We have

This has spherical symmetry, so it's easiest to begin by calculating \( \Phi \) for all \( r > a \), simply by enclosing the smaller sphere with a Gaussian spherical surface and using \( \oint_{S} \Phi dA = Q_{\text{enclosed}} \)

where \( Q_{\text{enclosed}} \) refers to the free charge, which here is simply \( Q \). So we have

\[
\Phi = \frac{Q}{4\pi r^2} r \quad r > a
\]

(Inside the metal sphere, of course, \( E = \Phi = 0 \).)

Knowing \( \Phi \), it's a trivial matter to find \( E \):

\[
E = \begin{cases} 
\frac{Q}{4\pi \varepsilon_0 r^2} r & a < r < b \\
\frac{Q}{4\pi \varepsilon_0 r^2} r & r > b
\end{cases}
\]

The potential at \( r = 0 \) is therefore

\[
V(0) = -\int_{r_0}^{r} E \cdot dl = -\int_{0}^{a} \left( \frac{Q}{4\pi \varepsilon_0 r^2} r \right) dr - \int_{a}^{b} \left( \frac{Q}{4\pi \varepsilon_0 r^2} r \right) dr - \int_{b}^{0} \left( \frac{Q}{4\pi \varepsilon_0 r^2} r \right) dr
\]

\[
= \frac{Q}{4\pi} \left( \frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon_0 a} - \frac{1}{\varepsilon_b} \right)
\]

b) The polarization in the dielectric is \( \vec{P} = \varepsilon_0 \varepsilon \vec{E} \), so

\[
\vec{P} = \varepsilon_0 \varepsilon \frac{Q}{4\pi r^2} r \quad \text{so}
\]

\[
\sigma_b = \vec{P} \cdot \vec{E} = \begin{cases} 
\frac{\varepsilon_0 \varepsilon \frac{Q}{4\pi r^2}}{\varepsilon_b} & \text{at } r = b \\
-\frac{\varepsilon_0 \varepsilon \frac{Q}{4\pi r^2}}{\varepsilon_a} & \text{at } r = a
\end{cases}
\] (and with \( \varepsilon_a = \frac{\varepsilon_b}{\varepsilon_0 + 1} \))
Electromagnetism

Exam 2 Solutions

1. a) This and part b) were done in class.

2. Because the current is steady, we have a time-independent situation, so \( \frac{\partial u}{\partial t} = 0 \), where \( u \) is energy density due to the \( E \) and \( B \) fields. That means that energy flux through any surface is given by \( \int \vec{S} \cdot \hat{n} \, da \), where \( \vec{S} = \vec{E} \times \vec{H} \) is the surface Poynting vector.

Outside the wire, we can write \( \vec{H} = \frac{\vec{B}}{\mu_0} \), so \( \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \). For a steady current \( I \), the \( \vec{B} \)-field encircles the wire with a magnitude \( |\vec{B}| = \frac{\mu_0 I}{2\pi r} \) at a perpendicular distance \( r \) from the wire. The electric field is gotten simply from Ohm’s Law \( V = IR \): a potential difference \( V \) along a segment of the wire requires an electric field \( E = \frac{V}{l} \), where \( l \) is the length of the wire segment. Then \( \int 0 \, E \times B = -\frac{\mu_0 I}{2\pi r} \hat{r} \), where \( \hat{r} \) points radially away from the wire. Surrounding the wire with a cylindrical surface of radius \( r \) and length \( l \) gives \( \int \vec{S} \cdot \hat{n} \, da = -I^2 R \hat{\hat{r}} \). That is, the
$E$ and $B$ fields associated with the current flow provide an energy flow $I^2 R$ into the wire, which exactly balances losses due to Joule heating, thereby maintaining the steady state (of course, the actual work is done by the battery or other power source.)

b) If $T_{ij}$ is the stress tensor, then the net force on the upper hemisphere is

$$F_i = \int_{\text{upper hemisphere}} d^3x \varepsilon_{ij} T_{ij} = \int_S \mathbf{n} \cdot T_{ij}$$

where $S$ is a surface containing enclosing a volume containing all the charge in the upper hemisphere and no other charge. This leaves us considerable freedom in choosing such a surface; as discussed in class, perhaps the easiest is simply to use the entire $xy$ plane (which contains the equator of the sphere). Other choices can be used and work equally well.

With our choice, we have $\mathbf{n} = -\hat{z}$ on the $xy$ plane, $E_z = 0$ by symmetry, so $I$ takes a very simple form:

$$I = \varepsilon_0 \begin{bmatrix} E_x^2 - E_z^2 & E_x E_y & 0 \\ E_x E_y & E_y^2 - E_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
and

\[
T \cdot n = \varepsilon_0 \begin{bmatrix} \frac{E_x^2}{2} & E_x E_y & 0 \\ E_x E_y & E_y^2 - \frac{E_z^2}{2} & 0 \\ 0 & 0 & -\varepsilon_0/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

\[= \varepsilon_0 E_z^2 / 2,\]

so that

\[
F = \frac{\varepsilon_0}{2} \int |E|^2 \, dx \, dy = \frac{\varepsilon_0}{2} \int_0^\infty (2\pi r \, dr) |E|^2 \, z
\]

From Gauss' Law,

\[E(r < R) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^3} \]

and

\[E(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^3} \]

so

\[E = \frac{3Q}{64 \pi \varepsilon_0 R^3} \]

2. This is the first half of Jackson 7.41, it does for metal what we did in class for dielectrics. The fact that the incident wave is normal to the flat surface of the metal makes the problem relatively straightforward.
In vacuum, we have \( k = \omega/c \). In the metal, we have
\[ k_t = \sqrt{n_0 \varepsilon(\omega)} \omega \]
Since (as stated in the problem)
\[ \varepsilon(\omega) = \varepsilon + i \sigma/\omega \]
we have
\[ k_t = \sqrt{\varepsilon \varepsilon_0 + i \omega \sigma} \]
giving
\[ \frac{E_0 e^k}{E_0} = \frac{1 - \frac{\varepsilon}{\varepsilon_0} + \frac{i \sigma}{\varepsilon_0 \omega}}{1 + \frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}} \]

Note that when \( \sigma = 0 \) we recover
\[ \frac{E_0 e^k}{E_0} = \frac{1 - \sqrt{\varepsilon}}{1 + \sqrt{\varepsilon}} \]
which we found for dielectrics.

Show that when \( \sigma \to \infty \), the reflection coefficient \( R = |E_0 e^k/E_0|^2 \) becomes
\[ R = 1 - 2 \varepsilon \frac{\omega}{c} \]
where \( \varepsilon = \varepsilon_{\text{metal}} \) is the skin depth.

b) Faraday's law is \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)
Using the wave form, given, this gives
\[ i k \times E_0 = i \omega \mathbf{B}_0 \Rightarrow i k \times E_0 = i \omega \mathbf{B}_0 \]
This can be rewritten (useful for the next part)
\[ \mathbf{B}_0 = \frac{k}{\omega} \hat{k} \times \hat{E}_0 = \frac{1}{c} \hat{k} \times \hat{E}_0 \]
since \( \omega = kc \) in vacuum.
The energy stored per unit volume in the electromagnetic fields is

\[ u = \frac{1}{2} \left( \varepsilon_0 |E|^2 + \frac{1}{\mu_0} |B|^2 \right) \]

We just saw that \(|B|^2 = \frac{1}{c^2} |E|^2 = \mu_0 \varepsilon_0 |E|^2\) in vacuum, so \(\varepsilon_0 |E|^2 = \frac{1}{\mu_0} |B|^2\); the electric and magnetic contributions to the energy are equal (in vacuum).

ii) The energy flux density transported by the fields is given by the Poynting vector \(\mathbf{S} = \frac{1}{\mu_0} (E \times B) = \epsilon_0 c \mathbf{E} \mathbf{E}^*\); not surprisingly, the energy density times the wave velocity \(c\). But we need to determine \(u\) and \(\mathbf{S}\); we've done several times in class, the average over one cycle is \(\langle u \rangle = \frac{1}{2} \varepsilon_0 |E|^2\), so \(\langle S \rangle = \frac{1}{2} \varepsilon_0 \varepsilon_0 E^2 \mathbf{k}\).

We also showed in class that the momentum flux is \(\langle \mathbf{p}^2 \rangle = \frac{1}{c^2} \langle S \rangle\), so \(\langle \mathbf{p} \rangle = \frac{1}{2c} \varepsilon_0 \varepsilon_0 E^2 \mathbf{k}\).
3. We have

\[ E_i(z, t) = E_{0i} \hat{x} e^{i(kz - \omega t)} \]
\[ B_i(z, t) = \frac{1}{c} E_{0i} \hat{y} e^{i(kz - \omega t)} \]

The incident wave is

\[ E_r(z, t) = E_{0r} \hat{x} e^{-i(kz + \omega t)} \]
\[ B_r(z, t) = -\frac{1}{c} E_{0r} \hat{y} e^{-i(kz + \omega t)} \]

The reflected wave is

because \( k_i = k_r \) (why?).

The transmitted wave is

\[ E_t(z, t) = E_{0t} \hat{x} e^{i(k_2z - \omega t)} \]
\[ B_t(z, t) = \frac{1}{v} E_{0t} \hat{y} e^{i(k_2z - \omega t)} \]

where \( v = c/n \).

The boundary condition gives...
\[ E_{oi} + E_{or} = E_{ot} \]
\[ \frac{1}{2}(E_{oi} - E_{or}) = \frac{1}{V} E_{ot} \quad \text{(since } \mu = \mu_0) \]

Solving gives

\[ E_{or} = \left( \frac{1 - c/V}{1 + c/V} \right) E_{oi} = \left( \frac{1 - n}{1 + n} \right) E_{oi} \]
\[ E_{ot} = \frac{2}{1 + n} E_{oi} \]

The reflection coefficient is then

\[ R = \frac{I_r}{I_i} = \left( \frac{E_{or}}{E_{oi}} \right)^2 = \left( \frac{1 - n}{1 + n} \right)^2 \]

and the transmission coefficient is

\[ T = \frac{I_t}{I_i} = n \left( \frac{E_{ot}}{E_{oi}} \right)^2 = \frac{4n}{(1 + n)^2} \]

and \( R + T = 1 \), as expected. The factor of \( n \) in the formula for \( T \) comes from the formula for intensity: \( I = \frac{1}{2} c V E_x^2 \).

b) With no free charges and \( J = \sigma E \), Maxwell's equations are

\[ \nabla \cdot E = 0 \quad \nabla \times E = - \frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \quad \nabla \times B = \mu_0 \frac{\partial E}{\partial t} + \mu_0 \sigma \frac{\partial E}{\partial t} \]

Taking the curl of the first curl equation gives

\[ \nabla \times \nabla \times E = \mu_0 \frac{\partial \nabla \times E}{\partial t} + \mu_0 \sigma \frac{\partial \nabla \times E}{\partial t} \]
and the same for \( B \). These still admit plane-wave solutions

\[ E(2, t) = E_0 \ e^{i(kx - \omega t)} \]
and similarly for $\vec{B}$, but with a complex wave number

$$k^2 = \mu_0 \omega^2 + i \omega \sigma$$

or

$$k = k_0 + i \kappa$$

with

$$k_0 = \omega \sqrt{\frac{\varepsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2} + 1 \right]}^{1/2}$$

$$\kappa = \omega \sqrt{\frac{\varepsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2} - 1 \right]}^{1/2}$$

Inside the metal the wave then has the form

$$\vec{E}(z, t) = \vec{E}_0 e^{-k^2 t} e^{i(k_0 z - \omega t)}$$

and similarly for $\vec{B}$, so the attenuation factor or "skin depth" is given by $1/\kappa$. 
4. a) Since $d << \lambda$ and the antenna is an oscillating electric dipole, we can use the electric dipole approximation. The formulas for the $E$ and $H$ fields are given; we need only find $\vec{p}$.

We have for the current density

$$\vec{J}(x,t) = I(z)\, \delta(x)\delta(y) \, e^{-i\omega t} \hat{z}$$

where $I(z) = I_0 (1 - \frac{z^2}{d^2})$ when $|z| < d/2$.

But $\vec{p} = \int d^3x' \vec{x}' \rho(x',t)$, so we need $\rho$.

We get this from the continuity eq'n.

Since $\vec{J}(x,t) = J(x) e^{-i\omega t}$, we must have $\vec{\rho}(x,t) = \rho(x) e^{-i\omega t}$, so from

$$\nabla \cdot \vec{J}(x) = \frac{d\vec{p}}{dt} = 0,$$

we get $\vec{\nabla} \cdot \vec{J}(x) = i\omega \rho(x)$.

So

$$\vec{\nabla} \cdot \vec{J}(x) = I_0 \left(-\frac{2}{d}\right) \delta(x)\delta(y) \Theta \left(\frac{d}{2} - |z|\right)$$

and so

$$\rho(x) = \frac{-I_0}{i\omega \left(\frac{2}{d}\right)} \delta(x)\delta(y) \Theta \left(\frac{d}{2} - |z|\right).$$

This gives

$$\vec{p} = \frac{iI_0}{2\omega} \hat{z} e^{-i\omega t}$$

Inserting this into the formulas for $E$ and $H$, and using $\frac{d\vec{p}}{dt} = \langle E \times H \rangle \hat{r}$

gives

$$\frac{d\vec{p}}{dt} = \frac{I_0^2}{8\varepsilon_0} \frac{1}{128\pi^2} (kd)^2 \sin^2 \Theta$$

where $\Theta$ is the angle between $\hat{z}$ and $\hat{n}$, the unit vector from the origin to the observation point, and

$$P = \frac{1}{2}\pi \int d^2 P \sin \Theta d\Theta = \frac{I_0^2 \, k^2 \, d^2}{4 \, 8 \, \pi}$$

Recall that $k = \omega/c$. 

6) An oscillating current loop is just an oscillating magnetic dipole.

\[ \vec{\mu} = \pi a^2 I_0 e^{-i\omega t} \]

Using the magnetic dipole radiation formulas gives

\[ \frac{dP}{d\Omega} = \frac{\mu_0}{\varepsilon_0} \frac{(ka)^4}{32} I_0^3 \sin^2 \Theta \]

and

\[ P = \frac{\mu_0}{\varepsilon_0} \frac{\pi}{12} I_0^3 (ka)^4 \]

\[ \frac{P_{mag.\ dip.}}{P_{el.\ dip.}} \sim (ka)^3 \ll 1 \]

Electric dipole radiation, when present, is always stronger than all other kinds when \( a \ll \lambda \).
5. Since the motion is nonrelativistic, we can use the Larmor formula

\[ P = \frac{M_0 q^2 a^3}{6 \pi c} \]  

If the acceleration is constant, the time it takes to come to rest is \( t = \frac{v_0}{a} \). The total energy radiated is then

\[ U_{rad} = Pt = \frac{M_0 q^2 a^3 v_0}{6 \pi c} \frac{1}{a} = \frac{M_0 q^2 v_0 a}{6 \pi c} \]

The initial energy was \( \frac{1}{2} m v_0^2 \), so the fraction of initial energy lost to radiation is

\[ f = \frac{U_{rad}}{U_{cin}} = \frac{M_0 q^2 a}{3 \pi m v_0 c} \]

b) Since here \( d = \frac{1}{2} at^2 = \frac{v_0^2}{2a} \), we have \( a = \frac{v_0^2}{2d} \). Then

\[ f = \frac{M_0 q^2 \frac{v_0^3}{2d}}{3 \pi m v_0 c} \]  

which gives \( f \approx (10^{-5}) \times \) (gravitational unit) 

\[ f = \frac{\left(4\pi \times 10^{-9}\right) \left(1.6 \times 10^{-19}\right)^2 \left(10^5\right)}{6 \pi \left(9.11 \times 10^{-31}\right) \left(3 \times 10^8\right) \left(3 \times 10^{-9}\right)} = 2 \times 10^{-10} \]

So radiative losses due to collisions in an ordinary wine are negligible.