PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2006

READ INSTRUCTIONS CAREFULLY

1. **ANSWER TWO OF THE THREE PROBLEMS.**

2. You have 3 hours to complete the examination.

3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.

4. Show **ALL** your work.
Problem I (50 pts)

Three point particles, with masses $m$, $3m$, and $2m$, are constrained to remain on a straight line, with neighboring masses connected by springs of unstretched length $a$ and force constant $6mv^2$. The first particle is attached to a piston and forced to oscillate as $F \sin \omega t$.

(a) (10 pts) Choose suitable generalized coordinates and write down the Lagrangian for this system.

(b) (5 pts) Derive the equations of motion.

(c) (20 pts) Find the most general solution of the equations for non-resonant $\omega$.

(d) (15 pts) Find the 2 values of $\omega$ for which the solution contains an oscillatory contribution whose amplitude grows with time. Calculate this contribution in one of the cases.

Problem II (50 pts)

A particle of mass $m$ moves along a straight line with potential energy $V(x) = \frac{1}{2}kx^2 + \epsilon x^4$, where $x$ is the particle's position. Throughout this question you are to regard $\epsilon x^4$ as a small perturbation on the harmonic oscillator potential, and keep only terms up to first order in $\epsilon$. You may want to use some integral formulas provided at the end of the exam.

(a) (10 pts) Write down the Hamiltonian $H(x, p)$ for the system. Is $H(x, p)$ conserved? (Explain) Is $H(x, p)$ equal to the total energy? (Explain)

(b) (10 pts) The particle is initially set in motion with total energy $E$. Draw a graph of the trajectory $C$ in phase space.

(c) (15 pts) Let $J$ be the area enclosed by $C$. Calculate $J$ as a function of $E$(remember: always calculate up to first order in $\epsilon$). Calculate the oscillation period as a function of $J$.

(d) (10 pts) We wish to find a canonical transformation, from variables $x, p$ to a new canonical pair $\theta, J$, where $J$ is the phase-space area of part (c). Without explicitly calculating the transformation equations, explain how they can be found. Hint: writing $p$ as a function of $x, J$ follows easily from preceding sections.

(e) (5 pts) With the new canonical variables of (d), what is the new Hamiltonian? What are the equations of motion for $\theta, J$?
Problem III (50 pts)

A particle of mass $m$ is constrained to move without friction on the surface of a sphere of radius $a$, in the presence of a uniform gravitational field (acceleration $g$).

(a) (10 pts) Using ordinary spherical coordinates $\theta$ and $\phi$, calculate the Lagrangian and Hamiltonian of the system.

(b) (10 pts) Find two independent conserved quantities. Prove that they are constant in time using Hamilton’s equations of motion.

(c) (15 pts) Use the Hamilton-Jacobi method to obtain new canonical coordinates $Q_1$ and $Q_2$ which are constants of the motion. Identify $Q_1$ and $Q_2$ in terms of the conserved quantities of part (b).

(d) (15 pts) In order for $Q_1$ and $Q_2$ to serve as legitimate canonical coordinates, they must have vanishing Poisson bracket and be functionally independent (i.e. at each point, the coordinate curves for $Q_1$ and $Q_2$ must intersect transversally). Where in the phase space does at least one of these conditions fail? Describe the corresponding particle motion on the sphere.

Integral formulas

\[
\int_0^1 (1 - x^2)^{\frac{1}{2}} \, dx = \frac{\pi}{4}
\]

\[
\int_0^1 x^n (1 - x^2)^{-\frac{1}{2}} \, dx = \frac{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)}{n \Gamma\left(\frac{n}{2}\right)}
\]
Solution to Problem 1

\[ x_i = F \sin \omega t \]

(a) Let \( x_1, x_2, x_3 \) be the coordinates of the 3 particles. Introduce

\[ \xi_1 = x_2 - F \sin \omega t - a \]
\[ \xi_2 = x_3 - x_1 - a \]
\[ \xi_3 = x_3 - x_2 - a \]

Then

\[ L(\xi_1, \xi_2, \xi_3, \dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, t) = \text{Kinetic Energy} + \text{Potential Energy} \]

\[ L = \frac{m}{2} \left[ \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 \right] + \frac{\omega^2}{2} (\xi_1^2 + \xi_2^2 + \xi_3^2) - \frac{F}{\omega} (\xi_2 + \xi_3 + \xi_1) - \frac{a}{\omega} (\xi_3 - \xi_1 - \xi_2) - 6 \nu^2 (\xi_2^2 + \xi_3^2) \]

(b) \[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\xi}_k} \right) - \frac{\partial L}{\partial \xi_k} = 0 \quad \text{for} \ k = 1, 2, 3 \]

\[ \frac{d}{dt} \left( 6 (\dot{\xi}_2 + F \omega \cos \omega t) + 4 (\dot{\xi}_2 + \dot{\xi}_3 + F \omega \cos \omega t) \right) + 12 \nu^2 \ddot{\xi}_1 = 0 \]

\[ \frac{d}{dt} \left( 4 (\dot{\xi}_2 + \dot{\xi}_3 + F \omega \cos \omega t) \right) + 12 \nu^2 \ddot{\xi}_2 = 0 \]

\[ \ddot{\xi}_1 + 2 \ddot{\xi}_2 + 6 \nu^2 \xi_2 \Rightarrow \ddot{\xi}_3 + 2 \ddot{\xi}_3 + 6 \nu^2 \xi_3 \Rightarrow F \omega^2 \sin \omega t \]

(c) (i) Particular solution from Ansatz \( (\xi_1 \xi_3) = (A \mathbf{t}) \sin \omega t \)

\[ M \cdot \mathbf{A} = \begin{pmatrix} 6 \nu^2 - 5 \omega^2 & -2 \omega^2 \\ -2 \omega^2 & 3 \nu^2 - \omega^2 \end{pmatrix} \mathbf{A} = F \omega^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[
\det M = (6\nu^2 - 5\omega^2)(3\nu^2 - \omega^2) - 2\omega^4 \\
= 3\nu^4 - 2\nu^2\omega^2 + 18\nu^2 = 3(\omega^2 - \nu^2)(\omega^2 - 6\nu^2)
\]

Resonance condition: \( \omega^2 = \nu^2 \) or \( 6\nu^2 \)

For \( \omega^2 \neq \nu^2, 6\nu^2 \),

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \frac{F_{\omega^2}}{\det M} \begin{pmatrix}
3\nu^2 - \omega^2 & 2\omega^2 \\
\omega^2 & 6\nu^2 - 5\omega^2
\end{pmatrix} \begin{pmatrix}
5
\
1
\end{pmatrix}
\]

\[
= \frac{F_{\omega^2}}{3(\omega^2 - \nu^2)(\omega^2 - 6\nu^2)} \begin{pmatrix}
15\nu^2 - 3\omega^2 \\
6\nu^2
\end{pmatrix}
\]

\[
= \frac{F_{\omega^2}}{(\omega^2 - \nu^2)(\omega^2 - 6\nu^2)} \begin{pmatrix}
5\nu^2 - \omega^2 \\
2\nu^2
\end{pmatrix}
\]

(d) Resonant solution for \( \omega = \nu \)

Ansatz for particular solution (the homogeneous eqn. solutions are the same as in (c)):

\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = \begin{pmatrix}
G \\
H
\end{pmatrix} \sin \omega t + \begin{pmatrix}
M \\
N
\end{pmatrix} \cos \omega t
\]

To avoid terms in the equations \( \propto \cos \omega t \), we must have

\[
\begin{pmatrix}
-\frac{\omega^2}{2} & \frac{\omega^2}{2} \\
1 & -2
\end{pmatrix} \begin{pmatrix}
M \\
N
\end{pmatrix} = 0 \implies M = 2N
\]

The remaining terms are all proportional to \( \sin \omega t \), giving

\[
\nu^2 \begin{pmatrix}
\frac{\omega^2}{1} & -1 \\
-\frac{\omega^2}{2} & \frac{\omega^2}{2}
\end{pmatrix} G - 2\nu N = 0
\]

\[
\begin{pmatrix}
5 \\
1
\end{pmatrix} = -\nu^2
\]
Solutions to Dynamics Prelim, Problems 2 and 3

July 13, 2006

Problem 2

(a) 
\[ L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 - \epsilon x^4 \]
\[ H(x, p) = \frac{p^2}{2m} + \frac{k}{2} x^2 + \epsilon x^4 \]
\[ \frac{\partial H}{\partial t} = 0 \implies \dot{H} = 0. \]

Kinetic energy \( K \) homogeneous quadratic in \( \dot{x} \) implies \( H = K + V \).

(b) The curve \( C \) is a perturbed ellipse with equation
\[ \frac{p^2}{2m} + \frac{k}{2} x^2 + \epsilon x^4 = E. \]

(c) The area inside \( C \), using reflection symmetry, is
\[ J = 4\sqrt{2m} \int_0^{x_{\text{max}}(\epsilon)} p(E, x) \, dx, \]
where
\[ p(E, x) = \sqrt{E - \frac{k}{2} x^2 - \epsilon x^4}, \]
\[ x_{\text{max}}(\epsilon) = \text{root of } E - \frac{k}{2} x^2 - \epsilon x^4 = \sqrt{\frac{2E}{k}} \left(1 - \frac{2E \epsilon}{k}\right). \]

Expanding \( J \) to first order in \( \epsilon \),
\[ J = 4\sqrt{2m} \int_0^{x_{\text{max}}(0)} \sqrt{E - \frac{k}{2} x^2} \, dx - 2\sqrt{2m} \epsilon \int_0^{x_{\text{max}}(0)} x^4 \, dx \]
\[ \frac{x^4 \, dx}{\sqrt{E - \frac{k}{2} x^2}} \]

1
with \( x_{\text{max}}(0) = \sqrt{\frac{2E}{k}} \). Using the integrals provided at the end of the exam,

\[
J = 2\pi E \sqrt{\frac{m}{k}} - 3\pi \sqrt{-}\frac{m}{k} \frac{E^2}{k^2} \epsilon,
\]
giving

\[
E = \frac{1}{2\pi} \sqrt{\frac{k}{m}} J + \frac{3\epsilon}{4\pi} \sqrt{\frac{k}{m}} \frac{J^2}{k^2}.
\]
The oscillation period is then

\[
\tau = \frac{1}{\frac{dE}{dJ}} = 2\pi \sqrt{\frac{m}{k}} (1 + \frac{3\epsilon J}{k^2})^{-1} = 2\pi \sqrt{\frac{m}{k}} (1 - \frac{3\epsilon J}{k^2})
\]
In terms of the energy, this is

\[
\tau = \tau_0 (1 - \frac{3\epsilon \tau_0}{k^2} E), \quad \tau_0 = 2\pi \sqrt{\frac{m}{k}}.
\]

(d)

\[
p(J, x) = \sqrt{E(J) - \frac{k}{2} x^2 - \epsilon x^4},
\]
with \( E(J) \) from part (c). This suggests introducing a generating function of type \( F_2 \) (old position variable, new momentum variable), i.e.

\[
F(x, J) = \int p(J, x)dx,
\]
which is designed to give

\[
p(J, x) = \frac{\partial F}{\partial x}.
\]
The new position variable, canonically conjugate to \( J \), is then

\[
\theta = \frac{\partial F}{\partial J}.
\]

(e) The new Hamiltonian is the old one written in terms of the new variables, plus \( \frac{\partial F}{\partial J} \) (the latter vanishes here). Hence the new Hamiltonian is just \( E(J) \). Hamilton's eqns. for the new Hamiltonian are

\[
\dot{J} = 0, \quad \dot{\theta} = \frac{dE}{dJ} = 2\pi \sqrt{\frac{m}{k}} \left(1 + \frac{3\epsilon J}{k^2}\right)
\]
Problem 3

(a) The $\theta = 0$ axis is chosen opposite to the direction of the gravitational field, i.e. "upward".

\[ L = \frac{ma^2}{2}(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mga \cos \theta \]

Introducing conjugate momenta

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2 \dot{\theta}, \]

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi}, \]

we get for the Hamiltonian,

\[ H = \frac{1}{2ma^2} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + mga \cos \theta. \]

(b) Hamilton's equations yield

\[ \dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \]

\[ \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0. \]

Thus $p_\phi$, the vertical component of angular momentum is conserved. So is the total energy, $H$ itself, since, from Hamilton's equations,

\[ \dot{H} = \frac{\partial H}{\partial \theta} \dot{\theta} + \frac{\partial H}{\partial \phi} \dot{\phi} + \frac{\partial H}{\partial p_\theta} \dot{p}_\theta + \frac{\partial H}{\partial p_\phi} \dot{p}_\phi = 0. \]

(c) Introduce the H-J generating fcn. $S(\theta, \phi, Q_1, Q_2, t)$ satisfying

\[ \frac{1}{2ma^2} \left[ \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] + mga \cos \theta + \frac{\partial S}{\partial t} = 0 \]

In standard fashion we separate out the $t$ dependence:

\[ S(\theta, \phi, Q_1, Q_2, t) = W(\theta, \phi, Q_1, Q_2) - Q_1 t \]
\[
\frac{1}{2ma^2} \left[ \left( \frac{\partial W}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial W}{\partial \phi} \right)^2 \right] + mga \cos \theta = Q_1
\]

Now we separate the \( \theta \) and \( \phi \) dependence, setting

\[
W = W_\theta(\theta, Q_1, Q_2) + W_\phi(\phi, Q_1, Q_2).
\]

so that

\[
\frac{1}{2ma^2} \left[ \left( \frac{dW_\theta}{d\theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{dW_\phi}{d\phi} \right)^2 \right] + mga \cos \theta = Q_1.
\]

The differential equation separates into the pair

\[
\frac{dW_\phi}{d\phi} = Q_2, \quad \left( \frac{dW_\theta}{d\theta} \right)^2 = 2ma^2 Q_1 - \frac{Q_2^2}{\sin^2 \theta} - 2m^2 ga^3 \cos \theta
\]

with the solutions

\[
W_\phi = Q_2 \phi, \quad W_\theta = \int \sqrt{2ma^2 Q_1 - Q_2^2/\sin^2 \theta - 2m^2 ga^3 \cos \theta} \, d\theta.
\]

The constants of the motion \( Q_1 \) and \( Q_2 \) are, respectively, the total energy and the \( z \)-component of the angular momentum.

(d) The matrix of partial derivatives is

\[
\begin{pmatrix}
\frac{\partial Q_1}{\partial \theta} & \frac{\partial Q_1}{\partial \phi} & \frac{\partial Q_1}{\partial p_\theta} & \frac{\partial Q_1}{\partial p_\phi} \\
\frac{\partial Q_2}{\partial \theta} & \frac{\partial Q_2}{\partial \phi} & \frac{\partial Q_2}{\partial p_\theta} & \frac{\partial Q_2}{\partial p_\phi}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial Q_1}{\partial \theta} & 0 & \frac{p_\phi}{ma^2} & \frac{-p_\theta}{ma^2 \sin^2 \theta} \\
0 & \frac{1}{ma^2} & 0 & 0
\end{pmatrix}.
\]

Functional dependence requires that the 4D phase-space gradient vectors of \( Q_1(\theta, \phi, p_\theta, p_\phi) \) and \( Q_2(\theta, \phi, p_\theta, p_\phi) \) be proportional. This happens only when

\[
p_\theta = 0, \quad \dot{p}_\theta = 0,
\]

i.e. for an orbit which always stays at the same height. The new coordinates are functionally independent everywhere else.

The Poisson bracket of \( Q_1 \) with \( Q_2 \) is

\[
\frac{\partial Q_1}{\partial \theta} \frac{\partial Q_2}{\partial p_\theta} - \frac{\partial Q_2}{\partial \theta} \frac{\partial Q_1}{\partial p_\theta} + \frac{\partial Q_1}{\partial \phi} \frac{\partial Q_2}{\partial p_\phi} - \frac{\partial Q_2}{\partial \phi} \frac{\partial Q_1}{\partial p_\phi}
\]

. Consulting the matrix of partial derivatives, we see that this vanishes everywhere.
Electromagnetism Preliminary Exam

1. Calculate the light pressure force $F$ acting on a mirror of area $A = 1 \text{m}^2$ facing the Sun on Earth. The luminosity of the Sun is $L = 3.8 \times 10^{33} \text{erg/s}$, the distance is $r = 1 \text{AU}=1.5 \times 10^{13} \text{cm}$, neglect atmospheric absorption.

2. A proton of energy $E$ collides with a neutron at rest. Calculate the threshold energy $E$ for neutral pion production, $p + n \rightarrow p + n + \pi^0$. ($m_p \approx m_n = 940 \text{MeV}$, $m_{\pi} = 135 \text{MeV}$)

3. Spherical coordinates $(r, \theta, \phi)$. There is a uniformly charged semi-sphere $r < a$, $\theta < \pi/2$. The total charge of the semi-sphere is $Q$. Calculate the electrostatic potential at a distant point $(r, \theta, \phi)$ to first two non-vanishing orders in $1/r$.

4. A long straight wire with current $I$ is in vacuum, parallel to the plane surface of a large block of iron. The wire is at a distance $a$ from the surface. Calculate the force per unit length of the wire $f$. Is it attraction or repulsion?

5. Calculate, as a function of the impact parameter $b$, the energy lost by a non-relativistic particle of charge $e$ and mass $m$, passing near a heavy magnetic monopole of magnetic charge $g$. Assume small-angle scattering.

6. A perfectly conducting rod of length $L$ rotates in a plane around one of its ends, with angular velocity $\Omega \ll c/L$. Estimate the power radiated by the rod.
Answers and Solutions

1. \( F = \frac{A}{\pi r^2} \frac{k}{c} = 0.9 \text{dyne} \)

From \( E = cp \).

2. \( E = M + 2m_\pi + m_\pi^2/(2M) = 1220 \text{MeV} \), where \( M \equiv m_p \approx m_n \).

In CM, the energies of p and n are: \( E_{CM} = M + m_\pi/2 \). From \( p_p^2 p_n^2 = v^2 \), \( E_{CM}^2 + p_{CM}^2 = EM \), where \( p_{CM}^2 = E_{CM}^2 - M^2 \).

3. \( \phi = Q/r + (3/8)Qa \cos \theta/r^2 \)

To first two orders we have a charge and a dipole. The dipole moment is \( d = Q < z > \), and
\[ < z > = \int r^2 dr \sin \theta d\theta \cos \theta / \int r^2 dr \sin \theta d\theta = (3/8)a. \]

4. attraction, \( f = I^2/(c^2 a) \)

\( B_t/\mu \) continuous, \( \mu \gg 1 \) gives \( B_t \approx 0 \) right above the surface. This means that the outside field of the surface current is given by an image current \( I \), at a distance \( a \) below the surface.

5. \( E = \frac{\pi}{4} \frac{a^3 e^2 v}{m^2 c^6 b^4} \).

The particle moves approximatively in a straight line with constant velocity \( x = vt, y = b \).

The force acting on the particle is
\[ F = \frac{e}{c} |\mathbf{v} \times \mathbf{B}| = \frac{e^2 b v}{c} (v^2 t^2 + b^2)^{-3/2}. \]

The radiated power is
\[ P = \frac{2e^2 F^2}{3m^2 c^5} = \frac{2e^4 g^2 b^2 v^2}{3m^2 c^5} (v^2 t^2 + b^2)^{-3}. \]

The energy loss is \( E = \int_{-\infty}^{+\infty} dt P(t) \).

6. \( P \sim m^2 \Omega^8 L^8/(c^3 e^2) \)

The centrifugal force acting in the comoving frame on electrons inside the rod is \( F \sim m \Omega^2 L \). It is balanced by electric field \( E \sim (m/e) \Omega^2 L \). The charge distribution creating this electric field will have a dipole \( d \sim (m/e) \Omega^2 L^4 \).
Answer all three questions. Show all work.
Percentage points assigned to each part of a problem is indicated in parentheses.
Time of exam: 3 hours.

1. The Schrödinger equation \( (H_0 + \lambda V)|n\rangle = E_n|n\rangle \) may be solved to any given order in the perturbation \( V \) by introducing the expansions \( |n\rangle = \sum_{m=0}^\infty \lambda^m E_n^{(m)} \) and

\[
E_n = \sum_{m=0}^\infty \lambda^m E_n^{(m)}, \text{ with } (H_0 - E_n^{(0)}|n^{(0)}\rangle = 0. \text{ The unperturbed states are nondegenerate, with } E_0^{(0)} \text{ the lowest level. The normalization conditions are }
\langle n^{(0)}|n^{(0)}\rangle = \langle n^{(0)}|\rangle = 1. \text{ The parameter } \lambda \text{ is set equal to unity in the following.}
\]

a) (16 pts) Derive the following relations by requiring validity of the perturbation expansions in each order.

i) \( \langle n^{(0)}|n^{(m)}\rangle = 0, \ m > 0. \)

ii) \( (H_0 - E_n^{(0)}|n^{(0)}\rangle = -Q_0|n^{(0)}\rangle, \ Q_0 = 1 - |n^{(0)}\rangle \langle n^{(0)}|n^{(0)}\rangle. \)

iii) \( E_n^{(2)} = \langle n^{(0)}|V|n^{(0)}\rangle \) and \( E_n^{(0)} = \langle n^{(0)}|E_n^{(0)} - H_0|n^{(0)}\rangle. \)

iv) \( E_n^{(2)} \), the second-order level shift for the ground state, is negative.

b) (14 pts) With \( |n^{(1)}\rangle \) an approximation to \( |n^{(0)}\rangle \), orthogonal to \( |n^{(0)}\rangle \), verify the identity

\[
E_n^{(2)} = \langle n^{(1)}|V|n^{(0)}\rangle + \langle n^{(0)}|V|n^{(1)}\rangle + \langle n^{(1)}|H_0 - E_n^{(0)}|n^{(1)}\rangle.
\]  \( (3) \)

Now write \( \langle n^{(0)}| = \langle n^{(0)}| + \langle n^{(1)}| \delta n^{(1)}|, \text{ where } \langle \delta n^{(1)}|n^{(0)}\rangle = 0. \text{ Let } E_n^{(2)} \text{ be defined by replacing } \langle n^{(0)}| \text{ with the trial bra } \langle n^{(1)}| \text{ in the above identity. Show that the error } \)

\( E_n^{(2)} - E_n^{(2)} \text{ is of second order assuming } \langle \delta n^{(1)}| \text{ is of first order. Under what circumstances will this variational principle actually be a minimum principle?} \)

c) (5 pts) Explain, without performing any calculations, how a knowledge of the wave function to order \( m \) in perturbation theory can be used to determine the energy correctly to order \( 2m + 1. \)
2. Consider a model with an s-wave scattering matrix defined as \( S(k) = f(k)/f^*(k) \), with 
\[ f(k) = (k + i\kappa)/(k - i\alpha) \]
and with \( k, \alpha, \text{and} \ \kappa \) all real.

a) (10 pts) Show that the s-wave phase shift \( \delta \) satisfies the relation 
\[ k \cot \delta = -\frac{1}{A} + \frac{1}{2} r_0 k^2. \]
Determine the scattering length \( A \) and effective range \( r_0 \) in terms of \( \kappa \) and \( \alpha \).

b) (5 pts) Eliminate \( \alpha \) from the above relations to find a connection among \( A, \kappa, \) and \( r_0 \).
Show that this implies an alternative “effective range expansion” 
\[ k \cot \delta = -\kappa + \frac{1}{2} r_0 (k^2 + \kappa^2). \]

c) (5 pts) With \( \alpha = 0 \) and \( \kappa > 0 \), explain the physical significance of \( \kappa \) from an examination of the scattering matrix \( S(k) \).

d) (15 pts) An absolute definition of \( \delta(k) \) may be imposed by requiring that it vanish for \( k \to \infty \). Show that this convention is consistent with the dynamics of this model and the physical interpretation of the phase shift. This absolute definition implies certain properties of \( \delta(0) \). Show that with \( \alpha > 0 \)

i) \( \delta(0) = \pi \) for \( \kappa > 0 \).

ii) \( \delta(0) = \pi/2 \) for \( \kappa = 0 \). Explain why in this case the behavior of \( S(k) \) at threshold is referred to as a “zero-energy resonance.”

iii) With \( \kappa < 0 \) and \( \alpha > |\kappa| \), \( \delta(0) = 0 \); \( \delta(k) \) increases to a maximum less than \( \pi \) and then decreases to zero at infinite energy.
3. An electron of mass $m$, charge $e$ and momentum $\hbar\mathbf{q}$, interacts with a field consisting of $N_{k\lambda}$ photons in a mode $(\mathbf{k}\lambda)$ with wave vector $\mathbf{k}$, angular frequency $\omega$ and (real) polarization vector $\lambda$. The electron emerges with momentum $\hbar\mathbf{q}_f$, with the field now consisting of $N_{k\lambda} - 1$ photons in mode $(\mathbf{k}\lambda)$ and one in mode $(\mathbf{k}'\lambda')$. The interaction is $H' = e^2 A^2 / (2mc^2)$, with the vector potential represented by the expansion

$$A(\mathbf{r}) = (2\pi\hbar c^2)^{1/2} \sum_{\mathbf{k}\lambda} \left[ a_{k\lambda} \lambda \frac{\exp(\mathbf{i}\mathbf{k}\cdot\mathbf{r})}{(\omega V)^{1/2}} + a_{k\lambda}^\dagger \lambda' \frac{\exp(-\mathbf{i}\mathbf{k}\cdot\mathbf{r})}{(\omega V)^{1/2}} \right],$$

where $V$ is the volume of the box used to normalize the states and

$$a_{k\lambda} |N_{k\lambda}, \ldots, N_{k\lambda}, \ldots\rangle = \sqrt{N_{k\lambda}} |N_{k\lambda}, \ldots, N_{k\lambda} - 1\ldots\rangle.$$

a) (10 pts) Evaluate the matrix element of $H'$ taken between the initial and final states. Integrate over the volume of the box using suitably normalized electron wave functions. (Ignore spin and treat the electron motion nonrelativistically.)

b) (15 pts) The transition probability per unit time in first order of perturbation theory is determined by the “golden rule”

$$w_{i\rightarrow f} = \frac{2\pi}{\hbar} \delta(E_f - E_i) \langle \psi_f | H' | \psi_i \rangle^2.$$

With the sum over $\mathbf{K}'$ replaced by an integration in the continuum limit, determine the integral of $w_{i\rightarrow f}$ over all $\mathbf{K}'$ in a small solid angle $d\Omega'$. 

c) (5 pts) The differential cross section is obtained by dividing the above result by the photon flux and summing over all final electron momenta $\hbar\mathbf{q}_f$. Show that this calculation leads to the expression

$$\frac{d\sigma_{i\rightarrow f}}{d\Omega'} = \left( \frac{e^2}{mc^2} \right)^2 \frac{\omega'}{\omega} (\lambda' \cdot \lambda' \lambda)^2.$$
(1.45) \( |n\rangle = \langle n^{(w)}\rangle + \lambda \langle n^{(u)}\rangle + \lambda^2 \langle n^{(2)}\rangle + \ldots \)  Project onto \( \langle n^{(w)}\rangle \) and require coefficient of \( \lambda^m \) vanish.

a) i) Insert expansion of \( |n\rangle \) into
\[
(H_0 + \lambda V) |n\rangle = (E_n^{(w)} + \lambda E_n^{(u)} + \lambda^2 E_n^{(2)} + \ldots) |n\rangle
\]

First order: \( H_0 |n^{(w)}\rangle + V |n^{(u)}\rangle = E_n^{(w)} |n^{(w)}\rangle + E_n^{(u)} |n^{(u)}\rangle \)
Project onto \( \langle n^{(w)}\rangle \) using \( \langle n^{(w)}|H_0|n^{(w)}\rangle = E_n^{(w)} \langle n^{(w)}|n^{(w)}\rangle \)
which vanishes by (i). This gives \( E_n^{(u)} = \langle n^{(w)}|V|n^{(u)}\rangle \)

Then
\[
(H_0 - E_n^{(w)}) |n^{(w)}\rangle + V |n^{(u)}\rangle - \langle n^{(w)}|V|n^{(u)}\rangle |n^{(w)}\rangle = 0
\]
as required in (ii).

a) ii) Second order:
\[
H_0 |n^{(w)}\rangle + V |n^{(u)}\rangle = E_n^{(w)} |n^{(w)}\rangle + E_n^{(u)} |n^{(u)}\rangle + E_n^{(2)} |n^{(2)}\rangle
\]
Project onto \( \langle n^{(w)}\rangle \) giving \( \langle n^{(w)}|V|n^{(u)}\rangle = E_n^{(2)} \)
using \( \langle n^{(w)}|n^{(w)}\rangle = 0 \). To obtain the second part of (ii) write
\[
\langle n^{(w)}|V|n^{(u)}\rangle = \langle n^{(w)}|V|n^{(w)}\rangle
\]
\[
= -\langle n^{(u)}|H_0 - E_n^{(w)}|n^{(w)}\rangle \text{ using (ii)}
\]

a) iv) The expectation value of \( H_0 - E_n^{(w)} \) is positive in the space of functions orthogonal to the unperturbed ground state. But for the ground state, \( n=0 \), \( \langle 0^{(w)}|0^{(w)}\rangle = 0 \) so
\[
E_n^{(2)} = \langle 0^{(u)}|E_n^{(w)} - H_0|0^{(u)}\rangle \quad \text{[from (iii)]}
\]
is negative.
The natural text is as follows:

a) \( f(k) = |f(k)| \, e^{\text{id}} \), then \( S(k) = e^{2 \text{id}} \)

\[ f(k) = \frac{(k+c)(k+cd)}{k^2 + ad} = \frac{k^2 - kd + ik(k+kd)}{k^2 + ad} \]

\[ \tan \delta = \frac{k(k+kd)}{k^2 - kd}, \quad k \cos \delta = \frac{k^2 - kd}{k+kd} = -\frac{1}{4} + \frac{1}{4} \frac{\rho_0 k^2}{A} \]

\[ A = \frac{k+kd}{k^2 - kd}, \quad \rho_0 = \frac{2}{k+kd} \]

b) \( \frac{1}{A} + \frac{1}{2} \rho_0 k^2 = \frac{k^2}{k+kd} + \frac{k}{k+kd} = k \left( \frac{1}{k+kd} \right) = 1 \)

so \( \frac{1}{A} = k - \frac{1}{2} \rho_0 k^2 \) so

\[ k \cos \delta = -k + \frac{1}{2} \rho_0 (k^2 + k^2) \]

c) With \( \delta = 0 \), \( S(k) = \frac{k+ck}{k-ck} \) has a simple pole

at \( k = i \alpha \), or \( \epsilon = -\frac{k^2}{2m}, k^2 \), corresponding to

a bound state.

d) \( S(k) \to 1, k \to \infty \) so \( \delta = 0 \) (mod \( \pi \)), consistent

with \( \psi \sim \sin(kr+\delta) \), \( r \to \infty \)

\[ i \frac{d}{dk} \tan \delta = \frac{1}{\cos^2 \delta} \frac{d \delta}{dk} = -\frac{1}{(k^2 - kd)^2} \left( k^2 + k^2 \right) \]

so \( \delta \) rises monotonically from zero at infinite \( k \)

and reaches \( \pi \) at \( k = 0 \)

i) \( k = 0 \), \( \tan \delta = \frac{\alpha}{k} \), \( \delta \) rises monotonically and

reaches \( \pi/2 \) at \( k = 0 \), \( A = \infty \) for \( k = 0 \), \( \delta = 4 \pi A^2 \)

\( \alpha - 1 \left( k^2 - k \alpha \right) \frac{1}{(k^2 + k^2)^2} \)

ii) \( k < 0 \) \( \tan \delta (0) = 0 \) \( i \frac{d}{dk} \tan \delta = \frac{-1}{(k^2 + k \alpha)^2} \)

\( \delta(k) \) has positive slope at \( k = 0 \), rises until

\( k^2 = 1k \alpha \), where it reaches a maximum,

then falls monotonically to zero for \( k \to \infty \).

[Part d illustrates Levinson's Theorem]
3. a) Write $\mathbf{A}^2 = \overrightarrow{A} \cdot \overrightarrow{A}$ and insert a complete set of photon states between the two vectors. Only two types of terms appear, giving the same result. 1) First destroy a $\overrightarrow{F_c}$ photon in the initial state. The intermediate state has one photon missing in that mode. Then create a $\overrightarrow{F_c}$ photon. The spatial integration gives
\[ \int d^3r \, e^{-i(q - \mathbf{r})} = e^{i(\mathbf{q} - \mathbf{r})} \cdot \overrightarrow{r} \cdot \frac{1}{\sqrt{V}} = \delta_{q + \overrightarrow{F_c}, \overrightarrow{r} + \overrightarrow{F_c}}. \]

The matrix element is
\[ 2 \left( 2\pi \hbar c^2 \right)^2 \sqrt{\frac{N_{F_c}}{\hbar}} \sqrt{\frac{1}{\omega}} \overrightarrow{r} \cdot \sum_{q + \overrightarrow{F_c}} \delta_{q + \overrightarrow{F_c}, \overrightarrow{r} + \overrightarrow{F_c}} \left( \frac{e^2}{2mc^2} \right). \]

b) \[ \frac{1}{V} \sum_{k'} \rightarrow \int d^3k' = \int d\omega' d\mathbf{k}' \frac{1}{(2\pi)^3}. \]

\[ \int d\omega' \delta(\omega' + \mathbf{k}' - \omega - \mathbf{k}) \rightarrow \frac{1}{V} \sum_{k'} \delta_{k + \overrightarrow{F_c}, k + \overrightarrow{F_c}} \rightarrow 1. \]

\[ \frac{d\omega}{dq} = \frac{2\pi}{\omega} \frac{1}{\omega} \frac{N_{F_c}}{V} \left( \frac{2\pi \hbar c^2}{\omega} \right)^2 \frac{1}{\omega} = \frac{4\pi}{2mc^2} \]

\[ \frac{d\sigma_{i+\overrightarrow{F}}}{d\omega'} = \left( \frac{c N_{F_c}}{V} \right) \frac{d\sigma_{i+\overrightarrow{F}}}{d\omega'} \quad \text{;} \quad \frac{c N_{F_c}}{V} = \text{flux}. \]

c) \[ \frac{d\sigma_{i+\overrightarrow{F}}}{d\omega'} = \left( \frac{e^2}{2mc^2} \right)^2 \frac{1}{\omega'} \left( \overrightarrow{k} \cdot \overrightarrow{q} \right)^2. \]

Find $\omega'$ from $\overrightarrow{F_c}$, direction of $\overrightarrow{F_c}$ using
\[ \frac{p_{F_c}^2}{2m} + \omega' = \frac{\hbar^2 \overrightarrow{q}^2}{2m} + \omega, \quad \overrightarrow{F_c} + \overrightarrow{q} = \overrightarrow{F_c} + \overrightarrow{q}. \]