1. In fact, $Q$ and $P$ are the classical equivalent of quantum creation and annihilation operators.

a) 
\[
\left[ Q, P \right]_{q,p} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = \left( \frac{1}{\sqrt{2mw}} \right)^2 (mw - i^2 mw) = 1
\]  
(1)

b) Since we want $F_1(q, Q)$, we write $p$ and $P$ in terms of $q$ and $Q$:
\[
\frac{\partial F_1}{\partial q} = p = -i\sqrt{2mw} Q + imw q, \quad \frac{\partial F_1}{\partial Q} = -P = iQ - i\sqrt{2mw} q
\]  
(2)

Integrating each and requiring consistency leads to
\[
F_1(q, Q) = \frac{i}{2} mw q^2 + \frac{i}{2} Q^2 - i\sqrt{2mw} q Q
\]  
(3)

c) Since the transformation is time independent the new Hamiltonian $K$ equals the old $H = \frac{1}{2} mw^2 q^2 + \frac{E}{2m}$ in the new variables, which then reads
\[
K = \frac{1}{2} mw^2 (Q - iP)^2 \frac{1}{2m} \frac{(Q + iP)^2}{2m} \frac{(-1)}{2} mw = -iw Q P
\]  
(4)

which leads to Hamilton’s equations
\[
\dot{Q} = \frac{\partial K}{\partial P} = -iw Q, \quad \dot{P} = -\frac{\partial K}{\partial Q} = iw P
\]  
(5)

which are trivially solved by $Q(t) = Q_0 e^{-iwt}$ and $P(t) = P_0 e^{iwt}$.

d) It’s easy to see what’s going on without doing much calculation. Since $Q \rightarrow Q e^{iwt}$ and $P \rightarrow P e^{-iwt}$, the final solution is going to be $Q(t) = Q_0$ and $P(t) = P_0$. The reason this happens from the Hamiltonian point of view is that now the new Hamiltonian vanishes:
\[
K = H + \frac{\partial F_1}{\partial t} = 0
\]  
(6)

where the new generating function is simply found from b) from the mapping $Q \rightarrow Q e^{iwt}$.

2.

a) The unperturbed action-angle variables $\theta, J$ for the 1D harmonic oscillator are well-known:
\[
q = \sqrt{\frac{2J}{mw_0}} \sin \theta, \quad p = \sqrt{2mw_0J} \cos \theta,
\]  
(7)

so the unperturbed Hamiltonian is simply $H_0 = w_0 J$. The perturbed Hamiltonian in these variables reads
\[
\Delta H = \epsilon \left( \alpha \frac{4J^2}{m^2w_0^2} \sin^4 \theta - \beta \sqrt{8mw_0J^3} \sin \theta \cos^2 \theta \right)
\]  
(8)

b) The first-order correction to the energy is simply the angular average of the first-order perturbed Hamiltonian,
\[
E_1 = \langle \Delta H \rangle = \epsilon \left( \alpha \frac{4J^2}{m^2w_0^2} \langle \sin^4 \theta \rangle - \beta \sqrt{8mw_0J^3} \langle \sin \theta \cos^2 \theta \rangle \right) = \epsilon \frac{3J^2}{2m^2w_0^2}
\]  
(9)

where $\langle \ldots \rangle \equiv \int_0^{2\pi} \ldots \frac{d\theta}{2\pi}$, and thus $\langle \sin^4 \theta \rangle = 3/8$ and $\langle \sin \theta \cos^2 \theta \rangle = 0$. 

c) The first-order correction to the frequency is given by
\[ w_1 = \frac{\partial E_1}{\partial J} = \epsilon \alpha \frac{3J}{m^2 w_0^3} \] (10)

3. We take the z-axis to coincide with the velocity of the inner sphere to take advantage of axial symmetry. The velocity potential satisfies the Laplace equation \( \nabla^2 \phi_v = 0 \) with boundary conditions,
\[
\left( \frac{\partial \phi_v}{\partial r} \right)_{r=a} = u \cos \theta, \quad \left( \frac{\partial \phi_v}{\partial r} \right)_{r=b} = 0
\] (11)
where \( v = \frac{\partial \phi_v}{\partial r} + \frac{1}{r} \frac{\partial \phi_v}{\partial \theta}. \)

a) The solution of Laplace obeys,
\[
\phi_v(r, \theta) = \sum_{\ell=0} A_\ell r^\ell + B_\ell r^{\ell+1} P_\ell(\cos \theta) \] (12)
Using orthogonality of Legendre polynomials the boundary condition at \( r=a \) gives:
\[
B_0 = 0, \quad A_1 = 2B_1 a^3 + u, \quad \ell A_\ell = (\ell + 1) \frac{B_\ell}{a^{2\ell+1}} \quad (\ell > 1)
\] (13)
while from the boundary condition at \( r=b \) we get
\[
B_1 = 2B_1 b^3, \quad \ell A_\ell = (\ell + 1) \frac{B_\ell}{b^{2\ell+1}} \quad (\ell > 1)
\] (14)
so we have \( A_1 = 2B_1 / b^3 = u a^3 / (a^3 - b^3) \) while all others vanish, and then the desired solution is
\[
\phi_v(r, \theta) = \frac{u a^3}{(a^3 - b^3)} \left( r + \frac{b^3}{2r^2} \right) \cos \theta
\] (15)

b) Taking the limit \( b \to \infty \) of Eq. (15), we get \( \phi_v = -(u a^3 / 2r^2) \cos \theta \), whereas for a dipole \( \phi_v = -\hat{\mu} \cdot r / (4\pi r^3) \), so that indeed \( \mu = 2\pi u a^3 \).

4. This is a well known problem for viscous fluids.

a) Choosing cylindrical coordinates, by symmetry \( \mathbf{v} = v_\phi(r) \hat{\phi} \), Navier-Stokes reads (radial and azimuthal)
\[
-r^2 \frac{\partial^2 v_\phi}{\partial r^2} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \quad 0 = \eta \left( \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} \right)
\] (16)

b) The second gives right away an ODE for \( v_\phi \) of the Euler form (as provided in the Hint):
\[
r^2 \frac{\partial^2 v_\phi}{\partial r^2} + r \frac{\partial v_\phi}{\partial r} - v_\phi = 0
\] (17)
with general solution \( v_\phi(r) = Ar + B/r \). The boundary conditions of the problem are no-slip, \( v_\phi(a) = 0 \) and \( v_\phi(b) = \Omega b \), thus the velocity field is
\[
\mathbf{v} = \frac{\Omega b^2}{b^2 - a^2} \left( r - \frac{a^2}{r} \right) \hat{\phi}
\] (18)
c) The viscous stress is $\sigma'_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$. Due to symmetry the only nontrivial components are

$$
\sigma'_{r\phi} = \sigma'_{\phi r} = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right) = \frac{2\eta \Omega a^2 b^2}{(b^2 - a^2)r^2} (19)
$$