Current enters an infinite plane conducting sheet at a point P and leaves at infinity. A circular hole, not including P, is cut somewhere in the sheet. Show that the potential difference between two points on the edge of the hole is double what it would have been in the absence of the hole.

\[ \nabla \cdot \mathbf{E} = \sigma \]

**Solution**

\[ \Phi_0 = \frac{\Phi_0}{\epsilon_0} \Phi_1 = \frac{\Phi_0}{\epsilon_0} \]

\[ \Phi_0 = \frac{1}{2} \ln \left( \frac{r^2 + \eta^2 - 2r \eta \cos \theta}{\eta^2} \right) + \frac{2}{\epsilon_0} \mathbf{B} \cdot \mathbf{r} \]

\[ \phi_1 = \phi_0 + \phi_1 = \frac{\Phi_0}{\epsilon_0} \ln \left( \frac{r^2 + \eta^2 - 2r \eta \cos \theta}{\eta^2} \right) + \frac{2}{\epsilon_0} \mathbf{B} \cdot \mathbf{r} \]

\[ A_0 = -\frac{1}{\epsilon_0} \mathbf{E}_0 \cdot \mathbf{r}, \quad A_1 = \frac{1}{\epsilon_0} \mathbf{E}_1 \cdot \mathbf{r} + B \]

\[ \Delta A = 0 \text{ at } \theta = 0 \]

\[ B \mathbf{r} = \frac{2}{\pi} \mathbf{A} \]

\[ \mathbf{B}_0 = 0, \quad \mathbf{A}_0 = -\frac{1}{\epsilon_0} \mathbf{E}_0, \quad \mathbf{A}_1 = \frac{1}{\epsilon_0} \mathbf{E}_1 \]

and \( \phi_1 = -\frac{1}{\epsilon_0} \mathbf{E}_1 \cdot \mathbf{r} = \frac{1}{\epsilon_0} \mathbf{E}_0 \cdot \mathbf{r} + \frac{2}{\epsilon_0} \mathbf{B} \cdot \mathbf{r} \)

so that \( \phi_1 \bigg|_{\theta = 0} = 2 \phi_0 \bigg|_{\theta = 0} \).
2.

A transmission line consists of two parallel perfect conductors of arbitrary but constant cross-section. The surrounding medium has constants $\varepsilon$ and $\mu$. Show that the product of the inductance per unit length and capacitance per unit length satisfies

$$L \ C = (\varepsilon \mu) / c^2.$$

**Solution**

D = B = 0 inside. Look at 2D.

On boundary and outside, have two separate structures

$$\nabla \cdot \mathbf{D} = \frac{4\pi I}{c}$$

$$\nabla \times \mathbf{D} = 0$$

so can make correspondence

$$\mathbf{H} = \frac{3}{2} \times \mathbf{D}, \quad \mathbf{j} = \mathbf{c} \mathbf{P}, \quad I = c Q / \varepsilon \mu.$$

$$U_m = \frac{4\pi}{3} \int \mathbf{H} \cdot d\mathbf{r} \quad U_e = \frac{1}{8\pi \varepsilon} \int \mathbf{j} \cdot d\mathbf{S}$$

and

$$U_m / U_e = \frac{3}{2}$$

But

$$U_m / U_e = \frac{1}{2} \frac{L I^2}{I / Q c} = \frac{L C (I / Q)^2}{2} = \frac{L C c^2}{2}$$

$$\therefore L \ C = \varepsilon \mu / c^2$$
Maxwell's equations in free space, but with sources, take on a simple form in terms of the vector $Q = E + i H$. If the current density $J$ is generalized to $j + i j_m$, interpret the $j_m$ that now appears.

**Solution**

$$Q = E + i H \quad \nabla \cdot E = 4\pi \rho, \quad \nabla \cdot H = 0; \quad \nabla \times Q = 4\pi j$$

$$\nabla \times E = -\frac{1}{c^2} \frac{d}{dt} H, \quad \nabla \times H = \frac{1}{c} \frac{d}{dt} E + \frac{4\pi}{c} i j_m$$

Since $j + i j_m = 0$, then $j = -i j_m$, and $j_m$ is the magnetic current density. The current density $j_m$ is then given by:

$$\nabla \times Q = 4\pi \left[ (j + i j_m) + \frac{4\pi}{c} i j_m \right]$$

$$\nabla \times E = \frac{4\pi}{c} (j + i j_m), \quad \nabla \times H = \frac{4\pi}{c} j_m, \quad \nabla \times H = \frac{1}{c} \frac{d}{dt} E + \frac{4\pi}{c} i j_m$$

When $j_m$, $j_m$ are magnetic charge, current density.
Show that a parallel pair of perfectly conducting solid cylinders of any given cross-sectional shape can transmit an E.M.-transverse (TEM) wave at any frequency \( \omega \).

\[
H = (H_x, H_y, 0) \propto (\varepsilon \frac{\omega^2}{c^2} - \omega^2), \quad E = (E_x, E_y, 0) \propto (\mu \frac{\omega^2}{c^2} - \omega^2)
\]

\( h = \omega / c \)

\[
\begin{align*}
\frac{\partial E_x}{\partial t} - \varepsilon \frac{\partial H_y}{\partial x} - \mu \frac{\partial H_x}{\partial y} &= 0, \\
\frac{\partial H_x}{\partial t} + \varepsilon \frac{\partial E_y}{\partial x} + \mu \frac{\partial E_x}{\partial y} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial E_x}{\partial x} &= 0, \\
\frac{\partial E_y}{\partial y} &= 0, \\
\frac{\partial^2 E_x}{\partial x^2} &= 0
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot E &= 0, \\
\nabla \times H &= 0
\end{align*}
\]

For \( h = \omega / c \) and \( \varepsilon \frac{\omega^2}{c^2} - \omega^2 = 0 \)

\[
\begin{align*}
\gamma_1 = \mu_1 \text{ for 1 conductor} \\
\text{electrostatic problem}
\end{align*}
\]

Solution
Two equal and opposite charges +q and -q are connected to the ends of a rigid uncharged rod of length d. The rod rotates in the x-y plane with constant angular velocity \( \omega \) about the z axis, which passes through its center. Recall that the non-relativistic acceleration field of a single charge q is given far away by

\[
E = \left( \frac{q}{c^2} \right) \hat{n} \times \left( \hat{n} \times \hat{v} \right)/R.
\]

Compute (non-relativistically) the instantaneous radiated power \( dP(t)/dt \) in the radiation zone. Integrate over solid angle \( \Omega \) to show that the total radiated power \( P(t) \) is in fact independent of time.

**Solution**

In the radiation zone, we can ignore the different locations of the two charges and use linear superposition to write \( i = 1, 2 \)

\[
dP(t)/d\Omega = \frac{c}{4\pi} | R \sum \mathbf{E}_i | = \left( \frac{1}{4\pi c^3} \right) \left( \hat{n} \times \mathbf{v}_i \right)^2.
\]

Now take q: \( \hat{n}_1 = \frac{1}{2} \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right), \quad \mathbf{v}_1 = -\frac{d\mathbf{r}_1}{dt} \left( \frac{\hat{x} \cos \theta + \hat{y} \sin \theta}{\hat{z}} \right) \)

- q: \( \hat{n}_2 = -\frac{1}{2} \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right), \quad \mathbf{v}_2 = \frac{d\mathbf{r}_2}{dt} \left( \frac{\hat{x} \cos \theta + \hat{y} \sin \theta}{\hat{z}} \right) \)

or \( \mathbf{v}_1 + \mathbf{v}_2 = -q \frac{d}{dt} \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \)

or with \( \hat{n} = \hat{n}_1 + \hat{n}_2, \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \)

\[
dP(t)/d\Omega = \frac{q^2}{16\pi^2 c^6} \left| \left( \hat{x} \cos \theta \omega + \hat{y} \sin \theta \omega + \hat{z} \right) \right|^2.
\]

Integrating over \( d\Omega = d\theta \, d\phi \), one gets

\[
P(t) = \frac{2q^2}{3c^3}, \quad \text{and ob. ~}\]
An electron $e^-$ with kinetic energy 1.0 MeV makes a head-on collision with a positron $e^+$ at rest. In the collision, the two particles annihilate each other and are replaced by two photons of equal energy, each traveling at angle $\Theta$ with the electron's direction of motion.

Determine the energy $E$, momentum $p$, and angle of emission $\Theta$ of each photon.

**Solution**

The incident electron, with rest mass $m = 0.511\text{MeV}/c^2$, has momentum $p$ along the positive $x$-axis and kinetic energy $K$. It follows that

$$p = \sqrt{K(K + 2m^2c^4)/c}$$

from which $p = 1.422\text{MeV}/c$ when $K = 1.000\text{MeV}$.

The total energy $E$ of the electron and the stationary positron before the collision is

$$E = K + 2mc^2 = 2.022\text{MeV}.$$  

The two photons emerge from the collision each with energy

$$E_\gamma = \frac{1}{2}E = 1.011\text{MeV}$$

as given by conservation of energy, each with magnitude of momentum

$$p_\gamma = E_\gamma/c = 1.011\text{MeV}/c.$$ 

The momentum vectors of the photons make angles $\pm\theta$ with the $x$-axis. Conservation of momentum in the $x$-direction is

$$p = 2p_\gamma \cos \theta$$

from which $\theta = 45.3^\circ$. 