Instructions

i) This is a no book exam. You have 3 hours to complete the exam.

ii) Do all problems, using separate booklets for each problem.

iii) Please show your work: credit will not be given otherwise!

1. [25 points] Consider the complex transformation of coordinates in phase space,

\[ Q(q,p) = \frac{mwq + ip}{\sqrt{2mw}}, \quad P(q,p) = i \frac{mwq - ip}{\sqrt{2mw}} = iQ^* \]  

a) Verify that this transformation is canonical. [5 points]

b) Find the generating function of first type \( F_1(q, Q) \). [10 points]

c) Consider the one-dimensional harmonic oscillator with mass \( m \) and frequency \( w \). Find the new Hamiltonian in the new canonical variables and solve the Hamilton equations in these variables. [10 points]

2. [20 points] A perturbed one-dimensional harmonic oscillator has the Hamiltonian

\[ \mathcal{H} = \frac{1}{2}(p^2 + q^2) + \epsilon q^2 p^2 \]
a) Introduce action-angle variables of the unperturbed problem $\theta_0, J_0$ by defining

$$q = \sqrt{2J_0} \cos \theta_0, \quad p = -\sqrt{2J_0} \sin \theta_0, \quad (3)$$

and verify that this transformation is canonical. Find the transformed Hamiltonian. [10 points]

b) Find the correction to the energy to first order in $\epsilon$. [5 points]

c) Find the correction to the frequency to first order in $\epsilon$. [5 points]

The following integral may be useful:

$$\int \sin^2(x) \cos^2(x) \, dx = \frac{x}{8} - \frac{1}{32} \sin(4x) \quad (4)$$

3. [35 points] Consider two incompressible fluids of density $\rho$ and $\rho'$ in a uniform gravitational field $\mathbf{g} = -g \hat{z}$. In equilibrium, the fluid with density $\rho$ occupies the region $-\infty < z < 0$, and the fluid with density $\rho'$ is in the region $0 < z < h'$. The two fluids are separated by the $xy$ plane at $z = 0$. When the fluids are perturbed, the flows are potential with waves that propagate in the $x$ direction, i.e., the velocity potentials are proportional to $\cos(kx - wt)$.

a) Write down appropriate solutions to the velocity potentials $\phi$ and $\phi'$, and assuming small perturbations (so quadratic terms can be neglected) show that continuity of the pressure at the interface between the fluids leads to [10 points]

$$g(\rho - \rho') \frac{\partial \phi}{\partial z} \bigg|_{z=0} = \rho \frac{\partial^2 \phi'}{\partial t^2} \bigg|_{z=0} - \rho' \frac{\partial^2 \phi}{\partial t^2} \bigg|_{z=0} \quad (5)$$

b) Find the analogous boundary condition at the free surface of the upper fluid (you may assume the fluid is in contact with air at atmospheric pressure). [10 points]

c) Using a) and b) find the dispersion relation for the waves. Interpret physically what happens in the case that i) $\rho' > \rho$ ii) $w^2 = kg$. [15 points]
4. **[20 points]** A river of depth $h_1$ and width $y \gg h_1$ has a riverbed that makes an angle $\alpha$ with the horizontal. The water has viscosity $\eta_1$ and density $\rho_1$ (assume water to be incompressible). Assume stationary flows.

a) write down the relevant Navier-Stokes equations for this (laminar) flow. [10 points]

b) Calculate the velocity and pressure profile. [10 points]