Part I: Classical Dynamics–Solutions

Problem 1 (33 pts)

A particle moves on the $x$ axis with Hamiltonian

$$H(x, p_x) = \frac{p_x^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$  

(a) (5 pts) The equations of motion (Hamilton’s equations) are:

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x,$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -x + x^3.$$  

(b) (5 pts) The equilibrium points of the system are found by setting the righthand sides of Hamilton’s equations in (a) equal to zero. We have a stable equilibrium at $x = 0, p_x = 0, H = 0$, and unstable equilibria at $x = \pm 1, p_x = 0, H = \frac{1}{4}$. Stability/instability is determined by expanding $H$ in Taylor series about the respective equilibrium points. To degree 2, the local $H$ is a harmonic oscillator for a stable equilibrium, an inverted oscillator for an unstable equilibrium.

(c) (8 pts)

(d) (5 pts) A particle is launched at $x = 0$ with initial velocity $1/\sqrt{2}$. The initial point is on the separatrix curve $H = 1/4$, directly above the origin. The initial momentum is positive and so the point moves to the right along
the separatrix, arriving at the unstable equilibrium point at \( x = 1, p = 0 \) after time
\[
\int_0^1 \frac{dx}{x} = \int_0^1 \frac{dx}{2(1 - x^2)} = \int_0^1 \frac{dx}{2(1 + x)(1 - x)} = \infty
\]

(e) (5 pts) We make the following transformation of phase space coordinates:
\[
(x, p_x) \mapsto (\theta, J),
\]
where
\[
x = \sqrt{2}J \sin \theta, \quad p_x = \sqrt{2}J \cos \theta.
\]
To prove that this is a canonical transformation, we show that the Poisson bracket of \( x \) with \( p_x \) is equal to 1 if \([\theta, J] = 1\).
\[
[x, p_x] = \frac{\partial x}{\partial \theta} \frac{\partial p_x}{\partial J} - \frac{\partial x}{\partial J} \frac{\partial p_x}{\partial \theta} = \sin^2 \theta + \cos^2 \theta = 1.
\]

(f) (5 pts)
\[
\dot{H}(\theta, J) = J - \epsilon J^2 \sin^4 \theta
\]
To get the new Hamiltonian, we need only to average over \( \theta \):
\[
\bar{H}(\bar{\theta}, \bar{J}) = \bar{J} - \frac{1}{2\pi} \int_0^{2\pi} \sin^4 \theta = \bar{J} - \frac{3}{8} \bar{J}^2 + O(\epsilon^2)
\]
The new Hamiltonian \( \bar{H} \) is a function only of \( \bar{J} \), with corrections of order \( \epsilon^2 \). The oscillation frequency, correct to first order in \( \epsilon \) is given by
\[
\bar{\omega} = \frac{\partial \bar{H}}{\partial \bar{J}} = 1 - \frac{3}{4} \bar{J} + O(\epsilon^2).
\]

Problem 2 (34 pts)
A dynamical system with two angular degrees of freedom has the Hamiltonian
\[
H(\phi_1, \phi_2, L_1, L_2) = L_1 + \frac{5}{3} L_2 + (3L_1 + 2L_2)^2 \sin(2\phi_1 - 3\phi_2).
\]
(a) (5 pts) Find a linear combination $F$ of $L_1$ and $L_2$ which commutes, in the sense of Poisson brackets, with $H$. Show that $F$ is a constant of the motion. It is sufficient to find $a, b$ such that

$$[aL_1 + bL_2, 2\phi_1 - 3\phi_2] = 0.$$  

From the elementary Poisson brackets,

$$-2a + 3b = 0.$$  

Hence we can take

$$F = c(L_1 + \frac{2}{3}L_2),$$  

where $c$ is an arbitrary numerical constant. Since $\frac{\partial F}{\partial t} = 0$, we have $\dot{F} = [F, H] = 0$.

(b) (5 pts) Show that $F/c$ and $H$ are functionally independent (i.e their phase-space gradient vectors are linearly independent) everywhere in the phase space. For dependence, all $2\times2$ minors of the rectangular matrix of partial derivatives,

$$\begin{pmatrix} \frac{\partial H}{\partial L_1} & \frac{\partial H}{\partial \phi_1} \\ \frac{2}{3} \frac{\partial H}{\partial L_2} & 0 \frac{\partial H}{\partial \phi_2} \end{pmatrix}$$

must vanish. In particular, one of the minors is

$$\frac{2}{3} \frac{\partial H}{\partial L_1} - \frac{\partial H}{\partial L_2} = \frac{2}{3} - \frac{5}{3} = -1,$$

and so the gradient vectors are linearly independent everywhere.

(c) (5 pts) Statements (a) and (b) imply that the topological structure of a compact, connected phase-space manifold $M_{h,f}$ on which $H$ and $F$ take the respective values $h$ and $f$ must be that of a 2-torus.

(d) (6 points) We make a canonical transformation

$$(\phi_1, \phi_2, L_1, L_2) \mapsto (\alpha_1, \alpha_2, I_1, I_2)$$

such that

$$\alpha_1 = \phi_1, \quad \alpha_2 = \phi_2 - \frac{2}{3}\phi_1,$$
This is accomplished by means of a generating function

\[ S(\phi_1, \phi_2, I_1, I_2) = I_1\phi_1 + I_2(\phi_2 - \frac{2}{3}\phi_1), \]

which gives

\[ L_1 = \frac{\partial S}{\partial \phi_1} = I_1 - \frac{2}{3}I_2, \]

\[ L_2 = \frac{\partial S}{\partial \phi_2} = I_2, \]

so that

\[ F = c(I_1 - \frac{2}{3}I_2 + \frac{2}{3}I_2) = cI_1, \]

as desired. Plugging the expressions for \( L_1, L_2, \phi_1, \phi_2 \) into \( H \) gives

\[ \bar{H}(\alpha_1, \alpha_2, I_1, I_2) = I_1 + I_2 - 9I_1^2 \sin 3\alpha_2. \]

(d) (5 points) To verify that the system is separable with respect to the coordinates \( \alpha_1, \alpha_2, I_1, I_2 \), we note that for fixed \( \bar{H} = h \) and \( I_1 = F/c = f \), we have that \( I_1 \) is a constant for \( 0 \leq \alpha_1 < 2\pi \), while \( I_2 \) is a function of \( \alpha_2 \) for \( 0 \leq \alpha_2 < 2\pi \), namely

\[ I_2 = h - f + 9f^2 \sin 3\alpha_2. \]

The manifold is thus diffeomorphic to a product of circles, hence to a 2-torus, as required by the Liouville-Arnold theorem.

(e) (8 points) We use the separability to introduce action-angle coordinates on the manifold \( M_{h,f} \). We have

\[ J_1 = \frac{1}{2\pi} \oint I_1 d\alpha_1 = f, \quad J_2 = \frac{1}{2\pi} \oint I_2 d\alpha_2 = h - f. \]

The generating function for the CT to action-angle coordinates is

\[ S(\alpha, J) = \int I_1(\alpha_1, J) d\alpha_1 + \int I_2(\alpha_2, J) d\alpha_2 = J_1\alpha_1 + J_2\alpha_2 - 3J_1^2 \cos 3\alpha_2. \]

and so we get

\[ \theta_1 = \frac{\partial S}{\partial J_1} = \alpha_1 - 6f \cos 3\alpha_2, \]

\[ \theta_2 = \frac{\partial S}{\partial J_2} = \alpha_2. \]
2. Spherical Charged Elastic Solid in a central E Field

A thin spherical solid shell with isotropic elasticity has an outer radius $R_2$ and an inner radius of $R_1$. It has a constant charge density $\rho$. In the center of the sphere is a point charge of charge $Q$ opposite to and much larger than the total integrated charge in the shell. Ignore the field generated by the shell and the variation in the electric field (from $Q$) in the thin shell.

a) (5 pts) Write the equations for elastic equilibrium of the spherical shell.
b) (10 pts) Find the form of the stresses and strains (don’t solve).
c) (5 pts) What are the boundary conditions?

d) (5 pts) Find the stresses and strains in the spherical shell.
e) (5 pts) How do the radial and tangential stresses compare?
f) (3 pts) Show that the forces between the two half shells (from the charge $Q$) are balanced by the stress on an equatorial cut.
2. Spherical Charged Elastic Solid in E field

\[ U_r(r) \to V \to \nabla \cdot V = 0 \to V = \frac{1}{E(1-\sigma)} \int \frac{1}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) \right) = -A \]

where spherical coord

\[ D(V) = \frac{1}{E(1-\sigma)} \int \frac{1}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) \right) = -A \]

\[ U_r = -\frac{1}{4} \frac{\partial U}{\partial r} = \frac{A}{2} + \frac{B}{3} - \frac{2C}{r^3} \]

\[ U_{\theta \theta} = U_{\phi \phi} = U_{rr}/r = \frac{-A}{4} + \frac{B}{3} + \frac{C}{r^3} \]

\[ \sigma_{rr} = \frac{E}{(1+\sigma)} \left[ U_{rr} + \frac{1}{2r^2} \left( U_{rr} + U_{\theta \theta} + U_{\phi \phi} \right) \right] = \frac{E}{(1+\sigma)(1-\sigma)} \left[ U_{rr} + 2U_{rr} + 2(\lambda + \mu) \right] \]

\[ \sigma_{\theta \theta} = \frac{E}{(1+\sigma)(1-\sigma)} \left[ U_{\theta \theta} + 2U_{rr} + 2(\lambda + \mu) \right] \]

\[ \sigma_{\phi \phi} = \frac{E}{(1+\sigma)(1-\sigma)} \left[ U_{\phi \phi} + 2U_{rr} \right] \]

\[ \text{B.C.} \quad \sigma_{rr} = 0 \text{ at } R_1 + R_2 \text{ (}\sigma_{\theta \theta}\text{ can be finite !)} \]

\[ \text{d)} \quad \text{Take } \sigma = 0 \to \sigma_{rr} = 0 \quad \sigma_{\theta \theta} = 0 \quad \sigma_{\phi \phi} = 0 \quad \text{if very thin } \]

\[ \frac{-A}{2} + \frac{B}{3} + \frac{2C}{R^2} = 0 \quad \Rightarrow \quad C = \frac{AR^2}{6} \]

\[ \frac{-A}{2} + \frac{B}{3} - \frac{2AR}{R^2} = 0 \quad \Rightarrow \quad B = 2AR \]

\[ \sigma_{\theta \theta} = \frac{E}{(1+\sigma)(1-\sigma)} \left( \frac{AR^2}{6} + 2AR + AR \right) = \frac{EAR}{2} \quad A = \frac{\rho Q}{2\pi \varepsilon_0 R^2} \]

\[ \frac{ER}{2} = \frac{\rho Q}{2\pi \varepsilon_0 R^2} \]
f) Thickness $T = R_2 - R_1$

Force down on half shell

\[ \text{Component along} \ \frac{2 \pi \rho \sin \theta \cos \theta}{4 \pi \epsilon_0 R^2} \int_0^{2 \pi} \sin \theta \cos \theta \, d\theta \]

\[ = \frac{2 \pi \rho \pi R^2}{4 \pi \epsilon_0} \left[ \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \]

\[ = \frac{2 \pi \rho \pi R^2}{8 \pi \epsilon_0} \text{Area of rim at equator} \]