New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2006

READ INSTRUCTIONS CAREFULLY

1. ANSWER TWO OF THE THREE PROBLEMS.

2. You have 3 hours to complete the examination.

3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.

4. Show ALL your work.
Problem I (50 pts)

Three point particles, with masses $m$, $3m$, and $2m$, are constrained to remain on a straight line, with neighboring masses connected by springs of unstretched length $a$ and force constant $6m\omega^2$. The first particle is attached to a piston and forced to oscillate as $F \sin \omega t$.

(a) (10 pts) Choose suitable generalized coordinates and write down the Lagrangian for this system.

(b) (5 pts) Derive the equations of motion.

(c) (20 pts) Find the most general solution of the equations for non-resonant $\omega$.

(d) (15 pts) Find the 2 values of $\omega$ for which the solution contains an oscillatory contribution whose amplitude grows with time. Calculate this contribution in one of the cases.

Problem II (50 pts)

A particle of mass $m$ moves along a straight line with potential energy $V(x) = \frac{1}{2}kx^2 + \epsilon x^4$, where $x$ is the particle’s position. Throughout this question you are to regard $\epsilon x^4$ as a small perturbation on the harmonic oscillator potential, and keep only terms up to first order in $\epsilon$. You may want to use some integral formulas provided at the end of the exam.

(a) (10 pts) Write down the Hamiltonian $H(x, p)$ for the system. Is $H(x, p)$ conserved? (Explain)

Is $H(x, p)$ equal to the total energy? (Explain)

(b) (10 pts) The particle is initially set in motion with total energy $E$. Draw a graph of the trajectory $C$ in phase space.

(c) (15 pts) Let $J$ be the area enclosed by $C$. Calculate $J$ as a function of $E$(remember: always calculate up to first order in $\epsilon$). Calculate the oscillation period as a function of $J$.

(d) (10 pts) We wish to find a canonical transformation, from variables $x, p$ to a new canonical pair $\theta, J$, where $J$ is the phase-space area of part (c). Without explicitly calculating the transformation equations, explain how they can be found. Hint: writing $p$ as a function of $x, J$ follows easily from preceding sections.

(e) (5 pts) With the new canonical variables of (d), what is the new Hamiltonian? What are the equations of motion for $\theta, J$?
**Problem III (50 pts)**

A particle of mass $m$ is constrained to move without friction on the surface of a sphere of radius $a$, in the presence of a uniform gravitational field (acceleration $g$).

(a) (10 pts) Using ordinary spherical coordinates $\theta$ and $\phi$, calculate the Lagrangian and Hamiltonian of the system.

(b) (10 pts) Find two independent conserved quantities. Prove that they are constant in time using Hamilton's equations of motion.

(c) (15 pts) Use the Hamilton-Jacobi method to obtain new canonical coordinates $Q_1$ and $Q_2$ which are constants of the motion. Identify $Q_1$ and $Q_2$ in terms of the conserved quantities of part (b).

(d) (15 pts) In order for $Q_1$ and $Q_2$ to serve as legitimate canonical coordinates, they must have vanishing Poisson bracket and be functionally independent (i.e. at each point, the coordinate curves for $Q_1$ and $Q_2$ must intersect transversally). Where in the phase space does at least one of these conditions fail? Describe the corresponding particle motion on the sphere.

**Integral formulas**

\[
\int_0^1 (1 - x^2)^{\frac{1}{2}} dx = \frac{\pi}{4}
\]

\[
\int_0^1 x^n(1 - x^2)^{-\frac{1}{2}} dx = \frac{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)}{n \Gamma\left(\frac{n}{2}\right)}
\]
Solution to Problem 1

\[ x_i = F \sin \omega t \]

(a) Let \( x_1, x_2, x_3 \) be the coordinates of the 3 particles. Introduce

\[
\begin{align*}
\delta x_2 &= x_2 - F \sin \omega t - a \\
\delta x_3 &= x_3 - x_2 - a
\end{align*}
\]

Then

\[
L(\delta x_1, \delta x_2, \delta x_3, t) = \text{Kinetic Energy } - \text{Potential Energy}
\]

\[
L = \frac{m}{2} \left[ F \omega^2 \cos \omega t \right.
\]

\[
+ 3 (\dot{\delta x}_2 + F \omega \cos \omega t)^2
\]

\[
+ 2 (\dot{\delta x}_2 + \dot{\delta x}_3 + F \omega \cos \omega t)^2
\]

\[
- 6 \gamma^2 (\dot{\delta x}_2 + \dot{\delta x}_3)^2 \left]
\right.
\]

(b) \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\delta x}_k} \right) - \frac{\partial L}{\partial \delta x_k} = 0 \quad , \quad k = 2, 3 \]

\[
\frac{d}{dt} \left( 6 (\dot{\delta x}_2 + F \omega \cos \omega t) + 4 (\dot{\delta x}_2 + \dot{\delta x}_3 + F \omega \cos \omega t) \right) + 12 \gamma^2 \delta x_2 = 0
\]

\[
\frac{d}{dt} \left( 4 (\dot{\delta x}_2 + \dot{\delta x}_3 + F \omega \cos \omega t) \right) + 12 \gamma^2 \delta x_2 = 0
\]

\[
\begin{align*}
\ddot{\delta x}_2 + 2 \gamma^2 \delta x_3 + 6 \gamma^2 \delta x_2 &= -5 F \omega^2 \sin \omega t \\
\ddot{\delta x}_3 + 2 \gamma^2 \delta x_3 + 8 \gamma^2 \delta x_2 &= F \omega^2 \sin \omega t
\end{align*}
\]

(c) (i) Particular solution from Ansatz \( \left( \begin{array}{c} \delta x_2 \\ \delta x_3 \end{array} \right) = \left( \begin{array}{c} A \\ B \end{array} \right) \sin \omega t \)

\[ M_0 \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{cc} 6 \gamma^2 - 5 \omega^2 & -2 \omega^2 \\ -2 \omega^2 & 3 \omega^2 - \omega^2 \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right) = F \omega \left( \begin{array}{c} 5 \\ 1 \end{array} \right) \]


\[ \text{det } M = (6v^2 - 5\omega^2)(3v^2 - \omega^2) - 2\omega^4 = 3\omega^4 - 21v^2\omega^2 + 18v^4 = 3(\omega^2 - v^2)(\omega^2 - 3v^2) \]

Resonance condition: \( \omega^2 = v^2 \) or \( 3v^2 \)

For \( \omega^2 \neq v^2, 3v^2 \),

\[
\begin{pmatrix} A \\ B \end{pmatrix} = \frac{F_0^2}{\text{det } M} \begin{pmatrix} 3v^2 - \omega^2 & 2\omega^2 \\ \omega^2 & 6v^2 - 5\omega^2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}
\]

\[
= \frac{F_0^2}{3(\omega^2 - v^2)(\omega^2 - 3v^2)} \begin{pmatrix} 15v^2 - 3\omega^2 \\ 6v^2 \end{pmatrix}
\]

\[
= \frac{F_0^2}{(\omega^2 - v^2)(\omega^2 - 3v^2)} \begin{pmatrix} 5v^2 - \omega^2 \\ 2v^2 \end{pmatrix}
\]

(a) Resonant solution for \( \omega = v \)

Ansatz for particular solution (the homogeneous eqn. solutions are the same as in (c)):

\[
\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} G \\ H \end{pmatrix} \sin \omega t + \begin{pmatrix} M \\ N \end{pmatrix} \cos \omega t
\]

To avoid terms in the equations \( \propto \cos \omega t \), we must have

\[
\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = 0 \Rightarrow M = 2N
\]

The remaining terms are all proportional to \( \sin \omega t \), giving

\[
v^2 \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} - 2vn \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} -v^2 \\ -v^2 \end{pmatrix}
\]
Solutions to Dynamics Prelim, Problems 2 and 3

July 13, 2006

Problem 2

(a)

\[ L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 - \epsilon x^4 \]

\[ H(x, p) = \frac{p^2}{2m} + \frac{k}{2} x^2 + \epsilon x^4 \]

\[ \frac{\partial H}{\partial \dot{t}} = 0 \implies \dot{H} = 0. \]

Kinetic energy \( K \) homogeneous quadratic in \( \dot{x} \) implies \( H = K + V \).

(b) The curve \( C \) is a perturbed ellipse with equation

\[ \frac{p^2}{2m} + \frac{k}{2} x^2 + \epsilon x^4 = E. \]

(c) The area inside \( C \), using reflection symmetry, is

\[ J = 4\sqrt{2m} \int_0^{x_{\text{max}}(\epsilon)} p(E, x) \, dx, \]

where

\[ p(E, x) = \sqrt{E - \frac{k}{2} x^2 - \epsilon x^4}, \]

\[ x_{\text{max}}(\epsilon) = \text{root of } E - \frac{k}{2} x^2 - \epsilon x^4 = \sqrt{\frac{2E}{k}} \left( 1 - \frac{2E}{k} \epsilon \right). \]

Expanding \( J \) to first order in \( \epsilon \),

\[ J = 4\sqrt{2m} \int_0^{x_{\text{max}}(0)} \sqrt{E - \frac{k}{2} x^2} \, dx - 2\sqrt{2m} \epsilon \int_0^{x_{\text{max}}(0)} \frac{x^4 \, dx}{\sqrt{E - \frac{k}{2} x^2}} \]
with \( x_{\max}(0) = \sqrt{\frac{2E}{k}} \). Using the integrals provided at the end of the exam,

\[
J = 2\pi E \sqrt{\frac{m}{k}} - 3\pi \sqrt{\frac{m}{k} E^2 k^2},
\]

giving

\[
E = \frac{1}{2\pi} \sqrt{\frac{k}{m} J + \frac{3\epsilon}{4\pi} \sqrt{\frac{k}{m}} k^2 J^2}.
\]

The oscillation period is then

\[
\tau = \frac{1}{\frac{dE}{dJ}} = 2\pi \sqrt{\frac{m}{k} (1 + \frac{3\epsilon J}{k^2})^{-1}} = 2\pi \sqrt{\frac{m}{k} (1 - \frac{3\epsilon J}{k^2})^{-1}}.
\]

In terms of the energy, this is

\[
\tau = \tau_0 (1 - \frac{3\epsilon \tau_0}{k^2} E), \quad \tau_0 = 2\pi \sqrt{\frac{m}{k}}.
\]

(d)

\[
p(J, x) = \sqrt{E(J) - \frac{k}{2} x^2 - \epsilon x^4},
\]

with \( E(J) \) from part (c). This suggests introducing a generating function of type \( F_2 \) (old position variable, new momentum variable), i.e.

\[
F(x, J) = \int p(J, x) dx,
\]

which is designed to give

\[
p(J, x) = \frac{\partial F}{\partial x}.
\]

The new position variable, canonically conjugate to \( J \), is then

\[
\theta = \frac{\partial F}{\partial J}.
\]

(e) The new Hamiltonian is the old one written in terms of the new variables, plus \( \frac{\partial F}{\partial \theta} \) (the latter vanishes here). Hence the new Hamiltonian is just \( E(J) \). Hamilton's eqns. for the new Hamiltonian are

\[
\dot{J} = 0, \quad \dot{\theta} = \frac{dE}{dJ} = 2\pi \sqrt{\frac{m}{k}} \left( 1 + \frac{3\epsilon J}{k^2} \right).
\]
Problem 3

(a) The $\theta = 0$ axis is chosen opposite to the direction of the gravitational field, i.e. "upward".

$$L = \frac{ma^2}{2} \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) - mga \cos \theta$$

Introducing conjugate momenta

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2 \dot{\theta},$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi},$$

we get for the Hamiltonian,

$$H = \frac{1}{2ma^2} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + mga \cos \theta.$$ 

(b) Hamilton’s equations yield

$$\dot{\hat{p}}_\theta = -\frac{\partial H}{\partial \theta},$$

$$\dot{\hat{p}}_\phi = -\frac{\partial H}{\partial \phi} = 0.$$ 

Thus $p_\phi$, the vertical component of angular momentum is conserved. So is the total energy, $H$ itself, since, from Hamilton’s equations,

$$\dot{H} = \frac{\partial H}{\partial \theta} \dot{\theta} + \frac{\partial H}{\partial \phi} \dot{\phi} + \frac{\partial H}{\partial p_\theta} \dot{p}_\theta + \frac{\partial H}{\partial p_\phi} \dot{p}_\phi = 0.$$ 

(c) Introduce the H-J generating fcn. $S(\theta, \phi, Q_1, Q_2, t)$ satisfying

$$\frac{1}{2ma^2} \left[ \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] + mga \cos \theta + \frac{\partial S}{\partial t} = 0$$

In standard fashion we separate out the $t$ dependence:

$$S(\theta, \phi, Q_1, Q_2, t) = W(\theta, \phi, Q_1, Q_2) - Q_1 t$$
\[
\frac{1}{2ma^2} \left[ \left( \frac{\partial W}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial W}{\partial \phi} \right)^2 \right] + mga \cos \theta = Q_1
\]

Now we separate the \( \theta \) and \( \phi \) dependence, setting

\[
W = W_\theta(\theta, Q_1, Q_2) + W_\phi(\phi, Q_1, Q_2).
\]

so that

\[
\frac{1}{2ma^2} \left[ \left( \frac{dW_\theta}{d\theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{dW_\phi}{d\phi} \right)^2 \right] + mga \cos \theta = Q_1.
\]

The differential equation separates into the pair

\[
\frac{dW_\phi}{d\phi} = Q_2, \quad \left( \frac{dW_\theta}{d\theta} \right)^2 = 2ma^2Q_1 - \frac{Q_2^2}{\sin^2 \theta} - 2m^2ga^3 \cos \theta
\]

with the solutions

\[
W_\phi = Q_2 \phi, \quad W_\theta = \int \sqrt{2ma^2Q_1 - \frac{Q_2^2}{\sin^2 \theta} - 2m^2ga^3 \cos \theta} \, d\theta.
\]

The constants of the motion \( Q_1 \) and \( Q_2 \) are, respectively, the total energy and the \( z \)-component of the angular momentum.

(d) The matrix of partial derivatives is

\[
\begin{pmatrix}
\frac{\partial Q_1}{\partial \theta} & \frac{\partial Q_1}{\partial \phi} & \frac{\partial Q_1}{\partial p_\theta} & \frac{\partial Q_1}{\partial p_\phi} \\
\frac{\partial Q_2}{\partial \theta} & \frac{\partial Q_2}{\partial \phi} & \frac{\partial Q_2}{\partial p_\theta} & \frac{\partial Q_2}{\partial p_\phi}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial Q_1}{\partial \theta} & 0 & \frac{p_\phi}{ma^2} & -\frac{p_\phi}{ma^2 \sin^2 \theta} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Functional dependence requires that the 4D phase-space gradient vectors of \( Q_1(\theta, \phi, p_\theta, p_\phi) \) and \( Q_2(\theta, \phi, p_\theta, p_\phi) \) be proportional. This happens only when

\[
p_\theta = 0, \quad \dot{p}_\phi = 0,
\]

i.e. for an orbit which always stays at the same height. The new coordinates are functionally independent everywhere else.

The Poisson bracket of \( Q_1 \) with \( Q_2 \) is

\[
\left\{ \frac{\partial Q_1}{\partial \theta}, \frac{\partial Q_2}{\partial \phi} \right\} - \left\{ \frac{\partial Q_2}{\partial \theta}, \frac{\partial Q_1}{\partial \phi} \right\} + \left\{ \frac{\partial Q_1}{\partial \phi}, \frac{\partial Q_2}{\partial p_\phi} \right\} - \left\{ \frac{\partial Q_2}{\partial \phi}, \frac{\partial Q_1}{\partial p_\phi} \right\}
\]

. Consulting the matrix of partial derivatives, we see that this vanishes everywhere.
READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.

2. You have 3 hours to complete the examination.

3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.

4. Show ALL your work.
Problem I (30 pts)

A particle of mass \( m \) moves on a frictionless hyperboloid of revolution under the influence of a uniform vertical gravitational field (acceleration \( g \)). The surface may be represented parametrically in 3-dimensional space as

\[
\begin{align*}
x &= a \sinh \xi \cos \phi, \\
y &= a \sinh \xi \sin \phi, \\
z &= a \cosh \xi,
\end{align*}
\]

with \( 0 \leq \xi < \infty, 0 \leq \phi < 2\pi \) and \( a \) a positive constant.

(a) (5 pts) In terms of the coordinates \( \xi \) and \( \phi \) and their time derivatives, write down the Lagrangian of the system. Simplify the notation by choosing units such that \( ma^2 = 1 \) and writing \( \gamma = g/a \).

(b) (5 pts) Calculate the momenta \( p_\xi \) and \( p_\phi \) conjugate to \( \xi \) and \( \phi \) and calculate the Hamiltonian \( H \) of the system.

(c) (5 pts) Show that \( p_\phi \) and \( H \) are constants of the motion. What is the dimensionality and topological structure of a manifold \( M_{L,E} \) on which \( p_\phi \) and \( H \) take on specific values \( L \) and \( E \), respectively? You may assume that the two quantities are functionally independent and that the manifold is bounded and connected.

(d) (5 pts) Show that on \( M_{L,E} \) the \( \xi \) motion is governed by a differential equation of the form

\[
\dot{u}^2 = \frac{f(u)}{2u^2 - 1},
\]

where \( f(u) \) is a cubic polynomial in the variable \( u = \cosh \xi \). Sketch the function \( f(u) \) in the “physical” region \( u \geq 1 \). Show that all orbits are bounded.

(e) (5 pts) In terms of the function \( f(u) \), calculate the period \( \tau \) of the \( \xi \) motion and the amount \( \Delta \phi \) by which the azimuthal angle \( \phi \) increases during time \( \tau \).

(f) (5 pts) Describe qualitatively the motion of a particle released from rest at time zero \( (\xi(0) = \xi_0 > 0, \phi(0) = \xi(0) = \phi(0) = 0) \). Sketch the graphs of \( \xi \) and \( \phi \) as functions of time.

Problem II (36 pts)

A two-dimensional nonlinear oscillator has the Hamiltonian

\[
H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + q_1^2) + \frac{1}{2}(p_2^2 + q_2^2) + \epsilon p_1 q_2.
\]

(a) (10 pts) Introduce, by means of a (complex) canonical transformation, new variables \( Q_1, Q_2, P_1, P_2 \) such that

\[
Q_k = \frac{1}{\sqrt{2}} \left(q_k + \frac{p_k}{i\omega_k}\right), \quad k = 1, 2.
\]
2. $P_k$ is a homogeneous linear function of $q_k$ and $p_k$.

3. The new Hamiltonian $K(Q_1, Q_2, P_1, P_2)$ takes the form

$$K = i\omega_1 Q_1 P_1 + i\omega_2 Q_2 P_2 + \epsilon V(Q_1, Q_2, P_1, P_2)$$

Calculate $P_k$.

(b) (10 pts) We now introduce, via a second canonical transformation, new canonical variables $\theta_k, J_k$, $k = 1, 2$, such that

$$J_k = iQ_k P_k, \quad k = 1, 2,$$

and each $\theta_k$ is an angular variable, i.e. a real number in the interval $[0, 2\pi)$. Express $q_k, p_k$, $k = 1, 2$ in terms of the transformed variables. Calculate the new Hamiltonian.

(c) (10 pts) By means of a final canonical transformation, introduce the perturbative action-angle variables $\bar{\theta}_k, \bar{J}_k$, $k = 1, 2$, such that the new Hamiltonian takes the form

$$\tilde{H}(\bar{\theta}_1, \bar{\theta}_2, \bar{J}_1, \bar{J}_2) = \tilde{H}_0(\bar{J}_1, \bar{J}_2) + O(\epsilon^2).$$

Calculate the oscillation periods to first order in $\epsilon$.

(d) (6 pts) It would seem that the technique of part (c) could be repeated again and again to remove the angular dependence to any desired order. Does this imply that the model is integrable? Explain briefly.

**Problem III (34 pts)**

A growing spherical bubble is at the origin of a system containing a viscous fluid.

(a) (6 pts) Assuming the flow is radial write the appropriate Navier-Stokes equation for the general case of a compressible fluid. Find the general expression for the stresses in the system.

Now assume that the system is incompressible.

(b) (4 pts) What is the radial form of the velocity field?

(c) (5 pts) What is the appropriate Navier-Stokes Equation for incompressible flow?

(d) (5 pts) If there is no explicit time dependence to the flow, aside from the growth of the bubble wall, find the radial variation of pressure in the fluid.

Take the velocity at a radius $b$ outside the initial bubble radius $R(t=0)$ as $v_0$

(e) (5 pts) When the bubble radius is $R$, what is the viscous dissipation in the system.

We want to look at the bubble wall and its motion.

(f) (4 pts) Find an expression for the position and velocity of the bubble wall as a function of time given the steady state velocity fields used above.

(g) (5 pts) What is the pressure difference between the gas in the bubble and the pressure in the adjacent fluid in the case of finite viscosity and zero surface tension? What is the pressure in the bubble? Is it time dependent?
Dynamics Prelim, Solutions to Probs. 1 + 2

1. (a) \[
x' = a \left( \cosh^2 \phi - \tanh \phi \sinh \phi \right)
\]
\[
y' = a \left( \cosh \phi \sinh \phi + \sinh \phi \cosh \phi \right)
\]
\[
z' = a \sinh \phi
\]
\[
|\mathbf{v}|^2 = x'^2 + y'^2 + z'^2 = a^2 \left[ (\cosh^2 \phi + \sinh^2 \phi) \dot{\phi}^2 + \sinh^2 \phi \dot{\phi}^2 \right]
\]
\[
L = \frac{1}{2} (\cosh^2 \phi + \sinh^2 \phi) \dot{\phi}^2 + \frac{1}{2} \sinh^2 \phi \dot{\phi}^2 - \lambda \cosh \phi
\]
where we have set
\[
\omega^2 = 1 \quad \text{and} \quad \lambda = \frac{a}{\omega^2}.
\]

(b) \[
P_\phi = \frac{\partial L}{\partial \dot{\phi}} = \sinh \phi \dot{\phi}, \quad P_\phi = \frac{\partial L}{\partial \dot{\phi}^2} = (\cosh^2 \phi + \sinh^2 \phi) \dot{\phi}
\]
\[
H = \phi P_\phi + \frac{1}{2} \frac{P_\phi^2}{\cosh^2 \phi + \sinh^2 \phi} + \frac{1}{2} \frac{P_\phi^2}{\sinh \phi} + \lambda \cosh \phi
\]

(c) \[
P_\phi = -\frac{\partial H}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad P_\phi, H \text{ are constant}
\]
\[
H = \frac{\partial H}{\partial t} = 0
\]

M_{\lambda, \phi} \text{ is a 2-dimensional torus (by Poincaré-Arnold Thm), provided it is connected and compact, and } E, L \text{ are functionally independent.}

(d) \[
E = \frac{1}{2} (\cosh^2 \phi + \sinh^2 \phi) \dot{\phi}^2 + \frac{1}{2} \frac{\dot{\phi}^2}{\sinh \phi} + \lambda \cosh \phi
\]
(energy conservation)
Introduce \( u = \cosh \xi \), \( \dot{u} = \sinh \xi \).

so that
\[
\xi^2 = \frac{\dot{u}^2}{u^2 - 1}
\]

Then, solving the energy conservation equation for \( u^2 \),
\[
\dot{u}^2 = 2(E - Ku)^2 - L^2 = \frac{f(u)}{2u^2 - 1}
\]

\[
f(u) = -L^2
\]

\[
f(u) \approx -2Ku
\]

for \( u \to +\infty \)

\( u \) oscillates periodically between \( u_1 \) and \( u_2 \) (2nd and 3rd zeros of \( f(u) \)), hence is bounded.

\[
three{d\xi}{dt} = \sqrt{\frac{f(u)}{2u^2 - 1}} \quad \Rightarrow \quad t = 2\int \frac{\sqrt{2u^2 - 1}}{f(u)} \, du
\]

\[
\frac{d\phi}{dt} = \frac{L}{u^2 - 1}
\]

\[
\frac{d\phi}{du} = \frac{d\phi}{dt} \cdot \frac{dt}{du} = \frac{L}{u^2 - 1} \sqrt{\frac{2u^2 - 1}{f(u)}} \quad \Rightarrow \quad \Delta \phi = 2L \int \frac{\sqrt{2u^2 - 1}}{u^2 - 1} \, du
\]

\( \phi = L - 0 \Rightarrow \phi = \frac{L}{\sinh \xi} = \frac{L}{-1 + u^2} = 0 \)

except at the bottom of the trajectory (\( \xi = 0 \), \( u = 1 \)) where \( \phi \) increases discontinuously by \( \pi \)
Motion is that of a "hyperbolic pendulum". Particle is instantaneously at rest at top of the swing \( t = 0, T, 2T, \ldots \) and has maximum speed at the bottom.

2. (a) \[ Q_k = \frac{\dot{x}}{\kappa} \left( \frac{\dot{x}}{\kappa} + \frac{P_k}{i\omega_k} \right) \]

\[ P_k = \kappa \dot{x} + \beta P_k \]

Canonical \( \Rightarrow \) \[ 1 = [Q_k, P_k] \] (Poisson bracket)

\[ = \frac{1}{\kappa} \left( \beta - \frac{\kappa}{i\omega_k} \right) \]

\[ \Rightarrow \quad P_k = \kappa \dot{x} + \left( \frac{\kappa}{i\omega_k} + \beta \right) P_k \]

Condition (3) \( \Rightarrow \) \[ \lambda = \frac{-\kappa \omega_k}{\sqrt{2}} \]

\[ P_k = \frac{1}{\kappa} \left( P_k - \kappa \dot{x} \right) \]

(b) \[ J_k = i Q_k P_k \]

\[ P_k = \frac{J_k}{i Q_k} \] suggests use of type \( F_k \) generating from

\[ = \frac{\partial}{\partial Q_k} F_k(Q_k, J_k) \]

\[ \Rightarrow \quad F_k = \frac{J_k}{i} \ln Q + G(J) \]
(b) cont.

\[ \beta_k = \frac{\partial F_k}{\partial Q_k} = \pm \frac{1}{2} \ln Q_k + G'(T_k) \]

We use the arbitrariness of \( G \) to eliminate the real part of \( \ln Q_k = \ln |Q_k| + i \text{Arg}(Q_k) \)

But \( |Q_k| = \sqrt{Q_k Q_k^*} = \sqrt{\frac{1}{2 \omega_k^2} \left( J_1^2 + \frac{\omega_k^4}{2} \right)} = \sqrt{\frac{1}{2 \omega_k^2}} \)

Hence

\[ G'(T_k) = -i \ln \sqrt{\frac{1}{2 \omega_k^4}} \]

\[ G(T_k) = -i \int \ln \sqrt{\frac{1}{2 \omega_k^4}} \, dT_k \]

With this choice,

\[ Q_k = \sqrt{\frac{1}{2 \omega_k^4}} e^{-i \theta_k} \]

\[ P_k = -i \omega_k Q_k^* = -i \sqrt{\frac{1}{2 \omega_k^4}} e^{-i \theta_k} \]

\[ p_k = \frac{1}{\sqrt{2}} (P_k + i \omega_k Q_k) = -\sqrt{\frac{1}{2 \omega_k^4}} \sin \theta_k \]

\[ q_k = \frac{1}{\sqrt{2}} (Q_k - \frac{P_k}{i \omega_k}) = \sqrt{\frac{1}{2 \omega_k^4}} \cos \theta_k \]

\[ H = \omega_1 T_1 + \omega_2 T_2 + \omega_4 T_1 T_2 \left( \frac{\omega_4}{\omega_1} \right) \sin^2 \theta \cos \theta \]

(c) Making a canonical transformation with generating function

\[ F_1(\theta_1, \theta_2, T_1, T_2) = \theta_1 T_1 + \theta_2 T_2 + \epsilon S(\theta_1, T_1, T_2) \]

we can eliminate all oscillatory terms in \( H \), being left with
\[ \bar{H} = \omega_1 \bar{J}_1 + \omega_2 \bar{J}_2 + 4e \bar{J}_1 \bar{J}_2 \left( \frac{\omega_1}{\omega_2} \right) \left( \sin^2 \theta \times \cos^2 \theta \right) \]
\[ + O(e^2) \]
\[ = \omega_1 \bar{J}_1 + \omega_2 \bar{J}_2 + e \bar{J}_1 \bar{J}_2 \left( \frac{\omega_1}{\omega_2} \right) + O(e^2) \]

\begin{align*}
\dot{\bar{\omega}}_1 &= \frac{\partial \bar{H}}{\partial \bar{J}_1} = \omega_1 + e \bar{J}_2 \left( \frac{\omega_1}{\omega_2} \right) = \omega_1 \left( 1 + \frac{e \bar{J}_2}{\omega_2} \right) \\
\dot{\bar{\omega}}_2 &= \frac{\partial \bar{H}}{\partial \bar{J}_2} = \omega_2 + e \bar{J}_1 \left( \frac{\omega_1}{\omega_2} \right) = \omega_2 \left( 1 + \frac{e \bar{J}_1}{\omega_2} \right) \\
\bar{\tau}_1 &= \frac{\bar{\tau}_1}{\omega_1} = \frac{2\pi}{\omega_1} \left( 1 - \frac{e \bar{J}_2}{\omega_2} \right) + O(e^2) \\
\bar{\tau}_2 &= \frac{\bar{\tau}_2}{\omega_2} = \frac{2\pi}{\omega_2} \left( 1 - \frac{e \bar{J}_1}{\omega_2} \right) + O(e^2) \\
\end{align*}

(a) The model is typically not integrable for 2 degrees of freedom. The perturbation series can be constructed to arbitrary order in \( e \), using the same technique as in (c). However, the series in \( e \) for \( \bar{H}, \bar{\omega}_1, \) etc. are not expected to converge.
III a) Only \( \omega r = \omega r \)

\[ \sigma_{rr} = -P + 2\gamma \frac{\partial \omega r}{\partial r} \quad \text{only stress in radial direction} \]

Navier-Stokes

\[ \frac{\partial \omega r}{\partial t} + \omega r \frac{\partial \omega r}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \gamma \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \omega r}{\partial r} \right) - \frac{2\omega r}{r^2} \right] \]

\[ \rightarrow \quad \text{b) Incompressible} \Rightarrow \frac{\partial \omega r}{\partial r} + 2\frac{\omega r}{r} = 0 \]

\[ \Rightarrow \omega r = \frac{\lambda}{r^2} \]

c) \[ \frac{\partial}{\partial r} \left( r \omega r \right) = \frac{\partial}{\partial r} \left( \frac{A}{r} \right) = - \frac{A}{r^2} \]

\[ \frac{\partial^2}{\partial r^2} (r \omega r) = \frac{2A}{r^3} \quad \frac{2\omega r}{r^2} = - \frac{2A}{r^4} \]

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r \omega r)}{\partial r} \right) = \frac{2A}{r^5} \]

\[ \frac{\gamma}{\rho} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r \omega r)}{\partial r} \right) - \frac{2\omega r}{r^2} \right] = 0 \]

Navier-Stokes Incompressible

\[ \frac{\partial \omega r}{\partial t} + \omega r \frac{\partial \omega r}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} \]

\[ \rightarrow \quad \text{Below} \quad \omega r = \frac{\lambda}{r^2} \]

d) \text{Steady state} \quad \frac{\partial \omega r}{\partial t} = 0 \quad \omega r \frac{\partial}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial r} \]

\[ \frac{\partial}{\partial r} \left( \frac{A}{r^2} \right) = - \frac{2A^2}{r^5} = -\frac{1}{\rho} \frac{\partial P}{\partial r} \]

\[ P - P_\infty = \int_{r}^{R} \frac{4\mu A^2}{r^5} dr = -2\frac{A^2}{r^4} \quad \Rightarrow \quad P = P_\infty - \frac{\rho A^2}{2r^4} \]
\[ v_0 = \frac{A}{b^2} \quad A = v_0 b^2 \]

e) \( \text{dissipation} = \frac{\partial E}{\partial t} = \frac{1}{2} \gamma \int (\dot{\gamma} r^2) \, dV \)

\[ \dot{\gamma} r = \frac{\partial v_r}{\partial r} = -2 \frac{A}{r^3} \]

\[ \frac{\partial E}{\partial t} = \frac{1}{2} \gamma \int_0^\infty \frac{4A^2}{r^6} 4\pi r^4 \, dr = 8\pi \gamma A^2 \int_0^1 \frac{1}{3} \frac{1}{r^3} \, dr = \frac{8\pi}{3} \gamma A^2 \]

\[ = \frac{8\pi}{3} \gamma \frac{A^2}{R^3} \]

f) \( v_R \frac{\partial R}{\partial t} = \frac{A}{R^2} \quad R^2 \frac{\partial R}{\partial t} = \Delta R \)

\[ R = \left( R_0^3 + 3 \Delta R \right)^{\frac{1}{3}} \]

g) \( \text{Equation stresses \textit{at} the bubble wall} \)

\( \text{Inside bubble} - P_i \)

\( \text{In fluid} \quad \sigma_{rr} = -P + 2\gamma \frac{d}{drr} \left|_R \right. \)

\[ P_i = P(1 + 3\gamma A) \frac{R}{R^3} \]

\[ = P_0 - \frac{\rho A^2}{2R} + 4\gamma A \frac{1}{R^3} \]

\( \text{Depends on Time!} \)
In spherical co-ordinates $r, \phi, \theta$ we have for the stress tensor

\[
\sigma_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r},
\]
\[
\sigma_{\phi\phi} = -p + 2\eta \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right),
\]
\[
\sigma_{\theta\theta} = -p + 2\eta \left( \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right),
\]
\[
\sigma_{r\theta} = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right),
\]
\[
\sigma_{r\phi} = \eta \left( \frac{\partial v_\phi}{\partial r} + \frac{1}{r} \frac{\partial v_r}{r \sin \theta} - \frac{v_\phi}{r} \right),
\]

(15.17)

while the equations of motion are

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_r^2 + v_\theta^2}{r}
\]

\[
= -\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \nu \left[ \frac{1}{r} \frac{\partial^2 (r v_r)}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} + \cot \theta \frac{\partial v_r}{\partial \theta} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} - \frac{2 v_r}{r^2} - \frac{2 \cot \theta}{r^2} + \frac{v_\theta}{r} \right],
\]

\[
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} - \frac{v_\theta^2}{r}
\]

\[
= -\frac{1}{\rho r} \frac{\partial \rho}{\partial \theta} + \nu \left[ \frac{1}{r} \frac{\partial^2 (r v_\theta)}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \cot \theta \frac{\partial v_\theta}{\partial \theta} - \frac{2 \cos \theta}{r^2} \frac{\partial v_\phi}{\partial \phi} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} - \frac{v_\theta}{r^2} \right],
\]

\[
\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi^2 \cot \theta}{r}
\]

\[
= -\frac{1}{\rho r \sin \theta} \frac{\partial \rho}{\partial \phi} + \nu \left[ \frac{1}{r} \frac{\partial^2 (r v_\phi)}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \cot \theta \frac{\partial v_\phi}{\partial \theta} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{2 \cos \theta}{r^2} \frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi}{r^2} \right].
\]

\[
\frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r \sin \theta} + \frac{\partial v_\phi}{r \sin \theta \sin \theta} + \frac{2 v_r}{r} + \frac{v_\phi \cot \theta}{r} = 0.
\]

(15.18)