Dynamics 2012 Prelim Exam

Instructions

i) This is a no book exam. You have 3 hours to complete the exam.

ii) Do all problems, using separate booklets for each problem.

iii) Please show your work: credit will not be given otherwise!

1. [35 points] Consider the complex transformation of coordinates in phase space,

$$Q(q,p) = \frac{mwq + ip}{\sqrt{2mw}}, \qquad P(q,p) = i\,\frac{mwq - ip}{\sqrt{2mw}} = i\,Q^* \tag{1}$$

a) Verify that this transformation is canonical. [5 points]

b) Find the generating function of first type $F_1(q,Q)$. [10 points]

c) Consider the one-dimensional harmonic oscillator with mass m and frequency w. Find the new Hamiltonian in the new canonical variables and solve the Hamilton equations in these variables. [10 points]

d) Consider now the extension of the transformation to a time-dependent one,

$$Q(q,p,t) = \frac{mwq + ip}{\sqrt{2mw}}\,\mathrm{e}^{iwt}, \qquad P(q,p,t) = i\,\frac{mwq - ip}{\sqrt{2mw}}\,\mathrm{e}^{-iwt} = i\,Q^* \tag{2}$$

Repeat part c). [10 points]

2. [30 points] Consider a weakly perturbed one-dimensional harmonic oscillator with perturbation Hamiltonian ($\epsilon \ll 1$)

$$\Delta H(q,p) \equiv \epsilon\; (\alpha\, q^4 - \beta\, q\, p^2) \tag{3}$$

a) Write down the full Hamiltonian in terms of the *unperturbed* action-angle variables. [10 points]

b) Use canonical perturbation theory to compute the correction to the energy to first-order in ϵ. [10 points]

c) Use canonical perturbation theory to compute the correction to the oscillation frequency to first-order in ϵ. [10 points]

3. [20 points] An ideal incompressible fluid is contained between two spheres of radii a and b ($b > a$). The outer sphere is at rest while the inner sphere moves with speed u.

a) Find the velocity potential at the time when the spheres are concentric. [15 points]

b) Take the limit $b \to \infty$ and show that the resulting field is that of a dipole with dipole moment $\mu = 2\pi u a^3$. [5 points]

4. [15 points] Consider the Couette stationary flow of a viscous incompressible fluid between concentric cylinders. The internal cylinder of radius a is at rest, while the outer cylinder of radius b rotates about the z-axis with constant angular velocity Ω.

a) write down the relevant Navier-Stokes equations for this (laminar) flow. [5 points]

b) Calculate the velocity field everywhere in the fluid. [5 points]

c) Calculate the viscous stress tensor everywhere in the fluid. [5 points]

Hint: you may need to use that solutions of differential equations of the form $\sum_n x^n (\frac{d^n y}{dx^n}) = 0$ are power-laws, i.e. $y \propto x^\alpha$. The Laplacian in cylindrical coordinates r, ϕ, z reads,

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \psi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{4}$$