New York University

Department of Physics

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2014

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. On the front cover of each booklet write your identification number.
4. Show ALL your work.

Part I: Classical Dynamics

Problem 1 (33 pts)

A system with two degrees of freedom has canonical phase-space coordinates

$$\xi = (\phi_1, \phi_2, L_1, L_2), \qquad 0 \leq \phi_1, \phi_2 < 2\pi,$$

and Hamiltonian

$$H = \omega_1 L_1 + \omega_2 L_2 + \epsilon L_1 L_2 \cos(\phi_1 + 3\phi_2).$$

(a) (5 pts) Write down Hamilton's equations of motion for this system.

(b) (14 pts) Find a scalar function K (a linear combination of L_1 and L_2) which has zero Poisson bracket with H and is independent of H (as a function of ξ), except on a manifold $\mathcal{E}$ of dimension less than 4. Find $\mathcal{E}$. What does the Liouville-Arnol'd Therorem have to say about the topological structure of sub-manifolds on which H and K assume specific values?

(c) (14 pts) With the aid of a suitable generating function, make a canonical transformation to phase-space coordinates for which the system is separable. Show that it is indeed separable. What are H and K in the new canonical coordinates?

Problem 2 (33 pts)

In this problem we again consider the 2 DOF system with Hamiltonian

$$H = H_0 + \epsilon H_1,$$

$$H_0 = \omega_1 L_1 + \omega_2 L_2, \quad H_1 = L_1 L_2 \cos(\phi_1 + 3\phi_2),$$

this time using canonical perturbation theory with parameter ϵ.

(a) (25 pts) Make a perturbative canonical transformation $(\phi_1, \phi_2, L_1, L_2) \to (\alpha_1, \alpha_2, I_1, I_2)$ such that the transformed Hamiltonian has the form (ignoring $O(\epsilon^3$ terms)

$$K = K_0(I_1, I_2) + \epsilon K_1(I_1, I_2) + \epsilon^2 K_2(\alpha_1, \alpha_2, I_1, I_2).$$

Specify explicitly the generating function, the transformation equations, and the functions K_0,K_1, and K_2.

(b) (8 pts) By performing a second perturbative canonical transformation $(\alpha_1, \alpha_2, I_1, I_2) \to (\theta_1, \theta_2, J_1, J_2)$, one could obtain a new Hamiltonian (ignoring $O(\epsilon^3)$ terms)

$$\bar{H} = \bar{H}_0(J_1, J_2) + \epsilon \bar{H}_1(J_1, J_2) + \epsilon^2 \bar{H}_2(J_1, J_2).$$

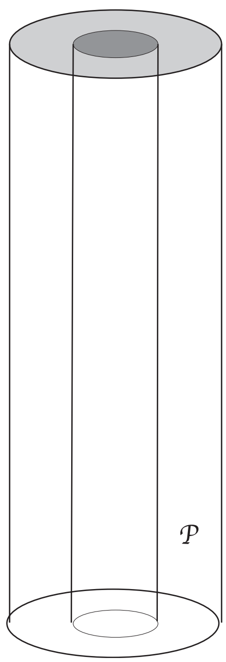
Without explicitly performing the canonical transformation, determine $\bar{H}_0$, $\bar{H}_1$, and $\bar{H}_2$.

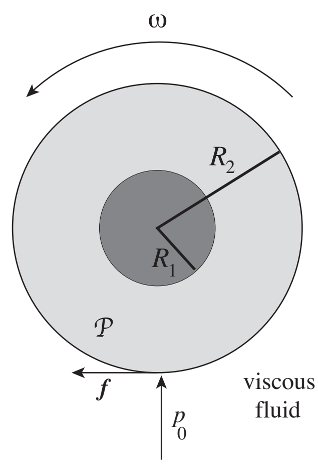
Part II: Continuum Mechanics

In dealing with the following problem, you may find useful some of the formulas in the Appendix.

Problem 3 (34 pts)

A long (effectively infinite) cylindrical pipe $\mathcal{P}$ of mass density ρ, inner radius R_1, and outer radius R_2, is immersed in a viscous fluid. The inner surface of $\mathcal{P}$ is rigidly attached to a cylindrical rod of fixed radius R_1 rotating at a uniform angular velocity ω. The fluid exerts a uniform pressure p_0, as well as a tangential frictional drag force f per unit area on the surface of $\mathcal{P}$. The elastic constants of $\mathcal{P}$ are E (Young's modulus) and σ (Poisson ratio). Using cylindrical coordinates r, ϕ, z in the (non-inertial!) rest frame of the $\mathcal{P}$, calculate the stress tensor (components σ_{ij}), strain tensor (components $u_{i,j}$), and deformation field (components u_i) throughout $\mathcal{P}$.





Cross-section

Appendix A

Miscellaneous elasticity formulas

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right).$$

$$\sigma_{ik} = \frac{\partial F}{\partial u_{ik}} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj})$$

$$u_{ik} = \frac{1}{9K}\sigma_{jj}\delta_{ik} + \frac{1}{2\mu}(\sigma_{ik} - \frac{1}{3}\delta_{ik}\sigma_{jj}).$$

$$F = \frac{1}{2}\sigma_{ik}u_{ik}$$

$$E = \frac{9K\mu}{3K+\mu}, \qquad \sigma = \frac{1}{2}\left(\frac{3K-2\mu}{3K+\mu}\right).$$

$$\frac{\partial \sigma_{ik}}{\partial x_k} + g_i = 0.$$

$$\begin{aligned} \sigma_{ik} &= \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{jj}\delta_{ik}\right), \\ u_{ik} &= \frac{1}{E}\left((1+\sigma)\sigma_{ik} - \sigma\sigma_{jj}\delta_{ik}\right) \end{aligned}$$

$$\nabla^2\mathbf{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\mathbf{u}) = -\frac{2(1+\sigma)}{E}\mathbf{g},$$

$$\nabla(\nabla\cdot\mathbf{u}) - \frac{1-2\sigma}{2(1-\sigma)}\nabla\times(\nabla\times\mathbf{u}) = -\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\mathbf{g}$$

Vector calculus in orthogonal coordinate systems

Consider orthogonal coordinates u_1, u_2, u_3 with arc length along the coordinate directions given by

$$ds_i = h_i(u_1, u_2, u_3)\, du_i.$$

Examples of such systems are *spherical coordinates* with

$$\begin{aligned} u_1 &= r, \qquad u_2 = \theta, \qquad u_3 = \phi \\ h_1 &= 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta, \end{aligned}$$

and *cylindrical coordinates*, with

$$\begin{aligned} u_1 &= \rho \qquad u_2 = \phi, \qquad u_3 = z \\ h_1 &= 1, \qquad h_2 = \rho, \qquad h_3 = 1, \end{aligned}$$

In these orthogonal coordinate systems, the gradient, divergence, curl and Laplacian are given, respectively, by

$$
\begin{aligned}
(\nabla V)_i &= \frac{\partial V}{\partial s_i} = \frac{1}{h_i}\frac{\partial V}{\partial u_i}, \\
\nabla \cdot \mathbf{A} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \text{cyclic permutations}\right) \\
(\nabla \times \mathbf{A})_3 &= \frac{1}{h_1 h_2}\left(\frac{\partial (h_2 A_2)}{\partial u_1} - \frac{\partial (h_1 A_1)}{\partial u_2}\right), \qquad \text{(cyclic permutations)}, \\
\nabla \cdot (\nabla V) &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \text{cyclic permutations}\right).
\end{aligned}
$$

Strain tensor in spherical and cylindrical coordinates

Spherical coordinates

$$
u_r r = \frac{\partial u_r}{\partial r}, \qquad u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad u_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}
$$

$$
u_{\theta\phi} = \frac{1}{2r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial \phi}, \qquad u_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right),
$$

$$
u_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right).
$$

Cylindrical coordinates

$$
u_{rr} = \frac{\partial u_r}{\partial r}, \qquad u_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, \qquad u_{zz} = \frac{\partial u_z}{\partial z},
$$

$$
u_{\phi z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}\right), \qquad u_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),
$$

$$
u_{r\phi} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}\right)
$$

Appendix B: Stress-strain relations in a Newtonian fluid

0.1 Rectangular coordinates

$$
\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)
$$