New York University

Department of Physics

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2016

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. On the front cover of each booklet write your identification number.
4. Show ALL your work.

Part I: Classical Dynamics

Problem 1 (30 pts)

The Hamiltonian of a nonlinear oscillator takes the form

$$H(\theta_1, \theta_2, J_1, J_2, t) = \nu_1 J_1 + \nu_2 J_2 + \epsilon\, a(J_1^2 + J_2^2)\sin^2(3\theta_1 - 4\theta_2) + \epsilon\, b(J_1^2 + J_2^2)^2 \cos^2 \omega t,$$

where ν_1, ν_2, a, b are constants and ϵ is a small positive perturbation parameter.

With the aid of a perturbative generating function, make a canonical transformation to new phase-space variables $\bar{\theta}_1, \bar{\theta}_2, \bar{J}_1, \bar{J}_2$ such that the new Hamiltonian $\bar{H}$ depends only on $\bar{J}_1$ and $\bar{J}_2$ (with no time dependence), apart from terms of order ϵ^2. Write $\bar{\theta}_i$ and $\bar{J}_i$, $i = 1, 2$ as functions of $\theta_j, J_j, t,\ j = 1, 2$, correct to order ϵ.

Problem 2 (40 pts)

A particle of unit mass moves on a torus with the Hamiltonian

$$H(\phi_1, \phi_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + a p_2 \cos 2\phi_1,$$

where $0 \le \phi_i < 2\pi,\ i = 1, 2$ and a is a positive constant.

(a) (10 pts) Calculate Hamilton's equations of motion. Identify a function of the phase-space variables which is independent of H and has zero Poisson bracket with H, so that by definition the system is integrable. Show that the system is also separable.

(b) (15 pts) Consider the three-dimensional manifold M_a where $p_2 = a$. Construct a reduced Hamiltonian, resembling that of a simple pendulum, which governs the time-evolution of ϕ_1 and p_1. Draw a phase portrait showing typical projected orbits in the ϕ_1, p_1 plane. As in the pendulum case, these should include fixed points, librational and rotational orbits, and separatrices. Which of the fixed points of the projected system are true equilibria of the full system with two degrees of freedom (see part (a))? Are the equilibria stable in the full phase space?

(c) (5 pts) Show that the points of M_a which have $p_1 = \phi_1 = 0$ are an invariant manifold topologically equivalent to a circle. Calculate explicitly the orbit, within M_a,

$$\xi(\phi_{20}, t) = (\phi_1(t), \phi_2(t), p_1(t)), \quad -\infty < t < \infty,$$

which passes through the point $\xi(\phi_{20}, 0) = (0, \phi_{20}, 0)$.

(d) (10 pts) Calculate explicitly the orbit, within M_a,

$$\eta(\phi_{21}, t) = (\phi_1(t), \phi_2(t), p_1(t)), \quad -\infty < t < \infty,$$

which passes through $\eta(\phi_{21}, 0) = (\pi/2, \phi_{21}, 2a)$. The following integrals may be helpful:

$$\int \frac{dx}{\sin x} = \ln\left(\tan\frac{x}{2}\right), \qquad \int dx\, \tanh^2 x = x - \tanh\, x.$$

Show that, for $t \to -\infty$, the orbit $\eta(\phi_{21}, t)$ tends asymptotically to the curve $\xi(\phi_{20}, t)$ for a suitable choice of ϕ_{20}.

Part II: Continuum Mechanics

In dealing with the following problem, you may find useful some of the formulas in the Appendices.

Problem 3 (30 pts)

An incompressible fluid with uniform density ρ and viscosity μ flows through a pipe with annular cross-section under the influence of a pressure gradient $-\Delta p/L$. The inner and outer boundaries are coaxial cylinders of radii R_1 and R_2, respectively. We assume no frictional drag at the inner boundary and a non-slip condition at the outer boundary. Determine the velocity field within the fluid, the drag force per unit length on the outer boundary, and the discharge (rate of mass flow). Compute the $R_1 \to 0$ limits to obtain these quantities for a cylindrical pipe. You may find useful the integral formula

$$\int dr\, r^{n-1} \ln(\frac{r}{c}) = \frac{r^n}{n^2}\left(n \ln(\frac{r}{c}) - 1\right).$$

Appendix A: Stress-strain relations in a Newtonian fluid

(cylindrical coordinates)

$$\begin{aligned}
\sigma_{rr} &= -p + 2\mu \frac{\partial v_r}{\partial r}, \\
\sigma_{\phi\phi} &= -p + 2\mu \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right), \\
\sigma_{zz} &= -p + +2\mu \frac{\partial v_z}{\partial z} \\
\sigma_{r\phi} &= \mu \left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) \\
\sigma_{\phi z} &= \mu \left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right) \\
\sigma_{zr} &= \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)
\end{aligned}$$

Appendix B: Equations of Motion in Rectangular and Cylindrical Coordinates

1. **Cartesian coordinates:**

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+F_x$$

together with similar equations for v and w.

2. Cylindrical polar coordinates (r = distance from axis, ϕ = azimuthal angle about axis, z = distance along axis):

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_z}{\partial z}=0$$

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\phi^2}{r}\right]=-\frac{\partial p}{\partial r}$$

$$+\mu\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right]+F_r$$

$$\rho\left[\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_r u_\phi}{r}+\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi}+u_z\frac{\partial u_\phi}{\partial z}\right]=-\frac{1}{r}\frac{\partial p}{\partial \phi}$$

$$+\mu\left[\frac{\partial^2 u_\phi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{\partial^2 u_\phi}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right]+F_\phi$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial p}{\partial z}$$

$$+\mu\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+F_z.$$