New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2011

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.
4. Show ALL your work.

Part I: Classical Dynamics

Problem 1 (33 pts)

A particle moves on the x axis with Hamiltonian

$$H(x, p_x) = \frac{p_x^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

(a) (5 pts) Write down the equations of motion of the system in phase space.

(b) (5 pts) Locate all equilibrium points of the system, determining which are stable and which are unstable.

(c) (8 pts) Draw a phase portrait of the system, including sketches of (i) equilibrium points, (ii) oscillatory orbits, (iii) orbits from $\pm\infty$ to $\pm\infty$, (iv) orbits from $\pm\infty$ to $\mp\infty$, (v) separatrices.

(d) (5 pts) A particle is launched at $x = 0$ with initial velocity $1/\sqrt{2}$. Show that the particle travels to the right, steadily slowing down and coming to a stop after time T, which could be infinite. Obtain an expression for T as a definite integral and determine whether it is finite or infinite.

(e) (5 pts) We make the following transformation of phase space coordinates:

$$(x, p_x) \mapsto (\theta, J),$$

$$x = \sqrt{2J}\sin\theta, \qquad p_x = \sqrt{2J}\cos\theta.$$

Prove that this is a canonical transformation and calculate the new Hamiltonian, $\hat{H}(\theta, J)$.

(f) (5 pts) To study small oscillations about stable equilibrium, write $\hat{H}$ as

$$\hat{H}(\theta, J) = \hat{H}_0(J) + \epsilon \hat{H}_1(\theta, J),$$

where ϵ is a perturbation parameter which is to be set equal to 1 at the end. In first-order canonical perturbation theory, we make another canonical transformation, to coordinates $(\bar{\theta}, \bar{J})$, such that the new Hamiltonian $\bar{H}$ is a function only of $\bar{J}$, with corrections of order ϵ^2. Calculate $\bar{H}$ and the oscillation frequency, correct to first order in ϵ. Note: it is not necessary to derive explicit formulas for the coordinate transformation.

Problem 2 (34 pts)

A dynamical system with two angular degrees of freedom has the Hamiltonian

$$H(\phi_1, \phi_2, L_1, L_2) = L_1 + \frac{5}{3}L_2 + (3L_1 + 2L_2)^2 \sin(2\phi_1 - 3\phi_2).$$

(a) (5 pts) Find a linear combination F of L_1 and L_2 which commutes, in the sense of Poisson brackets, with H. Show that F is a constant of the motion.

(b) (5 pts) Show that F and H are functionally independent (i.e their phase-space gradient vectors are linearly independent) everywhere in the phase space.

(c) (5 pts) What do the statements of (a) and (b) imply about the topological structure of a compact, connected phase-space manifold $M_{h,f}$ on which H and F take the respective values h and f? Note: this is a consequence of the Liouville-Arnold theorem.

(d) (6 points) We make a canonical transformation

$$(\phi_1, \phi_2, L_1, L_2) \mapsto (\alpha_1, \alpha_2, I_1, I_2)$$

such that

$$\alpha_1 = \phi_1, \qquad \alpha_2 = \phi_2 - \frac{2}{3}\phi_1, \qquad I_1 \propto F.$$

Construct a generating function for this transformation using the first two of the equations, imposing the third as a constraint. Calculate I_2 as a function of ϕ_1, ϕ_2, L_1, L_2. Show that the new Hamiltonian, $\bar{H}(\alpha_1, \alpha_2, I_1, I_2)$, takes the form

$$\bar{H}(\alpha_1, \alpha_2, I_1, I_2) = I_1 + I_2 - 9I_1^2 \sin 3\alpha_2.$$

(d) (5 points) Verify that the system is separable with respect to the coordinates $\alpha_1, \alpha_2, I_1, I_2$. Verify directly the topological structure of $M_{h,f}$ required by the Liouville-Arnold theorem.

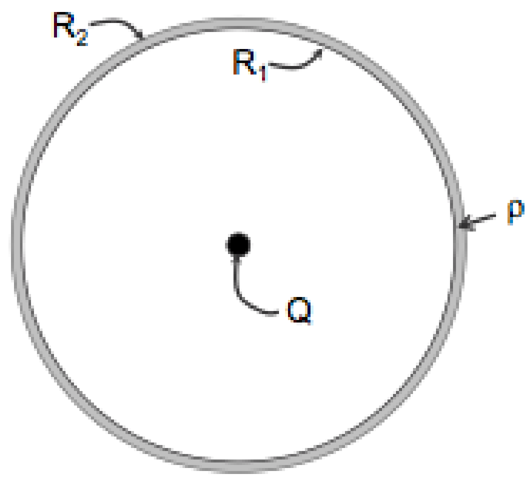
(e) (8 points) Introduce action-angle coordinates on the manifold $M_{h,f}$. Hint: first exploit the separability to construct the actions; then construct a suitable generating function.

⟨ Part II on next page ⟩

Part II : Continuum Mechanics

Problem 3 (33 pts)

Spherical Charged Elastic Solid in a central E Field



A thin spherical solid shell with isotropic elasticity has an outer radius R_2 and an inner radius of R_1. It has a constant charge density ρ. In the center of the sphere is a point charge of charge Q opposite to and much larger than the total integrated charge in the shell. Ignore the field generated by the shell and the variation in the electric field (from Q) in the thin shell.

a) (5 pts) Write the equations for elastic equilibrium of the spherical shell.
b) (10 pts) Find the form of the stresses and strains (don't solve).
c) (5 pts) What are the boundary conditions?

For the rest of the problem take the Poisson ratio as zero. (Also note that if Y(x) and Y(x+dx) are both zero then Y'(x)=0.)

d) (5 pts) Find the stresses and strains in the spherical shell.
e) (5 pts) How do the radial and tangential stresses compare?
f) (3 pts) Show that the forces between the two half shells (from the charge Q) are balanced by the stress on an equatorial cut.