1. An impenetrable sphere of radius $a$ carries a single charge $+e$. How many oppositely charged point particles will be attracted to the sphere at zero temperature? What is the total (net) charge of the final cluster of charges? How does this answer change if the point particles carry charge $-2e$? Assume that the sphere has the same dielectric constant as the surrounding medium.
**Solution:** We bring point charges to the sphere’s surface one at a time and measure the resulting change in potential energy, setting the reference point at infinite separations. Let us assume for now that the point particles carry charge \(-q_e\), where \(q\) is an integer. We set the reference for the electrostatic energy \((U = 0)\) to be the state with all charges at infinity.

**one particle:** This cluster is stable for all \(q\) because its energy,

\[
U = -\frac{q_e^2}{a} < 0
\]

is always lower than that for the reference state.

**two particles:** The lowest-energy configuration has the two point particles on opposite poles of the sphere:

\[
U = -\frac{2q_e^2}{a} + \frac{q_e^2}{a} = \frac{q_e^2}{2a} (q - 4) .
\]

This cluster is at least marginally stable for \(q \leq 4\).

**three particles:** The lowest energy state has the three point particles at the vertices of an equilateral triangle at the sphere’s equator:

\[
U = -\frac{3q_e^2}{a} + 3 \frac{q_e^2}{\sqrt{3}a} = \frac{\sqrt{3} q_e^2}{a} (q - \sqrt{3}) .
\]

This cluster is stable for \(q = 1\), but unstable for \(q \geq 2\). Thus the stable cluster for point particles with \(q = -2e\) has two bound particles and net charge \(Z = -3e\).

**four particles:** The lowest energy state has the four point particles on the vertices of a tetrahedron inscribed within the sphere. Thus, the separation between point particles on the surface is roughly \(1.6a\). So the energy of this cluster is

\[
U = -\frac{4q_e^2}{a} + 6 \frac{q_e^2}{1.6a} = \frac{q_e^2}{1.6a} (6q - 6.4) .
\]

This is still just barely stable for \(q = 1\)! The cluster has a total charge of \(Z = -3e\).

**five particles:** No five-particle cluster can be stable because 10 repulsive contributions would have to be balanced by only 5 attractive contributions. This would be possible only if the mean separation between point particles were greater than the diameter of the sphere. Hence the stable \(q = 1\) cluster also has \(Z = -3e\).
2. Suppose that a sphere of radius $R$ is composed of two conducting halves separated by an infinitesimally thin insulating ring at the equator. The top hemisphere is connected to a battery and thereby held at potential $V = V_0$. The bottom hemisphere is grounded, and thus is at potential $V = 0$. Under these conditions, the sphere will have a non-zero surface charge density $\sigma$.

Find the electric dipole moment

$$\vec{p} = \int \sigma \, \vec{r} \, dA$$

for this charge distribution.

**Solution:** The potential due to a dipole $\vec{p}$ is

$$V(r, \theta, \phi) = \frac{\vec{p} \cdot \hat{z}}{r^2} = \frac{p \cos \theta}{r^2}.$$  

To relate this to the potential on the surface of the sphere, we expand the potential outside the sphere in a general multipole series:

$$V(r, \theta, \phi) = \sum_{\ell,m} a_{\ell m} \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \phi).$$

Noting that $Y_{10}(\vec{r}) = \sqrt{\frac{3}{4\pi}} \cos \theta$, the dipole term yields

$$p = \sqrt{\frac{3}{4\pi}} a_{10}.$$  

At the surface of the sphere, the potential must satisfy

$$V(\vec{r}) = \begin{cases} V_0, & 0 \leq \theta < \pi/2 \\ 0, & \pi/2 < \theta \leq \pi \end{cases}$$

The dipole term then yields

$$a_{10} = R^2 \sqrt{\frac{3}{4\pi}} \int_0^{2\pi} d\phi \int_0^{\pi/2} V_0 \cos \theta d\theta$$

$$= \pi \sqrt{\frac{3}{4\pi}} R^2 V_0.$$  

From this, we obtain

$$p = \frac{3}{4} R^2 V_0.$$
3. A plane electromagnetic wave of angular frequency $\omega$ propagates along the $\hat{z}$ axis through an optically active medium. Such materials have the ability to rotate the plane of polarization of linearly polarized light about the direction of propagation. The polarization vector of the medium is given by

$$\vec{P} = \gamma \nabla \times \vec{E},$$

where $\vec{E}$ is the light’s electric field and $\gamma$ is a real constant.

(a) The wave will experience two different refraction coefficients. Find the two indexes of refraction.

(b) Find the electric field configurations that correspond to the two refractive indexes obtained in (a).

**Solution:** We use the constitutive equations and Maxwell’s equations to obtain a wave equation for the electric field.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\quad = \epsilon_0 \vec{E} + \gamma \nabla \times \vec{E}.$$  

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$
$$\quad = \mu_0 \vec{H}.$$  

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$
$$\quad = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \gamma \nabla \times \frac{\partial \vec{E}}{\partial t}.$$  

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$  

From these, and using the vector identity $\nabla \times \nabla \times \vec{E} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$, we obtain

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \gamma \mu_0 \nabla \times \frac{\partial^2 \vec{E}}{\partial t^2}.$$  

Remembering that the wave is propagating along $\hat{z}$, this wave equation has a plane wave solution

$$\vec{E} = \vec{E}_0 e^{ikz - i\omega t}$$  

whose components must satisfy

$$(k^2 - \omega^2/c^2) E_x + i\gamma \mu_0 \omega^2 k E_y = 0$$
$$-i\gamma \mu_0 \omega^2 k E_x + (k^2 - \omega^2/c^2) E_y = 0$$
**Solution:** Solutions to this set of equations exist for

\[ k_\pm = \frac{1}{2} \left[ (\gamma^2 \mu_0^2 \omega^4 + 4\omega^2/c^2)^{1/2} \pm \gamma \mu_0 \omega \right]. \]

We then use the definition of the refractive index, \( n_\pm = (c/\omega) k_\pm \) to obtain

\[ n_\pm = 1 \pm \frac{\gamma \mu_0 \omega}{2}. \]

The fields that correspond to these solutions are circularly polarized plane waves

\[ E_+ = E_x + i E_y \]
\[ E_- = E_x - i E_y. \]
4. If the proton is treated as a point charge, the ground state wavefunction of the electron in a hydrogen atom is

\[ \psi(\vec{r}) = \frac{1}{\sqrt{\pi a^3}} \exp \left( -\frac{r}{a} \right) \]

where \( a = \frac{\hbar^2}{(me^2)} \) is the Bohr radius, \( e \) is the magnitude of the electron charge, and \( m \) is the mass of the electron. The ground state energy is \( E = -\frac{me^4}{(2\hbar^2)} \).

It would be more accurate to treat the proton as a uniform ball of charge centered at the origin. Assuming the proton’s radius \( R \) to be much smaller than \( a \), calculate the corresponding correction to the ground state energy of hydrogen.

**Solution:** The electrostatic potential is unchanged outside the proton’s volume, but takes a new form inside:

\[ \phi(r) = \int_r^\infty E_r \, dr = \begin{cases} \frac{e}{r}, & r \geq R \\ \frac{3}{2} \frac{e}{R} - \frac{1}{2} \frac{e^2 r^2}{R^3}, & r < R \end{cases} \]

The change to the electron’s potential energy then is

\[ \Delta V = -e \Delta \phi = \begin{cases} 0, & r \geq R \\ -\frac{3}{2} \frac{e^2}{R} + \frac{1}{2} \frac{e^2 r^2}{R^3} + \frac{e^2}{r}, & r < R \end{cases} \]

Assuming \( R \ll a \), we compute the shift in the ground state energy with first-order perturbation theory:

\[ \Delta E = \langle \psi^{(0)} | \Delta V | \psi^{(0)} \rangle = \frac{1}{\pi a^2} \frac{4\pi}{\pi} \int_0^R r^2 \, dr \, e^{-2r/a} \left[ -\frac{3}{2} \frac{e^2}{R} + \frac{1}{2} \frac{e^2 r^2}{R^3} + \frac{e^2}{r} \right] \]

Because \( R \) is small, we may approximate \( \exp(-r/a) \approx 1 \). In that approximation,

\[ \Delta E = \frac{e^2}{2} \left[ -\frac{1}{2R} \left( \frac{2R}{a} \right)^3 + \frac{a^2}{40R^3} \left( \frac{2R}{s} \right)^5 + \frac{1}{a} \left( \frac{2R}{a} \right)^2 \right] \]

\[ = \frac{2}{5} \frac{e^2 R^2}{a^3} \]
5. A particle of charge $q$ and mass $m$ is affixed to the end of a spring of spring constant $k$. At time $t = 0$, the particle is displaced by a distance $x_0$ from its equilibrium position, after which it is released and allowed to oscillate. Assume that the subsequent motion is non-relativistic ($v \ll c$) and that $|q|$ is small enough that the particle’s oscillation is only weakly damped by emission of electromagnetic radiation (recall Larmor emission). Treating the motion as that of a damped harmonic oscillator,

$$m\ddot{x} + \gamma \dot{x} + kx = 0,$$

determine the value of the damping coefficient $\gamma$ in terms of $q$, $m$, and $k$.

**Solution:** The oscillation is damped by loss of energy to electromagnetic radiation. The power radiated by the accelerating charge is given by Larmor’s relation:

$$P = -\frac{2}{3} q^2 \ddot{x}^2 = \frac{2}{3} q^2 (\gamma \dot{x} + kx)^2.$$

This must be equal to the rate of change of energy:

$$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\dot{E}(t) = m \ddot{x} \dot{x} + k x \ddot{x} = \dot{x} (m \ddot{x} + k x)$$

The motion of a lightly damped harmonic oscillator is

$$x(t) = A e^{-\frac{1}{2} \tilde{\gamma} t} \cos \theta(t),$$

where $\tilde{\gamma} = \gamma/m$, $\theta(t) = \omega t + \delta$, and $\omega^2 \approx k/m$. Therefore,

$$\dot{x}(t) = A e^{-\frac{1}{2} \tilde{\gamma} t} \left[ -\frac{1}{2} \tilde{\gamma} \cos \theta - \omega \sin \theta \right]$$

$$\ddot{x}(t) = A e^{-\frac{1}{2} \tilde{\gamma} t} \left[ \left( \frac{\tilde{\gamma}^2}{4} - \omega^2 \right) \cos \theta + \omega \tilde{\gamma} \sin \theta \right]$$

By assumption, $\tilde{\gamma}^2 \ll 4 \omega^2$. Furthermore, we may average $P$ and $\dot{E}$ over one cycle, so that $\langle \cos^2 \theta \rangle = \langle \sin^2 \theta \rangle = 1/2$ and $\langle \cos \theta \sin \theta \rangle = 0$, where we have neglected the slow time dependence of $\exp(-\tilde{\gamma} t)$. Therefore, to first order in $\tilde{\gamma}/\omega$,

$$P = -\frac{1}{3} q^2 A^2 e^{-\tilde{\gamma} t} \omega^4$$

and

$$\dot{E} = -\frac{1}{2} A^2 e^{-\tilde{\gamma} t} \omega^2 \gamma.$$

Equating these, we obtain

$$\gamma = \frac{2}{3} q^2 \omega^2 = \frac{2}{3} q^2 \frac{k}{m}.$$