Dynamics 2017 Prelim Exam

Instructions

i) This is a no book exam. You have 3 hours to complete the exam.

ii) Do all problems, using separate booklets for each problem.

iii) Please show your work: credit will not be given otherwise!

1. [25 points] Consider the complex transformation of coordinates in phase space,

$$Q(q,p) = \frac{mwq + ip}{\sqrt{2mw}}, \qquad P(q,p) = i\,\frac{mwq - ip}{\sqrt{2mw}} = i\,Q^* \tag{1}$$

a) Verify that this transformation is canonical. [5 points]

b) Find the generating function of first type $F_1(q,Q)$. [10 points]

c) Consider the one-dimensional harmonic oscillator with mass m and frequency w. Find the new Hamiltonian in the new canonical variables and solve the Hamilton equations in these variables. [10 points]

2. [20 points] A perturbed one-dimensional harmonic oscillator has the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(p^2 + q^2) + \epsilon\, q^2\, p^2 \tag{2}$$

a) Introduce action-angle variables of the unperturbed problem θ_0, J_0 by defining

$$q = \sqrt{2J_0}\cos\theta_0, \qquad p = -\sqrt{2J_0}\sin\theta_0, \tag{3}$$

and verify that this transformation is canonical. Find the transformed Hamiltonian. [10 points]

b) Find the correction to the energy to first order in ϵ. [5 points]

c) Find the correction to the frequency to first order in ϵ. [5 points]

The following integral may be useful:

$$\int \sin^2(x)\cos^2(x)\,dx = \frac{x}{8} - \frac{1}{32}\sin(4x) \tag{4}$$

3.[35 points] Consider two incompressible fluids of density ρ and ρ' in a uniform gravitational field $\mathbf{g} = -g\,\hat{z}$. In equilibrium, the fluid with density ρ occupies the region $-\infty < z < 0$, and the fluid with density ρ' is in the region $0 < z < h'$. The two fluids are separated by the xy plane at $z = 0$. When the fluids are perturbed, the flows are potential with waves that propagate in the x direction, ie. the velocity potentials are proportional to $\cos(kx - wt)$.

a) Write down appropriate solutions to the velocity potentials ϕ and ϕ', and assuming small perturbations (so quadratic terms can be neglected) show that continuity of the pressure at the interface between the fluids leads to [10 points]

$$g(\rho - \rho')\frac{\partial\phi}{\partial z}\Big|_{z=0} = \rho'\frac{\partial^2\phi'}{\partial t^2}\Big|_{z=0} - \rho\frac{\partial^2\phi}{\partial t^2}\Big|_{z=0} \tag{5}$$

b) Find the analogous boundary condition at the free surface of the upper fluid (you may assume the fluid is in contact with air at atmospheric pressure). [10 points]

c) Using a) and b) find the dispersion relation for the waves. Interpret physically what happens in the case that i) $\rho' > \rho$ ii) $w^2 = kg$. [15 points]

4.[20 points] A river of depth h_1 and width $y \gg h_1$ has a riverbed that makes an angle α with the horizontal. The water has viscosity η_1 and density ρ_1 (assume water to be incompressible). Assume stationary flows.

a) write down the relevant Navier-Stokes equations for this (laminar) flow. [10 points]

b) Calculate the velocity and pressure profile. [10 points]