one-dimensional box of length \(L\), extending from \(x = 0\) to \(x = L\). The situation has been set up so that both electrons have the same spin state. Ignore the Coulomb interactions between electrons.

a) Write the ground state wavefunction \(\psi(x_1, x_2)\) for the two-electron system.

Let the one particle eigenstates be \(\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)\) with energy \(E_n = (\pi^2 \hbar^2 / 2mL^2)n^2\).

The ground state is then

\[
\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[ \psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1) \right] \chi_{s_1s_2} \tag{1}
\]

\[
= \left( \frac{\sqrt{2}}{L} \right) \left[ \sin(\pi x_1/L) \sin(2\pi x_2/L) - \sin(\pi x_2/L) \sin(2\pi x_1/L) \right] \chi_{s_1s_2}, \tag{2}
\]

where \(\chi_{s_1}\) is the spin part of the wavefunction and \(s_1 = s_2\). [Note: a number of people wrote \(\psi(x_1, x_2) = \sin(\pi x_1/L) + \sin(2\pi x_2/L)\), which makes no sense. The vector space of two-particle states is the tensor-product space of one-particle states.]

b) What is the ground state energy of the system?

The ground state energy is \(E_1 + E_2 = 5(\pi^2 \hbar^2 / 2mL^2)\).

c) What is the probability that both electrons are found in the same half of the box?

The probability that both particles are found in the left side of the box is

\[
P_L = \int_0^{L/2} dx_1 \int_0^{L/2} dx_2 |\psi(x_1, x_2)|^2 = \frac{1}{2} \int_0^{L/2} dx_1 \int_0^{L/2} dx_2 \left[ \psi_1^*(x_1)\psi_2^*(x_2) - \psi_1(x_1)\psi_2(x_2) \right] \chi_{s_1s_2} \chi_{s_1s_2} \tag{3}
\]

\[
= I_{11} - I_{12}^2, \tag{4}
\]

where \(I_1 = \int \psi_1^2(x) dx = 1/2\), \(I_2 = \int \psi_2^2(x) dx = 1/2\), \(I_{12} = \int \psi_1(x)\psi_2(x) dx = 4/3\pi\), so \(P_L = (1/2)^2 - (4/3\pi)^2 \approx 0.07\). The total probability is twice this or approximately 0.14.

Problem 2: The setup in problem 1 lends itself to analysis in terms of a 2nd quantized electron field.

a) Write the 2nd quantized electron field appropriate to this problem. Give a brief explanation of your reasoning. What commutation or anti-commutation relations do you impose on the creation and annihilation operators and why? Be explicit about the position dependence of the field. You need not determine the overall normalization factor.

\[
\Psi(x, t) = \sum_{s=\pm1/2} \sum_{n=1}^{\infty} b_{n,s} \psi_n(x)e^{-i\omega_n t} \chi_s = \sqrt{2/L} \sum_{n,s} b_{n,s} \sin(\pi kx/L)e^{-i\omega_n t} \chi_s, \tag{5}
\]
where $\chi_s$ is the spin wavefunction, and $\omega_n = E_n/h$. $b_{n,s}$ annihilate an electron in state $n$, $s$, and the hermitean conjugate operators create those states. The creation and annihilation operators anti-commute, e.g.,

$$\{b_{n,s}, b_{n',s'}^\dagger\} = \delta_{nn'}\delta_{ss'}.$$  \hspace{1cm} (6)

For all other anticommutators the rhs is 0.

b) Write the ground state of 2 electrons both with spin $+1/2$, in terms of creation and annihilation operators acting on the vacuum. Prove this is the lowest energy state of 2 electrons with identical spin.

$$b_{1,+}^\dagger b_{2,+}^\dagger |0\rangle.$$  \hspace{1cm} (7)

Due to anticommutation, the possible lower energy state satisfies

$$b_{1,+}^\dagger b_{1,+}^\dagger |0\rangle = -b_{1,+}^\dagger b_{1,+}^\dagger |0\rangle = 0.$$  \hspace{1cm} (8)

c) Derive an expression for the energy of a state, in terms of creation and annihilation operators. Point out how this determines the correct choice of commutation or anti-commutation relations you adopted in a).

The Hamiltonian of the system is

$$\mathcal{H} = \sum_n \hbar \omega_n b_{n,+}^\dagger b_{n,+}.$$  \hspace{1cm} (9)

To “derive” this, one could write

$$\mathcal{H} = \int dx \Psi^\dagger(x,t) H \Psi(x,t),$$  \hspace{1cm} (10)

where $H$ is the single-particle Hamiltonian. Since $\Psi(x,t)$ in the Heisenberg picture obeys the Schrödinger equation, we have

$$H \Psi(x,t) = i\hbar d\Psi(x,t)/dt = \sum_{n,s} \hbar \omega_n b_{n,s} \psi_n(x,t) \chi_s.$$  \hspace{1cm} (11)

Substituting this and the Hermitian conjugate of Eq. (5) into (10) and using the orthonormality of the $\psi_n$ yields Eq. (9).

Another derivation of Eq. (9) would be to start with Eq. (10) with $H = p^2/2m = -(\hbar^2/2m)(\partial^2/\partial x^2)$, so

$$\mathcal{H} = -\frac{\hbar^2}{2m} \int dx \Psi^\dagger(x,t) \frac{\partial^2}{\partial x^2} \Psi(x,t)$$  \hspace{1cm} (12a)

$$= -\frac{\hbar^2}{2m} \int dx \sum_{nn'} a_{n}^\dagger a_{n'} \psi_n^*(x) \psi_{n'}(x)$$  \hspace{1cm} (12b)
where $\psi_{n'}^\mu(x) = -\sqrt{2/L(n'/\pi/L)^2} \sin(n'/\pi x/L) = -(n'/\pi/L)^2 \psi_{n'}(x)$. So

$$\mathcal{H} = \sum_{nn'} a_n^\mu a_{n'} \frac{\hbar^2 n'^2}{2m} \frac{n^2 \pi^2}{L^2} \int dx \psi_{n'}^\mu(x) \psi_{n'}(x)$$

(13a)

$$= \sum_{nn'} a_n^\mu a_{n'} \frac{\hbar^2 n'^2}{2m} \frac{n^2 \pi^2}{L^2} \delta_{n',n}$$

(13b)

$$= \sum_n a_n^\mu a_n \frac{\hbar^2 \pi^2 n^2}{2mL^2}.$$  \hspace{1cm} (13c)

\textbf{Problem 3:}

\textit{a) Estimate } $|v|/c$ \textit{for an electron in the hydrogen atom. Is this sufficiently non-relativistic that the Dirac equation is irrelevant?}

$|v|/c \approx \alpha \approx 1/137$, where alpha is the fine-structure constant. How can we figure this out. If we use atomic units (remember from QM1), where $e = m = \hbar = 1$, and $c = 1/\alpha \approx 137$ where the fine-structure constant $\alpha = e^2/\hbar c$. In atomic units the Bohr radius $a_0 = 1$, so by the uncertainty principle, $\nu \approx 1$, so $|v/c| \approx 1/137$. Another way to make this estimate is to use the fact that the electron kinetic energy is of the same order as the binding energy (virial theorem), or $m_e v^2/2 \sim 13.6$ eV, so $(v/c)^2 \sim 13.6 eV/(m_e c^2) = 13.6/(5 \times 10^4)$.

\textit{b) If you believe the Dirac equation is relevant for the hydrogen atom, give an example of an effect for which the Dirac equation gives an } $O(1)$ \textit{change in the result without the Dirac equation. How is this consistent with your result in a)?}

Thus, relativistic effects are small in the $H$ atom. If you consider higher order effects in the hydrogen atoms, such as fine and hyperfine structure, these effects depend on the g-factor of the electron spin. A non-relativistic treatment of electron-spin gives a g-factor that differs by a factor of two from a treatment using the Dirac equation.

\textbf{Problem 4:} \textit{Suppose that a new type of (neutral) vector boson } $Z'$ \textit{is discovered at the LHC - much like a photon but massive: } $m_{Z'} = 1$ TeV. \textit{Its coupling is found to exactly parallel the coupling of the photon to electrically charged particles, but with strength } $e' \approx 3e$.

\textit{a) Write the interaction Hamiltonian between the } $Z'$ \textit{field and the electron and muon fields.}

$$\mathcal{H}_{\text{int}} = e' \bar{\Psi}_\mu \gamma^\lambda \Psi_\mu Z'_\lambda + e' \bar{\Psi}_e \gamma^\lambda \Psi_e Z'_\lambda.$$  \hspace{1cm} (14)

\textit{b) Draw the lowest order Feynman diagrams for } $e^+(p) e^-(q) \rightarrow \mu^+(p') \mu^-(q')$; \textit{write down the corresponding scattering amplitude.}

$$\mathcal{M} = \bar{u}(q') \gamma_\mu v(p') \bar{v}(p) \gamma^\mu u(q) \left( \frac{(-ie')^2}{(p+q)^2 - M^2 Z'} + \frac{(-ie)^2}{(p+q)^2} \right).$$  \hspace{1cm} (15)
c) This new interaction induces an effective 4-fermi interaction involving electrons and muons that is not present in the absence of the Z'. Write down the effective Hamiltonian for this interaction and give an expression for the coupling constant analogous to $G_F$ in terms of the fundamental parameters.

$$G_Z \Psi_{\mu} \gamma^\lambda \Psi_{\bar{\mu}} \psi_{\mu} \psi_{\bar{\mu}} \gamma^\lambda \psi_{\bar{\mu}}; \quad G_{Z'} = \frac{9e^2}{M_{Z'}^2}.$$ \hspace{1cm} (16)

d) Estimate at what CM energy in a future $e^+e^-$ collider, the existence of the $Z'$-induced interaction would make a $\geq 10\%$ change in the cross section for $e^+e^- \rightarrow \mu^+\mu^-$ compared to what it would be with only photons. (Ignore all other particles besides electrons, muons, photon and $Z'$.) Since there is no way to tell whether a photon or a $Z'$ mediates a given event, the amplitudes add and the cross section is proportional to the total amplitude squared. So to get a 10% change in the cross section, the $Z'$ amplitude should be $\approx 5\%$ the photon amplitude. From part b), that means

$$\frac{e'^2}{M_{Z'}^2} \approx 0.05 \frac{e^2}{E_{\text{CM}}^2},$$ \hspace{1cm} (17)

where $E_{\text{CM}}$ is the total CM energy. Using $e' = 3e$, this is satisfied when $E_{\text{CM}} = \frac{M_{\mu}}{\sqrt{10}} \approx M_{Z'}/13$.

e) Suppose a bremsstrahlung photon is emitted during the scattering in b), so the process becomes $e^+(p) e^-(q) \rightarrow \mu^+(p') \mu^-(q') \gamma(k)$. Draw the complete set of Feynman diagrams for this process; give the Feynman amplitude corresponding to one of them, specifying which diagram you are doing.

**Problem 5:** A hypothetical new attractive "fifth" force might help explain some puzzling aspects of Dark Matter interactions. Suppose you want to model the new force as mediated by exchange of a new weakly coupled scalar boson, $X$, analogous to the Yukawa interaction, taking the DM to be a neutral fermion with mass $m_{\text{DM}}$. To have the desired impact on cosmology, the range of the interaction must be of order 10 million light years or greater.

a) In order that the 5th force have such a long range, what order-of-magnitude limit can you put on the mass $m_X$ of the $X$ boson?
Figure 3: 4e: Bremsstrahlung diagrams for photon-mediated processes. There are also 4 corresponding diagrams for $Z'$ mediated processes. As an example, the momentum structure of one diagram is shown.

The range of a Yukawa potential is the inverse of the mass of the spin-0 particle which mediates it (in natural units), the $X$ in this case. The nuclear force, whose range is $1$ fm (the characteristic size of a nucleon or small nucleus), is mediated by the pion, whose mass is $100$ MeV. So we can solve this without having to remember Planck's constant as follows:

$$m_X \approx m_\pi \frac{1 \text{ fm}}{r_X} \approx 10^8 \text{eV} \frac{10^{-13} \text{cm}}{10^7 \times 3 \times 10^{10} \times \pi \times 10^3 \text{cm}} = 10^{-30} \text{eV}.$$  \hspace{1cm} (18)

b) What if the $X$ boson were a vector instead of a scalar but was otherwise identical in its mass and coupling strength – would that make a qualitative change in the force? Yes! If a “fifth force” were mediated by a vector instead of a scalar, it would be repulsive – just as we found in class that the Yukawa force mediated by a spin-0 particle is always attractive while a photon-mediated interaction between like charges is repulsive.

c) If you think of the gravitational interaction between two particles as being due to the exchange of a massless graviton, which you can take as a scalar boson for simplicity, how big (order of magnitude) does the coupling of the Dark Matter particle to the $X$ boson have to be, for the strength of the fifth force between DM particles to be comparable to their gravitational interaction, for distances small compared to the range of the fifth force?

You may find it useful to express $G_N$ in natural units, using $G_N \equiv M_F^{-2}$, where the Planck mass $M_F \approx 10^{18}$ GeV.

The gravitational potential between two DM particles is $G_N m_{DM}^2 r^{-1} \equiv m_{DM}^2 / M_F^2 r^{-1}$. With
a scalar boson exchange, at distances small compared to the range of the 5th force, the potential is $g_5^2 r^{-1}$, so we require $g_5 \approx M_{DM}/M_{Pl}$.

**Problem 6:**

a) *Outline the EPR argument leading to the conclusion in their 1935 paper that quantum mechanics is not complete.*

One can make the argument equally well in terms of position and momentum (as EPR did), or in terms of different components of spin. If we consider two spin \(1/2\) particles in a singlet state, then the results of measurements of these spins along the same direction (say by Alice and Bob, respectively), will be perfectly correlated. This allows Alice to predict Bob’s result with certainty for any direction he measures (say along \(x\)) if Alice also measures in that direction. This means that the \(x\)-component of spin of Bob’s particle is an *element of reality* (or that it has a definite value before Bob makes the measurement). Since Alice’s direction of measurement can’t possibly affect Bob’s result (locality), the \(x\)-component of spin of Bob’s particle must be an element of reality regardless of what direction Alice measures. The same argument applies to the \(y\)-component of spin of Bob’s particle. Thus, both the \(x\) and \(y\) components of spin of Bob’s particle are elements of reality. QM doesn’t allow both these quantities to have definite values, (uncertainty principle), therefore QM is incomplete. Any answer to this question must therefore involve the concepts of: *perfect correlation, element of reality, locality, and the non-definiteness in QM of quantities corresponding to non-compatible observables.*

b) *Briefly state the content and significance of John Bell’s work that lead to the “Bell Inequality”.*

Bell showed that any local hidden variable theory leads to an inequality involving measured quantities. This inequality is inconsistent with the predictions of quantum mechanics. Bell’s work showed that such hidden variable theories are subject to experimental test.

c) *In the discussion of the teleportation scheme in lecture, it was stated that if Alice’s particle were initially entangled with some fourth particle, then after teleportation, the state of Bob’s particle would be entangled with this fourth particle in exactly the same way as Alice’s original particle. Provide a proof or convincing argument for this statement.*

If Alice’s particle is entangled with a fourth particle, such a state can always be written

$$|\phi_{14}\rangle = a|\alpha_4\rangle \langle \uparrow_1| + b|\beta_4\rangle \langle \downarrow_1|$$

(19)

where \(|\alpha_4\rangle\) and \(|\beta_4\rangle\) are any two possible states of the fourth particle. [Note that the most general pure state of a two-particle system can be written \(a_{uu}|\uparrow_4\rangle \langle \uparrow_1| + a_{du}|\downarrow_4\rangle \langle \uparrow_1| + a_{ud}|\uparrow_4\rangle \langle \downarrow_1| + a_{dd}|\downarrow_4\rangle \langle \downarrow_1|\), which can also be expressed in the form of Eq. (19).]

The teleportation scheme outlined in the notes [starting with Eq. (6.18)] then goes through for this case if one simply makes the substitutions \(a \rightarrow a|\alpha_4\rangle\) and \(b \rightarrow b|\beta_4\rangle\).
A simple harmonic one-dimensional oscillator has energy levels given by $E_n = \hbar (n + \frac{1}{2})\omega$, where $\omega$ is the characteristic (angular) frequency of the oscillator and where the quantum number $n$ can assume the possible integral values $n = 0, 1, 2, \ldots$. Suppose that such an oscillator is in thermal contact with a heat reservoir at temperature $T$ low enough so that $kT/\hbar \omega \ll 1$.

(a) Find the ratio of the probability of the oscillator being in the first excited state to the probability of its being in the ground state.

(b) Assuming that only the ground state and first excited state are appreciably occupied, find the mean energy of the oscillator as a function of the temperature $T$.

Consider an ideal gas of $N$ electrons in a volume $V$ at absolute zero.

(a) Calculate the total mean energy $\bar{E}$ of this gas.

(b) Express $\bar{E}$ in terms of the Fermi energy $\mu$.

(c) Show that $\bar{E}$ is properly an extensive quantity, but that for a fixed volume $V$, $\bar{E}$ is not proportional to the number $N$ of particles in the container. How do you account for this last result despite the fact that there is no interaction potential between the particles?

Consider an ensemble of classical one-dimensional harmonic oscillators.

Let the displacement $x$ of an oscillator as a function of time $t$ be given by $x = A \cos (\omega t + \phi)$. Assume that the phase angle $\phi$ is equally likely to assume any value in the range $0 < \phi < 2\pi$. The probability $w(\phi)\,d\phi$ that $\phi$ lies in the range between $\phi$ and $\phi + d\phi$ is then simply $w(\phi)\,d\phi = (2\pi)^{-1}\,d\phi$. For any fixed time $t$, find the probability $P(x)\,dx$ that $x$ lies between $x$ and $x + dx$ by summing $w(\phi)\,d\phi$ over all angles $\phi$ for which $x$ lies in this range. Express $P(x)$ in terms of $A$ and $\omega$.

Consider a classical ideal gas in thermal equilibrium at temperature $T$ in a container of volume $V$ in the presence of a uniform gravitational field. The acceleration due to gravity is $g$ and directed along the $-z$ direction.

(a) Calculate the chemical potential $\mu$ of an element of volume $\delta V$ of such a gas as a function of the pressure $p$, the temperature $T$, and the height $z$.

(b) Show that the requirement that $\mu$ is constant implies immediately the law of atmospheres which gives the dependence of $p$ on $T$ and $z$.

The nuclei of atoms in a certain crystalline solid have spin one. According to quantum theory, each nucleus can therefore be in any one of three quantum states labeled by the quantum number $m$, where $m = \pm 1, 0$. This quantum number measures the projection of the nuclear spin along a crystal axis of the solid. Since the electric charge distribution in the nucleus is not spherically symmetrical, but ellipsoidal, the energy of a nucleus depends on its spin orientation with respect to the internal electric field existing at its location. Thus a nucleus has the same energy $E = \varepsilon$ in the state $m = 1$ and the state $m = -1$, compared with an energy $E = 0$ in the state $m = 0$.

(a) Find an expression, as a function of absolute temperature $T$, of the nuclear contribution to the molar internal energy of the solid.

(b) Find an expression, as a function of $T$, of the nuclear contribution to the molar entropy of the solid.

In some homogeneous substance at absolute temperature $T_s$ (e.g., a liquid or gas) focus attention on some small portion of mass $M$. This small portion is in equilibrium with the rest of the substance; it is large enough to be macroscopic and can be characterized by a volume $V$ and temperature $T$. Calculate the probability $P(V, T)\,dV\,dT$ that the volume of this portion lies between $V$ and $V + dV$ and that its temperature lies between $T$ and $T + dT$. Express your answer in terms of the compressibility $\kappa$ of the substance, its density $\rho$, and its specific heat per gram $c_v$ at constant volume.
1. **Solutions**

**Final - TSP, 2008**

1. \[ E_0 = \frac{k_0}{2}, \quad E_1 = \frac{3k_0}{2} \]

2. \[ P_0 = \frac{1}{Z} e^{-\beta E_0}, \]

3. \[ P_1 = \frac{1}{Z} e^{-\beta E_1}, \]

4. \[ \frac{P_1}{P_0} = e^{-k_0/T} \]

5. \[ Z = e^{-\beta E_0} + e^{-\beta E_1} \]

6. \[ E = \frac{\frac{k_0}{2} + 3 e^{-\beta k_0}}{1 + e^{-\beta k_0}} \]

2. **For** \( T = 0 \) \[ n(\varepsilon) = \begin{cases} 1, & \varepsilon < \mu \\ 0, & \varepsilon > \mu \end{cases} \]

\[ \mu = \varepsilon_F \]

\[ N = (2\pi \hbar)^3 \int_{-\infty}^{\infty} n(\varepsilon) g(\varepsilon) d\varepsilon \]

\[ = (2\pi \hbar)^3 \int_{0}^{\mu} g(\varepsilon) d\varepsilon \]

\[ g(\varepsilon) d\varepsilon = \frac{\sqrt{\varepsilon}}{(2\pi \hbar)^3} \frac{d^3 k}{(2\pi)^3} = \frac{4\pi k^2 V d k}{(2\pi)^3} \]

\[ N = \frac{1}{5} \frac{V}{\pi^2} k_F^3 = \frac{1}{3} \frac{V}{\pi^2} (\mu \cdot 2m)^{3/2} \]
\[ E = \sum_{N=0}^{N_{\text{max}}} \frac{\hbar^2 k_F^2 (N')}{2m} \]
\[ = \frac{\hbar^2}{2m} \left( \frac{3\pi}{V} \right)^{2/3} \frac{3}{5} N^{5/3} \]

\( b) \quad E = \frac{3}{5} \mu N \)

\( c) \quad E \text{ is extensive} \)

\[ x = A \cos (\omega t + \varphi) \]
\[ w(\varphi) \, d\varphi = \frac{1}{2\pi} \, d\varphi \]
\[ P(x) \, dx = 2 w(\varphi) \, d\varphi \]
\[ \frac{dx}{d\varphi} = A \sin (\omega t + \varphi) \]
\[ d\varphi = \frac{dx}{A \left[ 1 - \cos^2 (\omega t + \varphi) \right]^{1/2}} \]
\[ = \frac{dx}{(A^2-x^2)^{1/2}} \]

\[ P(x) \, dx = \frac{1}{2\pi} \frac{dx}{\sqrt{A^2-x^2}} \]
Energy of a particle

\[ \epsilon(p) = mgz + \frac{p^2}{2m} \]

\[ \mu = -T \ln \frac{Z}{N} \]

\[ Z = \sum_{p} e^{-\frac{\beta \epsilon(p)}{T}} = e^{-\beta mgz} \frac{V}{(2\pi m T)^{3/2}} \]

\[ \mu = -T \left\{ \frac{\ln V}{N} + \frac{3}{2} \ln T + \text{const} - mgz/T^2 \right\} \]

\[ PV = NT \]

\[ \mu = mgz - T \left\{ \frac{\ln T}{T} + \frac{3}{2} \ln T + \text{const} \right\} \]

\[ \frac{\partial \mu}{\partial T} = mgz + \frac{T}{P} \frac{\partial P}{\partial T} = 0 \]

\[ P = \rho_0 e^{-mgz/T} \]
\( m = -1, 0, 1 \)
\[ E_l = E_{-1} = \varepsilon, \quad E_0 = 0 \]
\[ E = \frac{\sum E_k e^{-\beta E_k}}{\sum e^{-\beta E_k}} \]
\[ = \frac{2 \varepsilon e^{-\beta \varepsilon}}{1 + 2 e^{-\beta \varepsilon}} \]

For \( \varepsilon \) mole
\[ E_{N_0} = N_0 \varepsilon \]
\[ S = k_B \ln Z + \beta \overline{E} \]
\[ = k_B \left( \ln (1 + 2 e^{-\beta \varepsilon}) + \frac{2 \varepsilon e^{-\beta \varepsilon}}{T(1 + 2 e^{-\beta \varepsilon})} \right) \]

\[ \Delta G = G_0(T,V) - G_{\text{min}} \]
\[ = \frac{1}{2} \left( \frac{\partial^2 G_0}{\partial T^2} \right)_V (\Delta T)^2 + \frac{1}{2} \left( \frac{\partial^2 G_0}{\partial V^2} \right)_T (\Delta V)^2 \]
\[ \Delta T = T - T_0, \quad \Delta V = V - V_0 \]
\[ T_0, V_0 \quad \text{at the equilibrium} \]
\[
\left( \frac{\partial G_0}{\partial T} \right)_V = \left( \frac{\partial E}{\partial T} \right)_V - T_0 \left( \frac{\partial S}{\partial T} \right)_V = 0
\]

From the 1st thermodynamic law,
\[
\left( \frac{\partial E}{\partial T} \right)_V = T \left( \frac{\partial S}{\partial T} \right)_V
\]
\[
\left( \frac{\partial G_0}{\partial T} \right)_V = \left( 1 - \frac{T_0}{T} \right) \left( \frac{\partial E}{\partial T} \right)_V
\]
\[
\left( \frac{\partial^2 G_0}{\partial T^2} \right)_V = \left( 1 - \frac{T_0}{T} \right) \left( \frac{\partial^2 E}{\partial T^2} \right)_V + \frac{T_0}{T^2} \left( \frac{\partial E}{\partial T} \right)_V
\]

For \( T = T_0 \),
\[
\left( \frac{\partial^2 G_0}{\partial T^2} \right)_V = \frac{1}{T_0} \left( \frac{\partial E}{\partial T} \right)_V = M C_v
\]

\[
\left( \frac{\partial G_0}{\partial V} \right)_T = \left( \frac{\partial E}{\partial V} \right)_T - T_0 \left( \frac{\partial S}{\partial V} \right)_T + p_0
\]

From the 1st thermodynamic law,
\[
\left( \frac{\partial E}{\partial V} \right)_T = T \left( \frac{\partial S}{\partial V} \right)_T - p
\]

At \( T = T_0 \):
\[
\left( \frac{\partial G_0}{\partial V} \right)_T = -p + p_0
\]
\[
\left( \frac{\partial^2 G_0}{\partial V^2} \right)_T = - \left( \frac{\partial p}{\partial V} \right)_T
\]

Compressibility
\[
\alpha = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T
\]
\[ P(T, V) = A \exp \left\{ -\frac{MCV(\Delta T)^2}{T_0^2} \right\} - \frac{1}{2T_0V_0}\sigma (\Delta V)^2 \right\} \]
\[ V_0 = \frac{M}{\sigma_0} \]
\[ a = \left( \frac{MC}{2\pi T_0^2} \right)^{1/2} \left( \frac{\sigma_0}{2\pi \times T_0 M} \right)^{1/2} \]