New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

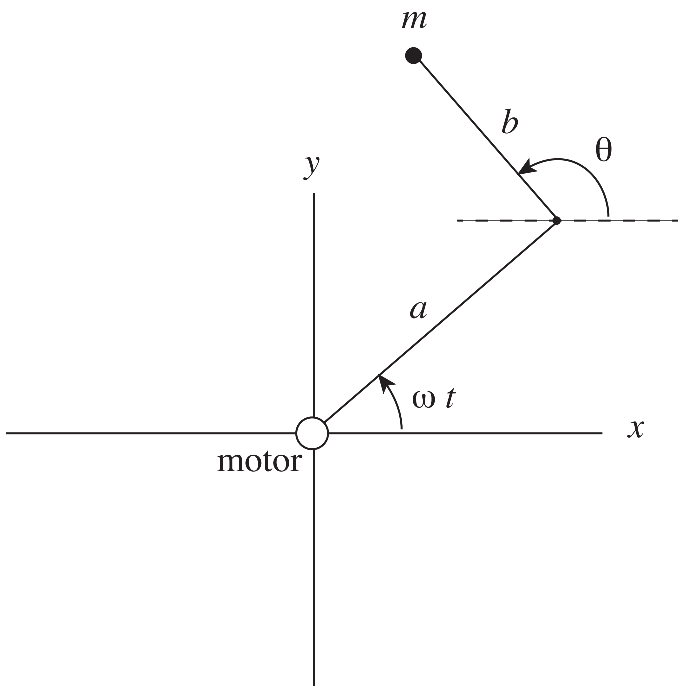
Fall, 2013

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.
4. Show ALL your work.

Part I: Classical Dynamics

Problem 1 (33 pts)



A rod of length a rotates at constant angular velocity ω in a horizontal frictionless plane. At the end of the rod is attached, by means of a frictionless hinge, a second rod, of negligible mass and length b, with a particle of mass m attached at the other end. The setup is shown in the accompanying figure.

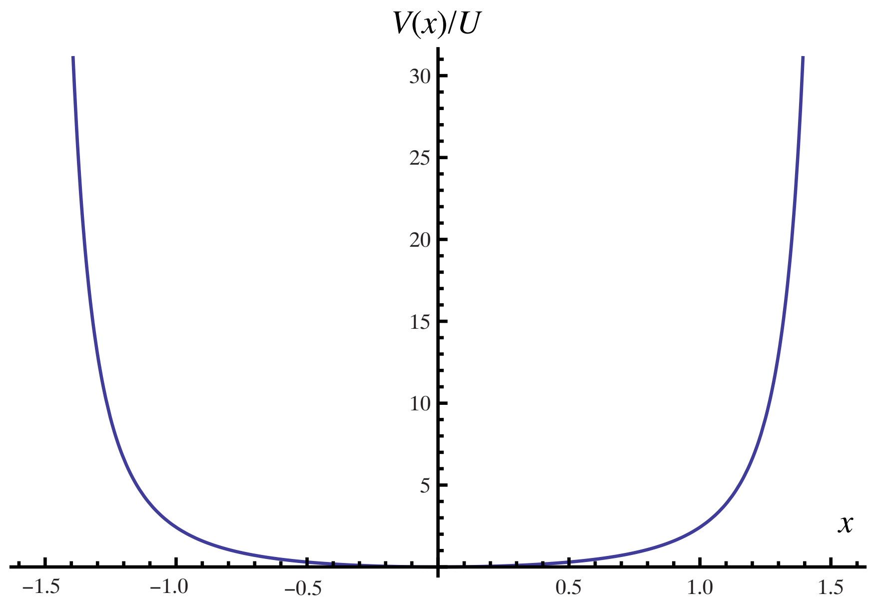
(a) (5 pts) Introducing as a generalized coordinate the angle θ between the second rod and the x direction, construct the Lagrangian for the system.

(b) (5 pts) Calculate the momentum variable p_θ conjugate to θ and construct the Hamiltonian function $H(\theta, p_\theta, t)$.

(c) (8 pts) With the aid of a suitable generating function, make a canonical transformation $(\theta, p_\theta) \to (q, p)$ such that the new Hamiltonian $K(q, p)$ has no explicit time dependence.

(d) (8 pts) Make a canonical transformation $(q, p) \to (Q, P)$ such that the new Hamiltonian $H'(Q, P)$ has no terms linear in P. Show that in the Q, P frame of reference, the system behaves like a simple pendulum of length b, mass m, and gravitational acceleration $g = a\omega^2$.

(e) (7 pts) Write down Hamilton's equations of motion in the (Q, P) frame. Find the stable and unstable fixed points of the system in this frame of reference, and describe the corresponding motions in the laboratory frame.



Problem 2 (33 pts)

A particle of mass $m = 1$ moves along the x axis with potential energy $V(x) = U \tan^2 x$.
(a) (7 pts) Write down the Hamiltonian $H(x, p)$ and sketch the phase-space orbits for several energy values, paying special attention to the intercepts with the x and p axes. Indicate with arrows the direction of motion on the orbits.

(b) (12 pts) For arbitrary positive energy, perform the canonical transformation to angle and action variables θ and J. You may find useful the indefinite integral

$$\int \frac{dq}{\sqrt{a^2 - \tan^2 q}} = \frac{\sin^{-1}\left(\frac{\sqrt{1+a^2}}{a} \sin q\right)}{\sqrt{1+a^2}}.$$

and the definite integral

$$\int_0^{\tan^{-1} a} dq \sqrt{a^2 - \tan^2 q} = \frac{\pi}{2}(\sqrt{1+a^2} - 1)$$

Write down the Hamiltonian $K(\theta, J)$ with respect to the action-angle variables. Calculate the oscillation frequency as a function of energy.

(c) (10 pts) The system is perturbed by adding to the Hamiltonian K a term $\epsilon J^3 \sin^2 \theta$. Using a suitable perturbative generating function, perform the canonical transformation to the action-angle variables θ', J' of the perturbed system, correct to first order in ϵ.

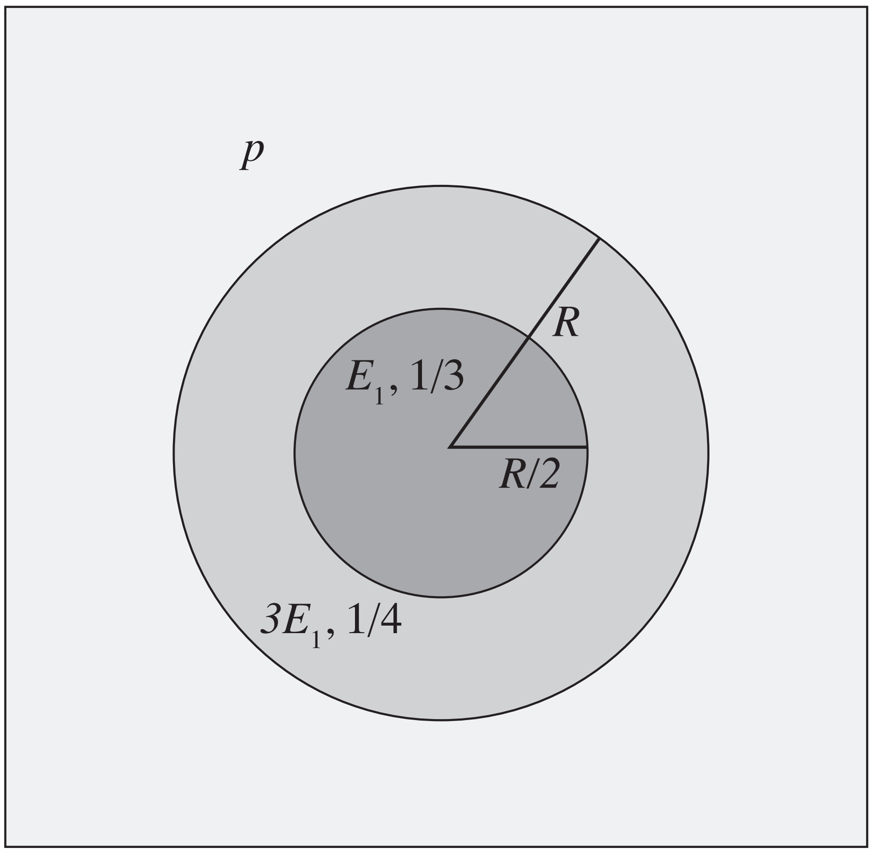
(d) (4 pts) Calculate the perturbed Hamiltonian $K'(J')$ and the oscillation frequency as a function of energy, $\omega'(E')$, correct to first order in ϵ.

Part II: Continuum Mechanics

In dealing with the following problem, you may find useful some of the formulas in the Appendix to this examination.

Problem 3 (34 pts)

A solid ball of radius R contains within it a spherical core of radius $\frac{1}{2}R$, as shown in the accomanying figure. The two concentric regions $0 < r < \frac{1}{2}R$ and $\frac{1}{2}R < r < R$ are characterized



by Young's moduli $E_1, 3E_1$ and Poisson ratios $\frac{1}{3}, \frac{1}{4}$, respectively. The surface at R is subjected to a uniform pressure p by the surrounding medium.

(a) (7 pts) In a spherical coordinate system, which components of the deformation field, strain tensor, and stress tensor are continuous at the boundary $r = \frac{1}{2}R$? Justify your answer. It may help to consider the balance of forces and torques on a small volume element in the boundary layer $\frac{1}{2}R - \epsilon < r < \frac{1}{2}R + \epsilon,\ \epsilon \ll \frac{1}{2}R$.

(b) (27 pts) Calculate the deformation field, strain tensor, and stress tensor everywhere within the ball. Calculate the pressure at the center of the ball.

Appendix

Miscellaneous elasticity formulas

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right).$$

$$\sigma_{ik} = \frac{\partial F}{\partial u_{ik}} = K u_{jj}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{jj})$$

$$u_{ik} = \frac{1}{9K}\sigma_{jj}\delta_{ik} + \frac{1}{2\mu}(\sigma_{ik} - \frac{1}{3}\delta_{ik}\sigma_{jj}).$$

$$F = \frac{1}{2}\sigma_{ik}u_{ik}$$

$$E = \frac{9K\mu}{3K+\mu}, \qquad \sigma = \frac{1}{2}\left(\frac{3K-2\mu}{3K+\mu}\right).$$

$$\frac{\partial \sigma_{ik}}{\partial x_k} + g_i = 0.$$

$$\begin{aligned} \sigma_{ik} &= \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{jj}\delta_{ik}\right), \\ u_{ik} &= \frac{1}{E}((1+\sigma)\sigma_{ik} - \sigma\sigma_{jj}\delta_{ik}) \end{aligned}$$

$$\nabla^2\mathbf{u} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\mathbf{u}) = -\frac{2(1+\sigma)}{E}\mathbf{g},$$

$$\nabla(\nabla\cdot\mathbf{u}) - \frac{1-2\sigma}{2(1-\sigma)}\nabla\times(\nabla\times\mathbf{u}) = -\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\mathbf{g}$$

Appendix

Vector calculus in orthogonal coordinate systems

Consider orthogonal coordinates u_1, u_2, u_3 with arc length along the coordinate directions given by

$$ds_i = h_i(u_1, u_2, u_3)\, du_i.$$

Examples of such systems are *spherical coordinates* with

$$\begin{aligned} u_1 &= r, \qquad u_2 = \theta, \qquad u_3 = \phi \\ h_1 &= 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta, \end{aligned}$$

and *cylindrical coordinates*, with

$$\begin{aligned} u_1 &= \rho \qquad u_2 = \phi, \qquad u_3 = z \\ h_1 &= 1, \qquad h_2 = \rho, \qquad h_3 = 1, \end{aligned}$$

In these orthogonal coordinate systems, the gradient, divergence, curl and Laplacian are given, respectively, by

$$\begin{aligned} (\nabla V)_i &= \frac{\partial V}{\partial s_i} = \frac{1}{h_i}\frac{\partial V}{\partial u_i}, \\ \nabla \cdot \mathbf{A} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \text{cyclic permutations}\right) \\ (\nabla \times \mathbf{A})_3 &= \frac{1}{h_1 h_2}\left(\frac{\partial (h_2 A_2)}{\partial u_1} - \frac{\partial (h_1 A_1)}{\partial u_2}\right), \qquad \text{(cyclic permutations)}, \\ \nabla \cdot (\nabla V) &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \text{cyclic permutations}\right). \end{aligned}$$

Strain tensor in spherical coordinates

$$u_{rr} = \frac{\partial u_r}{\partial r}, \qquad u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad u_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}$$

$$u_{\theta\phi} = \frac{1}{2r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial \phi}, \qquad u_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right),$$

$$u_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right).$$