New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

Fall, 2009

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.

2. You have 3 hours to complete the examination.

3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.

4. Show ALL your work.
Problem 1 (33 pts)

A particle moves in 3 dimensions with Hamiltonian (in spherical polar coordinates)

\[ H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + mgr \cos \theta. \]

(a) (6 pts) Derive Hamilton’s equations of motion for all 6 of the phase-space variables. Verify that \( \theta, p_\phi, \) and \( H \) are constants of the motion.

(b) (12 pts) Show that the radial motion is governed by an effective one-dimensional potential energy function of the form

\[ V_{\text{eff}}(r) = \frac{a}{r^2} + br, \]

where \( a \) and \( b \) are positive functions of the constants \( \theta, p_\phi, \) and \( H \). Let \( E \) be the energy and let \( V_0 \) be the minimum value of \( V_{\text{eff}} \). Describe the motion in configuration space and in phase space when (i) \( E > V_0 \), (ii) \( E < V_0 \), (iii) \( E = V_0 \).

(c) (10 pts) Prove the functional independence of the 3 constants \( \theta, p_\phi, \) and \( H \) everywhere in the phase space except for a manifold of dimension less than 6. Give the defining equations for this sub-manifold and describe precisely the orbits which inhabit it. What are the functional relations among the three integrals on the exceptional manifold? Relate your results to the results of part (b).

(e) (5 pts) Consider the sub-manifold \( \mathcal{M} \) of phase space for which \( \theta, p_\phi, \) and \( H \) assume specific (non-exceptional) values \( \theta_0, L, \) and \( E > V_0(\theta_0, L) \), respectively. Does \( \mathcal{M} \) have the topology of a 3-torus, as it must if it satisfies the integrability criteria of the Liouville-Arnol’d therorem. Explain.

Problem 2 (33 pts)

In parabolic coordinates \( u, v, \phi, \) the potential energy of a charged particle interacting with a point charge and, simultaneously a uniform electric field, is given by

\[ V(u, v, \phi) = -\frac{2\alpha}{u+v} + \beta(u-v), \]

while the kinetic energy takes the form

\[ T(u, v, \phi, \dot{u}, \dot{v}, \dot{\phi}) = \frac{m}{8} \left( \frac{u+v}{u} \dot{u}^2 + \frac{u+v}{v} \dot{v}^2 + 4uv \dot{\phi}^2 \right). \]

(a) (5 pts) Calculate the canonical momenta \( p_u, p_v, \) and \( p_\phi \) conjugate to \( u, v, \) and \( \phi, \) and show that the Hamiltonian of the system can be written

\[ H = \frac{1}{u+v} \left( \frac{2}{m} (up_u^2 + vp_v^2) + \frac{p_\phi^2}{2m} \left( \frac{1}{u} + \frac{1}{v} \right) - 2\alpha + \beta(u^2 - v^2) \right). \]
(b) (12 pts) By means of a generating function of the form

$$F_2(u, v, \phi, P_1, P_2, P_3, t) = W_u(u, P_1, P_2, P_3) + W_v(v, P_1, P_2, P_3) + P_3\phi - P_1t,$$

we construct a canonical transformation

$$(u, v, \phi, p_u, p_v, p_\phi) \rightarrow (Q_1, Q_2, Q_3, P_1, P_2, P_3),$$

such that the transformed Hamiltonian is identically equal to zero. This condition leads to a (Hamilton-Jacobi) partial differential equation with respect to $u$ and $v$, which can be reduced to separate differential equations for $W_u$ and $W_v$ by means of the method of separation of variables, with $P_2$ serving as the separation constant. Carry out this process and determine the functions $W_u$ and $W_v$ as unevaluated integrals of the form

$$W_u = \int\int_{u_0}^\infty \frac{du'}{u'} \sqrt{f(u', P_1, P_2, P_3)}, \quad W_v = \int_{v_0}^\infty \frac{dv'}{v'} \sqrt{g(v', P_1, P_2, P_3)},$$

where $f(u, P_1, P_2, P_3)$ and $g(v, P_1, P_2, P_3)$ are cubic polynomials in $u$ and $v$, respectively. Do not derive the transformation equations for $Q_i$, $i = 1, 2, 3$; they are not needed in what follows.

(c) (3 pts) What is the time dependence of the new $Q$ and $P$ variables?

(d) (6 pts) From here on we assume $P_3 = 0$.

Assume $P_1 < 0$, $P_2 > 0$, $\alpha > 0$, $2\alpha - P_2 > 0$ and $\beta$ a negative number sufficiently small to guarantee that $p_u(u)^2$ (resp. $p_v(v)^2$) (i) is concave upward, (ii) is singular at $u = 0$ (resp. $v = 0$), and (iii) has a simple zero $u_1$ (resp. $v_1$) on the $u$ (resp. $v$) axis. Find the asymptotic behavior of $p_u(u)$ (resp. $p_v(v)$) as $u$ (resp. $v$) tends to zero. Calculate $u_1$ and $v_1$. Show that the motion between $u = 0$ and $u = u_1$ (resp. $v = 0$ and $v = v_1$) is oscillatory, but not periodic.

(e) (7 pts) The results of (d) allow us, in principle, to make a canonical transformation to action-angle variables $J_i$ and $\theta_i$, $i = 1, 2, 3$. Trivially, $J_3 = P_3$. Calculate $J_1$ and $J_2$ (at $P_3 = 0$) as functions of $P_1, P_2$ (it is not necessary to evaluate the integrals). Calculate the frequencies of the $\theta_i$ oscillations as functions of $P_1$ and $P_2$. Note: to do this part, it is not necessary to explicitly invert the transformation equations.
Problem 3 (34 pts)

Note: useful formulas may be found on pp. 5-7.

A cylindrical system consists of a nondeformable center rod of radius $r_0$ attached to an isotropic elastic medium with outer radius $r_1$ surrounded by a viscous fluid which is inside a cylindrical wall of inner radius $r_2$ as shown in the figure below. The elastic material has density $\rho$, bulk modulus $B$ and shear modulus $\mu$. The fluid has density $\rho$ and viscosity $\eta$. The outer wall is rotating at frequency $\omega$. The innermost rod is held stationary.

Fluid part ($r_1$ to $r_2$)

a. (10 points) Find the velocity field and the pressure field in the fluid.
b. (3 points) What is the stress on the elastic cylinder’s surface (at $r_1$)

Elastic Part ($r_1$ to $r_0$)

c. (7 points) Find the strain field in the elastic medium given the stress $\sigma_{r\phi}$ at $r_1$
d. (3 points) Find the torque on the inner rod.
e. (6 points) How much power is needed to keep the outer cylinder rotating at $\omega$?
f. (5 points) Is the system stable or will distortions or secondary flows develop? (Two sentences or equations.)
Cylindrical polar coordinates \((R, \theta, z)\)

\[
\nabla \Phi = \frac{\partial \Phi}{\partial R} R + \frac{1}{R} \frac{\partial \Phi}{\partial \theta} \theta + \frac{\partial \Phi}{\partial z} z
\]

\[
\nabla \cdot \mathbf{F} = \frac{1}{R} \frac{\partial}{\partial R} (RF_R) + \frac{1}{R} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}
\]

\[
\nabla \times \mathbf{F} = \left[ \frac{1}{R} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right] \hat{R} + \left[ \frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R} \right] \hat{\theta} + \frac{1}{R} \left( \frac{\partial}{\partial R} (RF_\theta) - \frac{\partial F_R}{\partial \theta} \right) \hat{z}
\]

\[
\nabla^2 \Phi = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}
\]

\[
\nabla^2 \mathbf{F} = \left[ \nabla^2 F_R - \frac{1}{R^2} F_R - \frac{2}{R} \frac{\partial F_\theta}{\partial \theta} \right] \hat{R}
\]

\[
+ \left[ \nabla^2 F_\theta - \frac{1}{R^2} F_\theta + \frac{2}{R} \frac{\partial F_R}{\partial \theta} \right] \hat{\theta} + \nabla^2 F_z \hat{z}
\]

\[(\mathbf{B} \cdot \nabla) \mathbf{A} = [\mathbf{B} \cdot \nabla A_R - B_\theta A_\theta/R] \hat{R}
\]

\[
+ [\mathbf{B} \cdot \nabla A_\theta + B_\theta A_R/R] \hat{\theta} + \mathbf{B} \cdot \nabla A_z \hat{z}
\]
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v}.
\]

\[
\sigma_{rr} = -p + 2\eta \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right),
\]

\[
\sigma_{\theta r} = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\theta}{\partial r} \right),
\]

\[
\sigma_{rr} = -p + 2\eta \left( \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} \right),
\]

\[
\sigma_{zz} = -p + 2\eta \frac{\partial v_z}{\partial z},
\]

\[
\sigma_{\theta r} = \eta \left( \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial \phi} \right) \quad \sigma_{\theta r} = \eta \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \phi} + \frac{\partial v_r}{\partial r} \right).
\]

The three components of the Navier-Stokes equation and the equation of continuity are

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \phi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_r^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v_r}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2} \right),
\]

\[
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \phi} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\phi}{r} = - \frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \nu \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r^2} \right),
\]

\[
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v_z}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial \phi} \right),
\]

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{v_\theta}{\phi} + \frac{v_z}{z} = 0.
\]
\[
\begin{align*}
\frac{\partial u_r}{\partial r} - \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + u_r \frac{\partial u_r}{\partial r} + u_\phi \frac{\partial u_\phi}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \\
2 \frac{\partial u_\phi}{\partial r} - \frac{1}{r} \frac{\partial u_z}{\partial \phi} = \frac{\partial u_r}{\partial r} + \frac{\partial u_\phi}{\partial r}, \\
2 \frac{\partial u_r}{\partial r} - \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} = \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r}.
\end{align*}
\]

\(\mu = E/(1+\sigma), \quad K = E/(1-2\sigma)\). \hfill (5.9)

We shall write out here the general formulae of §4, with the coefficients expressed in terms of \(E\) and \(\sigma\). The free energy is

\[
F = \frac{E}{2(1+\sigma)} \left( u_{ik}^2 + \frac{\sigma}{1-2\sigma} u_{ii}^2 \right). \hfill (5.10)
\]

The stress tensor is given in terms of the strain tensor by

\[
\sigma_{ik} = \frac{E}{1+\sigma} \left( u_{ik} + \frac{\sigma}{1-2\sigma} u_{ij} \delta_{jk} \right). \hfill (5.11)
\]

Conversely,

\[
u_{ik} = \left[ (1+\sigma) \sigma_{ik} - \sigma \eta \delta_{ik} \right] / E. \hfill (5.12)
\]

\[
\frac{E}{2(1+\sigma)} \partial^2 u_i \frac{E}{2(1+\sigma)(1-2\sigma) \partial x_k \partial x_i} + u_i \partial^2 u_i + \rho g \partial_t = 0. \hfill (7.1)
\]

These equations can be conveniently rewritten in vector notation. The quantities \(\partial^2 u_i / \partial x_k \partial x_i\) are components of the vector \(\Delta \mathbf{u}\), and \(\partial u_i / \partial x_i \equiv \text{div} \mathbf{u}\). Thus the equations of equilibrium become

\[
\Delta \mathbf{u} + \frac{1}{1-2\sigma} \text{grad div} \mathbf{u} = -\rho g \frac{2(1+\sigma)}{E}. \hfill (7.2)
\]

It is sometimes useful to transform this equation by using the vector identity \(\text{grad div} \mathbf{u} = \Delta \mathbf{u} + \text{curl curl} \mathbf{u}\). Then (7.2) becomes

\[
\text{grad div} \mathbf{u} = \frac{1}{2(1-\sigma)} \text{curl curl} \mathbf{u} - \frac{1}{2(1-\sigma)} \text{curl curl} \mathbf{u} \left( \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)} \right). \hfill (7.3)
\]
Solutions to Problems 1 and 2 of the Dynamics Prelim (2009)

Problem 1

(a) (5 pts)

\[ \dot{r} = \frac{\partial H}{\partial p_r} = -\frac{v_r}{m}, \]
\[ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = 0, \]
\[ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{F_0}{mr^2 \theta} \]
\[ \ddot{r} = -\frac{\partial H}{\partial r} = \frac{v^2}{m r^2 \sin^2 \theta} - mg \cos \theta, \]
\[ \ddot{\theta} = -\frac{\partial H}{\partial \theta} = \frac{v^2 \cos \theta}{m r^2 \sin^2 \theta} + m g r \sin \theta \]
\[ \ddot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0. \]

From the equations of motion in (a), \( \theta \) and \( p_\phi \) are constant. Since \( H \) has no explicit time dependence, it also is constant.

(b) (pts) Consider an orbit for which \( H = E, \ p_\phi = L, \ \text{and} \ \theta = \theta_0 \). Then the definition of \( H \) gives

\[ E = \frac{1}{2} m r^2 + \frac{L^2}{2 m r^2 \sin^2 \theta_0} + m g r \sin \theta_0. \]  

(1)

This is the energy equation for a one-dimensional system with coordinate \( r \) restricted to positive values and a potential energy function

\[ V_{\text{eff}}(r) = \frac{a}{r^2} + b r, \quad a = \frac{L^2}{2 m r^2 \sin^2 \theta_0}, \quad b = m g \cos \theta_0. \]

This function is concave upward with a minimum \( V_0 = (2^{-2/3} + 2^{1/3}) a^{-1/3} b^{2/3} \) at \( r = r_0 = (2a/b)^{1/3} \).

The radial motion is found by integrating (1), considered as a first-order ODE for \( r \).

(i) For \( E > V_0, \ \dot{r}^2 = \frac{2}{m} (E - V_{\text{eff}}(r)) \) is positive between two neighboring zeroes \( r_1 \) and \( r_2 \) of the righthand side, and so there is an oscillatory solution between those extreme values. In the \( r, p_r \) plane (with \( E, L, \theta_0 \) fixed) the orbit is topologically equivalent to a circle. Having found \( r(t) \), one can then directly integrate the equations of motion for \( \phi(t) \) and \( p_\phi(t) \). During one period \( \tau_r \) of the \( r \) motion, the azimuth increases by \( \Delta \phi \), which is generically incommensurate with \( 2\pi \), so that over time the orbit covers quasiperiodically the portion \( r_1 \leq r \leq r_2 \) of the cone \( \theta = \theta_0 \) in configuration space. Similarly, during each time interval of length \( \tau_r \), the physically irrelevant coordinate \( p_\theta \) advances an amount \( \Delta p_\theta \), corresponding to an average velocity of \( \Delta p_\theta / \tau_r \) along the \( p_\theta \) axis.

(ii) For \( E < V_0, \ E - V_{\text{eff}} \) is strictly negative, and so there are no solutions to (1).

(iii) For \( E = V_0 \), we have \( r(t) = r_0 \) for all \( t \), while \( \phi \) and \( p_\phi \) increase at uniform rates. In configuration space, the orbit is a circle of radius \( r_0 \sin \theta_0 \) in a plane parallel to the \( x, y \) plane, at
a distance \( r_0 \cos \theta_0 \) above it. The orbit is a helix in the 3-dimensional \( x, y, p_\phi \) subspace of phase space.

(c) (12 pts) The integrals are functionally independent if all minors of the rectangular matrix of partial derivatives, \( \frac{\partial F_i}{\partial \xi_j}, F_i = \theta, p_\phi, H, \xi_j = r, \theta, \phi, p_r, p_\theta, p_\phi \), vanish.

\[
\begin{vmatrix}
0 & 0 & -\dot{p}_r \\
1 & 0 & -\dot{p}_\theta \\
0 & 0 & \frac{p_\phi}{m} \\
0 & 0 & \frac{p_\phi}{m^2 \sin \theta} \\
0 & 1 & \frac{p_\phi}{m^2 \sin \theta}
\end{vmatrix}
\]

The only possible nonzero minors have 1 in the first 2 columns and either \(-\dot{r}\) or \(p_r/m\) in the third column. Both vanish if and only if

\[ p_r = \dot{r} = 0, \]

which implies \( \dot{r} = 0 \) for all \( t \), with

\[ r \sin \theta = \left( \frac{p_\phi^2 \tan \theta}{m^2 g} \right)^{1/3}. \quad (2) \]

These exceptional orbits are just those of case (iii) in (b).

(d) (8 pts) Because of the unbounded \( p_\theta \) motion, the manifold \( \mathcal{M} \) is not compact, hence does not satisfy the hypotheses of the theorem and cannot be a 3-torus.

Problem II (33 pts)

(a) (5 pts)

\[ L = T - V, \quad p_u = \frac{\partial L}{\partial \dot{u}} = \frac{m(u + v)}{4u} \dot{u}, \quad p_v = \frac{\partial L}{\partial \dot{v}} = \frac{m(u + v)}{4v} \dot{v}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = muv. \]

\[ H = \dot{u}p_u + \dot{v}p_v + \dot{\phi}p_\phi - L = \frac{2}{m(u + v)} (up_u^2 + vp_v^2) + \frac{p_\phi^2}{2muv} - \frac{2\alpha}{u + v} + \beta(u - v) \]

(b) (12 pts)

\[ p_u = \frac{\partial F_3}{\partial u} = \frac{\partial W_u}{\partial u}, \quad p_v = \frac{\partial F_3}{\partial v} = \frac{\partial W_v}{\partial v}, \quad p_\phi = \frac{\partial F_3}{\partial \phi} = P_3. \]

The vanishing of the new Hamiltonian implies

\[ 0 = H(u, v, \phi, \frac{\partial W_u}{\partial u}, \frac{\partial W_v}{\partial v}, P_3) + \frac{\partial F_3}{\partial \epsilon} \]

\[ = \frac{1}{u + v} \left( \frac{2m}{u} \left( \frac{\partial W_u}{\partial u} \right)^2 + v \left( \frac{\partial W_v}{\partial v} \right)^2 \right) + \frac{p_\phi^2}{2m} \left( \frac{1}{u} + \frac{1}{v} \right) - 2\alpha + \beta(u^2 - v^2) - P_1(u + v), \]
hence
\[
\left( \frac{2u}{m} \left( \frac{\partial W_u}{\partial u} \right)^2 + \frac{P_3^2}{2mu} - 2\alpha - \beta u^2 - P_1u \right) = - \left( \frac{2v}{m} \left( \frac{\partial W_v}{\partial v} \right)^2 + \frac{P_3^2}{2mv} - 2\alpha - \beta v^2 - P_1v \right)
\]

For functions of \( u \) and \( v \) to be equal with independent \( u \) and \( v \), they must be equal to a constant, which we choose to be \( P_2 \). Thus,
\[
\left( \frac{\partial W_u}{\partial u} \right)^2 = \frac{2m}{u} \left( 2\alpha - P_2 + P_1u - \beta u^2 - \frac{P_3^2}{2mu} \right),
\]
\[
\left( \frac{\partial W_v}{\partial v} \right)^2 = \frac{2m}{v} \left( P_2 + P_1v + \beta v^2 - \frac{P_3^2}{2mv} \right),
\]
and so
\[
W_u = 2m \int_{u_0}^{u} \frac{du}{u} \left( \frac{P_3^2}{2m} + (2\alpha - P_2)u + P_1u^2 - \beta u^3 \right)^{\frac{1}{2}},
\]
\[
W_v = 2m \int_{v_0}^{v} \frac{dv}{v} \left( \frac{P_3^2}{2m} + P_2u + P_1v^2 + \beta v^3 \right)^{\frac{1}{2}}.
\]

(c) (8 pts) Since the Hamiltonian is identically zero, the equations of motion imply that all of the variables \( Q_1, Q_2, Q_3, P_1, P_2, P_3 \) are constant.

(d) (8 pts)
\[
p_u^2 = \left( \frac{\partial W_u}{\partial u} \right)^2 = \frac{2m}{u} \left( 2\alpha - P_2 + P_1u - \beta u^2 \right), \quad p_v^2 = \left( \frac{\partial W_v}{\partial v} \right)^2 = \frac{2m}{v} \left( P_2 + P_1u + \beta u^2 \right),
\]
\[
p_u \sim \sqrt{\frac{2m(2\alpha - P_2)}{u}}, \quad \text{as } u \to 0 \Rightarrow \dot{u} \sim \text{const} \frac{u}{u + v} \times \sqrt{u},
\]
\[
p_v \sim \sqrt{\frac{2mP_2}{v}}, \quad \text{as } v \to 0 \Rightarrow \dot{v} \sim \text{const} \frac{v}{u + v} \times \sqrt{v},
\]

Hence \( u = 0 \) (resp. \( v = 0 \)) is a turning point of the \( u \) (resp. \( v \)) motion. The other turning point is at \( u_1 \) (resp. \( v_1 \)) given by the quadratic equations
\[
u_1^2 - \frac{P_1}{\beta}u_1 - \frac{2\alpha - P_2}{\beta} = 0, \quad v_1^2 + \frac{P_1}{\beta}v_1 + \frac{P_2}{\beta} = 0.
\]
The relevant roots are
\[
u_1 = \frac{P_1}{2\beta} \left( 1 - \sqrt{1 + \frac{4(2\alpha - P_2)}{P_1^2}} \right), \quad v_1 = \frac{P_1}{2\beta} \left( 1 - \frac{4P_2}{P_1^2} \beta - 1 \right).
\]
The \( u \) (resp. \( v \) motion) is thus oscillatory, i.e. it goes back and forth between 0 and \( u_1 \) (resp. \( 0 \) and \( v_1 \)). Because \( \dot{u} \) (resp. \( \dot{v} \)) depends on \( v \) (resp. \( u \)), and the two motions are not typically
commensurate, the separate $u$ and $v$ motions are not periodic. The true periodicities are revealed in part (e) by a transformation to action-angle variables.

\[(e) \text{ (pts)} \]

\[
J_u = \frac{1}{2\pi} \oint p_u \, du = \frac{1}{\pi} \int_0^{\alpha_1} du \sqrt{\frac{2m}{u} \left(2\alpha - P_1 + P_1u - \beta u^2\right)^{\frac{3}{2}}}, \\
J_v = \frac{1}{2\pi} \oint p_v \, dv = \frac{1}{\pi} \int_0^{\beta_1} dv \sqrt{\frac{2m}{v} \left(P_2 + P_1v + \beta v^2\right)^{\frac{3}{2}}},
\]

The frequencies are given by the first-row matrix elements (recall $P_1$ is the energy) of

\[
\left( \begin{array}{cc}
\frac{\partial P_1}{\partial J_u} & \frac{\partial P_1}{\partial J_v} \\
\frac{\partial P_1}{\partial J_u} & \frac{\partial P_1}{\partial J_v}
\end{array} \right) = \left( \begin{array}{cc}
\frac{\partial J_u}{\partial P_1} & \frac{\partial J_v}{\partial P_1} \\
\frac{\partial J_u}{\partial P_1} & \frac{\partial J_v}{\partial P_1}
\end{array} \right)^{-1}
\]

The derivatives of $J_u$ and $J_v$ can be easily be performed on the integrands of the defining integrals.
\[ \text{Fluid: } \psi(r) = \psi_2, \quad U_r(r) = 0 \]

\[ \text{NS: eq.: } -\frac{\psi^2}{r} = -\frac{1}{\rho} \frac{\partial \rho}{\partial r} \]

\[ U = \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} \]

\[ \text{Try: } \psi = \frac{A}{r} + \frac{B}{r^2} \]

\[ \frac{\partial U}{\partial r} = A - \frac{B}{r^2}, \quad \frac{\partial^2 U}{\partial r^2} = B \frac{A}{r^3} \]

\[ \frac{2B}{r^3} + \frac{A}{r^3} - \frac{B r^2}{r^3} - \frac{A r}{r^3} = 0 \quad \text{OK} \]

\[ \omega_r = \frac{A r}{r^2} + \frac{B}{r^2}, \quad \omega = -\frac{B}{r^2} \frac{A}{r^2} = \frac{A^2}{r^4} \]

\[ \omega = \frac{A}{r^2} + \frac{B}{r^2} \]

\[ \psi = -\frac{B}{r^2} + \frac{B}{r^2} \]

\[ \text{b) Stress: } \sigma_{rr} = \frac{\partial^2 \psi}{\partial r^2} - \frac{\psi}{r} \]

\[ \sigma_{rr} = \gamma \left( -\frac{B}{r^2} - \frac{B}{r^2} + \frac{B}{r^2} - \frac{B}{r^2} \right) \]

\[ \sigma_{rr}(r) = \gamma \left(-\frac{B}{r^2} - \frac{B}{r^2} + \frac{B}{r^2} - \frac{B}{r^2} \right) \]

\[ \sigma_{rr}(r) = -2B \frac{B}{r^2} \]

\[ \text{c) } 0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial u}{\partial r} = \sigma_{rr} \]

\[ \frac{\partial^2 u}{\partial r^2} - \frac{\psi}{r^2} \]

\[ \frac{\partial^2 u}{\partial r^2} = \frac{C + \frac{2}{r^2}}{r^2} \]

\[ \frac{\partial u}{\partial r} = C - \frac{D}{r^2} \]

\[ \frac{\partial u}{\partial r} = C - \frac{D}{r^2} \]

\[ \frac{\partial u}{\partial r} = C + \frac{D}{r^2} \]
\[ 0 = \frac{E}{1 - \nu} \quad u_{\varepsilon \theta} = 2 \mu \quad u_{\varepsilon \phi} \]

\[ 2 U_{\varepsilon} = \frac{\partial U_{\varepsilon}}{\partial r} - \frac{U_{\varepsilon}}{r} = C - \frac{D}{r^2} - C - \frac{D}{r^2} = -2 \frac{D}{r^2} \]

\[ \sigma_{\varepsilon \phi} = -\frac{2 B \theta}{r^2} = -\mu \frac{2 D}{r^2} \]

\[ U_{\varepsilon}(R) = 0 \quad \text{if } \theta \leq \frac{D}{R} \]

\[ C = -\frac{D}{R^2} \]

\[ T = \text{constant} \text{ everywhere} \]

d) \[ T = 2 \pi r \sigma_{\varepsilon \phi} = 2 \pi r^2 \frac{D}{r^2} = -4 \pi \mu \frac{D}{r^2} = \text{independent of } r \]

e) \[ \frac{\partial U_{\varepsilon}}{\partial r} = 2 \pi \int_{r_1}^{r} \left( \frac{\partial U_{\varepsilon}}{\partial r} \right)^2 \mathrm{d}r \]

\[ = -2 \pi \frac{B^2}{2} \int_{r_1}^{r} \frac{1}{r^2} \mathrm{d}r \]

\[ = -2 \pi B^2 \left( \frac{1}{r} - \frac{1}{r_1} \right) \]

\[ = -2 \pi B^2 \frac{1}{r} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

f) System is stable against Taylor instability.

Pressure higher outside than inside.

Elastic system not under compression.
Electromagnetism (Prelim, 3 hours, need 5 correct answers to pass)

1. A spaceship travels at $v = 0.6c$ towards Earth. The cosmonauts are in TV contact with Earth, and the viewers on Earth can see the clock on the wall of the spaceship. How many rotations will the seconds hand do in one minute (by the Earth clock)?

2. (Greisen-Zatsepin-Kuzmin effect) A cosmic ray (proton) with energy $E$ collides with a CMB (cosmic microwave background, a big bang relic) photon of energy $T = 2.73K$ (Kelvin). Calculate the threshold energy $E$ (in eV) for neutral pion production ($p + \gamma \rightarrow p + \gamma + \pi^0$) ($m_p = 938\text{MeV}, m_{\pi}=135\text{MeV}$).

3. Reaction $p + p \rightarrow d + \pi$. One proton is at rest, find the energy $E$ of the moving proton, if the deuteron is produced at rest. ($m_p = 940\text{MeV}, m_d = 140\text{MeV}$, deuteron mass is close to two proton masses).

4. An electric dipole $d$ is at a distance $r$ from the surface of an infinite plane conductor. The dipole moment is parallel to the surface. Calculate the force $F$. Is it attraction or repulsion?

5. There is a charge $Q$ and a dipole $d$. The radius vector from the charge to the dipole is $r$. Calculate the force $F$ on the dipole.

6. An ultra-relativistic charged particle (mass $m$, charge $e$, Lorentz factor $\gamma \gg 1$) is moving past a heavy (not moving) magnetic monopole of magnetic charge $g$. The impact parameter $b$ is so large, that the particle is only slightly deflected when it passes near the monopole. Estimate the deflection angle $\chi$.

7. A nonrelativistic charged particle (mass $m$, charge $e$, velocity $v$) is moving past a heavy (not moving) charge $Q$. The impact parameter $b$ is so large, that the particle is only slightly deflected when it passes by the charge. Estimate the radiated energy $E$.

8. Spherical capacitor (concentric spheres of radii $R_1 < R_2$). The space between the spheres is filled with a medium of conductivity $\sigma$ and dielectric permittivity $\epsilon$. Calculate the discharge time $\tau$ (the time for the charge to decrease by factor $e$). (Note: the definition of conductivity of a dielectric is the same – the proportionality coefficient between current density and electric field.)

9. Give a rough estimate for the saturation magnetic field $B$ of a ferromagnetic solid. Both the formula and the number. Hint: Your answer should include only the electron mass $m$ and charge $e$ (plus $c$ and $\hbar$ for cgs units).

10. Bonus problem (correct answer counts as 3 normal problems). A $l = 10m$ long cylinder made of glass, $n = 1.5$, is rotating at angular velocity along the axis $\Omega = 1000s^{-1}$. Estimate the angle of rotation of the plain of polarization for the wave propagating along the cylinder. Both the formula and the number.

**ANSWERS (Only correct answers written on the back of this sheet will count)**
SOLUTIONS:

1. 2.

blueshift \( z = \gamma (1 + \beta) = 2 \).

2. \( E \approx \frac{m_p m_e}{2 \gamma} = 2.7 \times 10^{20} \text{eV} \).

invariant product of 4-momenta in lab and CM frames \( 2ET \approx m_p m_\tau \).

3. impossible

energy conservation requires \( (p^2 + m_p^2)^{1/2} + m_p = 2m_p + (p^2 + m_\tau^2)^{1/2} \)

4. \( F = \frac{3 \varepsilon_0 \vec{e}^2}{4 \pi \vec{r}^4} \), attraction

say, use a limiting procedure with two close charges to represent a dipole

5. \( \mathbf{F} = Q d - 3 \varepsilon_0 \vec{e}^2 \)

\( \mathbf{F} = -Q \mathbf{E} \) where \( \mathbf{E} \) is the electric field of the dipole at the charge.

6. \( \chi \sim \frac{e^2}{\gamma m c^2 b} \)

\( \chi \sim \delta p / p \), \( p \sim \gamma mc \), \( \delta p \sim eB\tau \), \( \tau \sim b / c \), \( B \sim g / b^2 \).

7. \( E \sim \frac{Q \varepsilon_0 \vec{e}^2}{m^2 c^3 \vec{r}^4} \)

\( E \sim P \tau \), \( \tau \sim b / c \), \( P \sim e^2 a^2 / c^3 \), \( a \sim F / m \), \( F \sim eQ / b^2 \).

8. \( \tau = \frac{e^2}{4 \pi \sigma} \)

\( \dot{Q} = -4 \pi \tau^2 j \), \( j = \sigma E \), \( E = \frac{Q \varepsilon_0}{\tau^2} \).

9. \( B \sim \frac{m^2 c^7}{\varepsilon_0 c^3} \sim 10 \text{ T} \)

\( B \sim \mu_B / a_B^3 \)

10. \( \alpha \sim (n - 1) \Omega l / c \sim 10^{-5} \)
Problem 1

Consider the initial state for a particle of mass \( m \) in one dimension \( \psi(x) = N \exp(-x/2a)^2 \). This represents a particle with \( \langle x \rangle = \langle p \rangle = 0 \) at some initial time \( t = 0 \).

- Find the value of \( N \) that normalizes the state.
- Compute the variance in position \( \sigma_x \) and momentum \( \sigma_p \) in this state. Do they satisfy the Heisenberg uncertainty principle?
- Write down a different, normalized initial state \( \psi_{x_0,p_0}(x) \) with the same \( \sigma_x \) and \( \sigma_p \), but with \( \langle x \rangle = x_0 \) and \( \langle p \rangle = p_0 \).

Problem 2

Find the energy eigenvalues for an isotropic harmonic oscillator in three dimensions with Hamiltonian \( H = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + \frac{m}{2} \omega^2 \mathbf{x} \cdot \mathbf{x} \). Find the degeneracy of the first 4 energy levels.

Problem 3

Consider a system of two spin 1/2 particles. Find the eigenvalues and eigenvectors of the operator \( \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} \), where \( \mathbf{S}^{(n)} \) is the spin angular momentum operator for the \( n \)th particle.

Problem 4

Consider a density matrix \( \rho \), with \( \text{Tr}(\rho) = 1 \).

- Prove that \( \text{Tr}(\rho^2) < 1 \) if and only if \( \rho \) cannot be written as the outer product of two pure states. You may assume \( \rho \) is diagonalizable.
• Prove that under unitary time evolution in the Schrödinger picture pure states cannot evolve to mixed states (and vice versa).

**Problem 5**

Consider a particle of mass $m$ moving in one dimension in a potential $V(x) = \frac{1}{2}m\omega^2x^2$, coupled weakly to a large heat bath at temperature $T$. You may assume a long time has passed since any external perturbation took place.

• Write the state (pure or mixed) that best describes the state of the particle.

• Normalize the state by computing the thermal partition function $Z$ for the particle as a function of $\omega$ and $T$. Your result should be in closed form (no infinite sums).

**Problem 6**

Estimate the ground state energy for a particle of mass $m$ moving in one dimension with potential $V(x) = \lambda x^4$. 
Quantum Prelim 2009 Solutions

1) \( \Psi = N e^{-\frac{(x)}{2a^2}} \)

\[
\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1 = \int \psi^2 \, e^{-\frac{x^2}{2a^2}} \, dx
\]

\[
= \psi^2 \sqrt{\frac{2\pi a^2}{}} \rightarrow \left[ N = e^{i\theta} \frac{1}{(2\pi a^2)^{1/2}} \right]
\]

\( \sigma_x = \langle x^2 \rangle - \langle x \rangle^2 \)

\( \langle x \rangle = 0 \) by symmetry, \( \langle x \rangle = -x \)

\( \langle x^2 \rangle = \int \psi^2 x^2 \, e^{-\frac{x^2}{2a^2}} = a^2 \)

\( \langle p \rangle = 0 \) by symm., \( \sigma_p = \langle p^2 \rangle - \langle p \rangle^2 \)

\( \langle p^2 \rangle = \left( i \hbar \frac{\partial}{\partial x} \right)^2 = -i^2 \psi^2 \int e^{-\frac{(x)}{2a^2}} \frac{\partial^2}{\partial x^2} e^{-\frac{(x)}{2a^2}} \)

\[
= \frac{i^2}{\hbar^2} \int \frac{\partial^2}{\partial x^2} \psi^2 \left( \frac{a^2}{2} \right) = \frac{\hbar^2}{4a^2}
\]

so

\[
\sigma_x \sigma_p = \frac{\hbar^2}{4}
\]
To get $\langle x \rangle$, $\langle p \rangle \neq 0$, can use translation operator

In $\mathbf{x}$: $e^{i\frac{p_0}{\hbar}x_0} \psi = e^{-\frac{i}{\hbar}p_0 x_0} \frac{\hat{p}}{\hat{x}} \psi (\psi = \psi(x-x_0))$

Shifts $x \to x - x_0$

In $\mathbf{p}$: $e^{i\frac{p_0}{\hbar}x} \psi$ shifts $\langle p \rangle \to \langle p + p_0 \rangle$.

So $\tilde{\psi} = e^{i\frac{p_0}{\hbar}x} e^{-\frac{(x-x_0)^2}{(2\sigma)^2}}$

2) Isotropic SHO in 3D

$H = H_x + H_y + H_z$  
$H_x = \frac{1}{2m} p_x^2 + \frac{m}{2} \omega^2 x^2$

$e^{+c}$.

$H_x$ has spectrum $\hbar \omega (n_x + \frac{1}{2})$

so $H$ has spectrum $\hbar \omega \left( \frac{3}{2} + n_x + n_y + n_z \right)$

$(n_x, n_y, n_z)$ triplet of integers $\geq 0$
2) Spectrum is :

<table>
<thead>
<tr>
<th>#</th>
<th>$E/(k_B)$</th>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$n_z$</th>
<th>degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$3/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$5/2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>$7/2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>$9/2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
\[ S^{(1)} = \frac{1}{2} \sigma \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ S^{(1)} = S^{(2)} = \frac{\hbar}{4} \left[ \sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)} \right] \]

\[ \sigma_x \sigma_y = - i \sigma_z \quad \sigma_x \sigma_y = \sigma_z \]

\[ S_x \sigma_y = \sigma_z \quad S_x \sigma_y = \sigma_x \]

\[ \sigma_x \sigma_y = - i \sigma_z \quad \sigma_x \sigma_y = \sigma_z \]

\[ \sigma_x \sigma_y = - i \sigma_z \quad \sigma_x \sigma_y = \sigma_z \]

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\[ \sigma_x \sigma_y = - i \sigma_z \quad \sigma_x \sigma_y = \sigma_z \]
\( \sigma |1-\rangle = \frac{k^2}{4} \left[ 2 |1-\rangle - 1|+\rangle \right] \)

\( \sigma |1+\rangle = \frac{k^2}{4} \left[ 2 |1+\rangle - 1|+\rangle \right] \)

\[
\begin{bmatrix}
1
\end{bmatrix}
\]

\[
\sigma \left( \frac{|1+\rangle + |1-\rangle}{\sqrt{2}} \right) = \frac{k^2}{4} \left( \frac{|1+\rangle + |1-\rangle}{\sqrt{2}} \right)
\]

\[
\sigma \left( \frac{|1+\rangle - |1-\rangle}{\sqrt{2}} \right) = \frac{k^2}{4} \left( \frac{-3 |1+\rangle + 3 |1-\rangle}{\sqrt{2}} \right)
\]

So e-values are
\[
\begin{bmatrix}
\frac{k^2}{4} & \frac{k^2}{4} & \frac{7}{4} & -3 \frac{k^2}{4}
\end{bmatrix}
\]
4) Prove: \( \text{Tr}(g^2) < 1 \iff g \neq |4\rangle \langle 4| \).

Assume \( g \neq |4\rangle \langle 4| \). Diagonalize \( g \).

Then \( g = \sum_i d_i |i\rangle \langle i| \)

\( \text{Tr}(g) = \sum_i d_i = 1 \). At least 2 \( d_i \neq 0 \) (since \( g \neq |4\rangle \langle 4| \)). Then by triangle inequality \( \text{Tr}(g^2) < 1 \).

Assume \( \text{Tr}(g^2) < 1 \). If \( g \neq |4\rangle \langle 4| \)

\( g = |4\rangle \langle 4| \), pick \( |4\rangle \) as a basis state. Then \( \text{Tr}(g^2) = \)

\[ \sum_i <i| (|4\rangle \langle 4| |i\rangle \langle i|) |4\rangle \]

\[ + \sum_i <i| (|4\rangle \langle 4| |i\rangle \langle i|) |4\rangle \]

\[ = 1 + 0, \text{ contradiction}. \]

\( \implies g \neq |4\rangle \langle 4| \) for any \( |4\rangle \).
Schrodinger picture

\[
| \psi(t) \rangle = e^{-i \frac{H t}{\hbar}} | \psi(0) \rangle
\]

\[
s \rightarrow e^s e^* e^{-i \frac{H t}{\hbar}} e^{+i \frac{H t}{\hbar}}
\]

\[
\text{Tr} (s^2) = \text{Tr} \left( e^{-i \frac{H t}{\hbar}} s(0) e^{+i \frac{H t}{\hbar}} e^{+i \frac{H t}{\hbar}} + i \frac{H t}{\hbar} \right)
\]

\[
= \text{Tr} \left( e^{-i \frac{H t}{\hbar}} s(0) g(0) e^+ \right)
\]

\[
= \text{Tr} (s(0)^2 e^+ e^-) = \text{Tr} (g(0)^2)
\]

Cyclicality of trace

So \( \text{Tr} (s^2) \) is \text{indep.} \ of \( t \), hence mixed cannot evolve to pure or pure to mixed.
5) Best state is thermal density matrix. In energy eigenbasis

\[ s = \sum_i \frac{e^{-\beta E_i}}{Z(\beta)} |E_i \rangle \langle E_i| \]

\[ Z(\beta) = \text{Tr} (\text{numerator}) = \sum_i e^{-\beta E_i} \]

For \( V = \frac{1}{2} m \omega^2 x^2 \) \( E_n = \hbar \omega (n + \frac{1}{2}) \)

So \( Z = \sum_n e^{-\beta \hbar \omega (n+\frac{1}{2})} \)

\[ = e^{-\frac{\beta \hbar \omega}{2}} \sum_{n=0}^{\infty} e^{-\beta \hbar \omega n} \]

\[ = e^{-\beta \hbar \omega / 2} \frac{1}{1 - e^{-\beta \hbar \omega}} = \frac{1}{2 \sinh \frac{\beta \hbar \omega}{2}} \]
6) Dimensional analysis

\[ [a^3] = [L] \rightarrow [a]^3 = \frac{\text{mass}}{(\text{length})^2 (\text{time})^2} \]

\[ [t] = \frac{\text{mass} \cdot \text{length}^2}{\text{time}} \quad [m] = \text{mass} \]

\[ \lambda^{a+b} m^c = \frac{M^a}{L^{2a-2a}} \frac{M^b L^{2b}}{T^b} M^c = M \frac{L^2}{T^2} \]

\[ \Rightarrow a + b + c = 1 \quad 2b - 2a = 2 \quad 2a + b = 2 \]
\[ \Rightarrow b = a + 1 \quad \Rightarrow 3a + 1 = 2 \]
\[ \Rightarrow a = \frac{1}{3} \]

\[ \frac{5}{3} + c = 1 \Rightarrow c = -\frac{2}{3} \]

\[ \text{So} \quad \lambda^{1/3} m^{1/3} \frac{1}{h^{1/3}} \]

\[ \text{Nirnal theorem} \]

\[ \text{CR:} \quad \sigma \approx \frac{1}{2} \quad \chi \left( \frac{L^2}{2m} \right) \approx 4 \chi \]
\[
\langle U \rangle \sim \frac{\hbar^{4/3}}{4^{4/3} m^{2/3}}\left(\frac{\gamma^{4/3}}{\gamma^{4/3}}\right)^{4/3}
\]

\[E = T + V, \quad 2T = 4V\]

\[= 3V, \quad \Rightarrow \]

\[E \sim \frac{3}{4^{4/3}} \frac{\hbar^{4/3}}{m^{2/3}}\left(\frac{\gamma^{4/3}}{\gamma^{4/3}}\right)^{4/3}
\]
QMII, FALL 2009, PRELIM EXAM
Natural units $\hbar = c = 1$ are used throughout.
Total Point Value: 40 Points

Problem 1

$N$ Fermions interact with the Hamiltonian

$$H = \sum_{n=1}^{N} a_n^\dagger a_n + \lambda (a_n a_{n+1} + a_{n+1}^\dagger a_n^\dagger + a_n^\dagger a_{n+1} + a_{n+1} a_n),$$

where the index $n$ is cyclically identified ($N + 1 \equiv 1$). The operators $a_n, a_n^\dagger$ obey canonical anti-commutation relations: $a_n a_m^\dagger + a_m^\dagger a_n = \delta_{m,n}$, while all other anti-commutators vanish.

a

Find all eigenvalues and their degeneracies for the "free" Hamiltonian at $\lambda = 0$.

[5 points]

b

To first order in $\lambda \ll 1$, find the split of the two lowest eigenvalues.

[5 points]

Solution

a

$E_n = n, \ 0 \leq n \leq N; \ N!/n!(N-n)!$.

b

The ground state $n = 0$ does not shift to first order in $\lambda$. The second lowest eigenvalue $n = 1$ is $N$-fold degenerate and the perturbation is diagonalized by the states $|k\rangle = \exp(ikn)a_0^\dagger|0\rangle$. $E_1 = 1 + 2\lambda \cos(k)$.

Problem 2

a

Find the transformation law of the Schrödinger wave function between a static frame and a non-relativistic inertial frames moving at constant speed $v$. Find the answer first in momentum representation then in coordinate representation.

[5 points]
b

The nucleus of a hydrogen atom in the ground state undergoes an elastic collision which gives it a final velocity \( v \ll 1/\text{am} \), where \( a \) is the Bohr radius and \( m \) is the electron mass. The duration \( \tau \) of the collision is much shorter than \( 1/E \) and \( a/v \).

Find the probability that the atom is in an excited state after the collision.

[5 points]

Solution

a

Primed quantities denote the moving frame. For an arbitrary system with momenta \( p_i, i = 1, \ldots, n \), \( p_i = p'_i + m_i v \) and the wave function solves the same equation written in new coordinates. So \( \psi'(p') = \psi(p) = \psi(p'_i - m_i v) \). In the Schrödinger (coordinate) representation:

\[
\psi'(x') = \exp(\text{i} m x'_v v) \psi(x'_i)
\]

b

The collision is sudden. Applying the formula in point (a) to \( \psi_1 = 2 \exp(-\tau/a)/\sqrt{4\pi a^3} \) we get a probability of excitation

\[
P = 1 - \frac{4\pi}{mv} \int_0^\infty drr \psi_1(r)^2 \sin(mvr) = 1 - (1 + m^2 v^2 a^2/4)^{-4}.
\]

Problem 3

The Dirac equation in an external electromagnetic field is

\[
[\gamma^m(\partial_m + ieA_m) - m]\psi = 0,
\]

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^\sigma = \begin{pmatrix} 0 & s \sigma \\ s \sigma & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.
\]

Define \( \phi' = \exp(i mt)\phi, \chi' = \exp(i mt)\chi \).

a

Show that when \( A_\mu = 0 \), in the non-relativistic limit \( (p \ll m) \) one has \( \phi' \approx \chi' \) and that \( \phi' \) approximately obeys the free Schrödinger equation

\[
i\partial_v \phi' = \frac{1}{2m} \nabla^2 \phi'.
\]

[5 points]
b
Show that the Schrödinger equation in an external electromagnetic field is
\[ i(\partial_0 + ieA_0)\phi' = \left[ -\frac{1}{2m}(\nabla + ie\vec{A})^2 + g\frac{e}{2m}\vec{B} \cdot \vec{\sigma} \right] \phi', \]
with $\vec{B}$ magnetic field. Find the gyromagnetic ratio $g$.
[5 points]

c
Dirac's equation can be modified by adding a Pauli term
\[ [\gamma^m(\partial_m + ieA_m) - m + i\alpha F_{mn}\gamma^m\gamma^n]\psi = 0. \]
Find the gyromagnetic ratio implied by the new equation as a function of $\alpha$.
[5 points]

Solution

a
Dirac's equation is
\[ (E - m)\phi + i\vec{p} \cdot \vec{\sigma} \chi = 0, \quad -(E + m)\chi + i\vec{p} \cdot \vec{\sigma} \phi = 0. \]
Hence $\chi = (E + m)^{-1}i\vec{p} \cdot \vec{\sigma} \phi$. Substituting into the first equation one gets $E^2 - p^2 = m^2$.
For $E = +\sqrt{p^2 + m^2} \approx m + p^2/2m$, $\chi \approx (2m)^{-1}i\vec{p} \cdot \vec{\sigma} \phi$ and the first equation reduces to the non-relativistic Schrödinger equation for $\phi'$.

b
Repeat with $\vec{p} \rightarrow \vec{P} = \vec{p} + e\vec{A}$; use $[P_i, P_j] = -i\epsilon_{ijk}B_k$, get $g = 2$.

c
$g = 2 + 4m\alpha/e$

Problem 4

A beam of particles of mass $m$ and intensity $I$ moves at speed $v$ in the direction $z$ in a diluted gas. The speed is such that the scattering amplitude of a single beam particle off a gas molecule is well approximated by the very low energy limit. Compute the amplitude in the scattering length approximation in terms of the attenuation coefficient $\kappa$ ($dI/dz = -\kappa I$).
[5 points]
Solution

The total cross section $\sigma$ is related to $\kappa$ by $\sigma = \kappa/vp$. The scattering amplitude is elastic and approximately given by $f \approx f_0 \approx (-1/a + ik)^{-1}$; the optical theorem gives $a = \sqrt{\sigma/4\pi}$. 
Statistical Physics
Prelim 2009

Closed Book
(each problem is worth 20 points)

1. A classical ideal gas in 3D starts at time 0 with a velocity distribution \( h(v) \), but is concentrated at the origin, \( r = 0 \). Find the time development of
   a) the spatial entropy \( s(t) = -\int n(r,t) \ln n(r,t) \, d^3r \), where \( n(r,t) \) is the particle density \( (15) \)
   b) the corresponding velocity space entropy \( (5) \)

2. A system of Hamiltonian \( H \) is in contact with a very large bath. In the combined system of total energy \( E_\alpha \), volume \( V_\alpha \), only energy and volume can be exchanged via this contact.
   a) applying the extended \( (H \leq E) \) microcanonical distribution to the combined system and bath, find the distribution attained by the system itself at fixed volume \( V \), and the corresponding partition function \( (10) \)
   b) show that the maximum of this function over \( V \) generates the Gibbs free energy \( (10) \)

3. A ring has 3 distinct sites to which molecules of species X can bind. At most one X can bind at each site. There is an energy of - A for each pair of adjacent occupied sites, and the ring can exchange molecules and energy with its environment.
   Find the mean number of occupied sites in terms of appropriate thermodynamic parameters.

4. A uniform, spin 1/2, fermion fluid of density \( n \) has Newtonian kinetic energy, and a pair interaction \( \phi(r-r') \) that is both long range and weak. Assuming that the \( \mu(n) \) relation is known, at what value of \( k \) does \( k \)-occupation become half-maximum?

5. The Vlasov equation for an N-particle classical fluid in an external potential is the one-body Liouville equation (normalized to \( N \)) in which the pair interaction of the other bodies is approximated by the mean field potential \( \int \phi(r-r') \, n(r',t) \, dr' \). The temperature tensor \( \Theta(r,t) \) is defined as the obvious generalization of the usual scalar temperature - is assumed known.
   Obtain the equations of density and current density conservation implied by the Vlasov equation.

PLEASE SHOW ALL WORK
(if in doubt as to terminology, state precisely what question you are answering)
1. \( u) f(\nu, \omega, 0) = \delta(\omega) \delta(\nu) \)
\[ + f(n, \nu, t) = \delta(n - 3w) \delta(\nu) \\
\quad = \frac{1}{3} \delta(n - 3w) \delta(\nu) \frac{d^3\nu}{d\nu} \]
\[ = \frac{1}{3} \delta(n - 3w) \]
\[ \sigma_n(t) = -\int \frac{1}{3} \delta(n - 3w) \left( \lambda n \delta(n - 3w) - 3 \lambda n t \right) d^3\nu = \frac{\lambda n}{3} (1 + 3 \lambda n t) \]
\[ \nu(n + t) = \nu(n), \quad \sigma_n(t) = -\int \frac{\lambda n}{3} \delta(n - 3w) d^3\nu = \frac{\lambda n}{3} \]

2. \( \Theta = H(V, q) + H_b(V_0 - V, \eta) \)
\[ = W \Theta \]
\[ W = \int \frac{1}{2} \omega^2 d\eta = \frac{1}{2} \omega^2 \frac{d^2V}{d\eta^2} \]
\[ = \frac{1}{2} \frac{\lambda n}{3} \nu \frac{dV}{d\nu} \]
\[ \beta = \frac{\lambda n}{3} \nu \frac{dV}{d\nu} \]
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3. \( \Omega = \Omega(V, \nu_2, \nu_3) = -A(\nu_1, \nu_2, \nu_3) \nu, \nu = \nu_1 \Omega \)
\[ = \frac{1}{2} (\nu_1 + \nu_2 + \nu_3) \frac{d\Omega}{d\nu_1} + \frac{1}{2} (\nu_1 + \nu_2 + \nu_3) \frac{d\Omega}{d\nu_2} \]
\[ = \frac{1}{2} (\nu_1 + \nu_2 + \nu_3) \frac{d\Omega}{d\nu_1} + \frac{1}{2} (\nu_1 + \nu_2 + \nu_3) \frac{d\Omega}{d\nu_2} \]
\[ = 1 + 3 \nu_1 \frac{d\Omega}{d\nu_1} + 3 \nu_2 \frac{d\Omega}{d\nu_2} + \frac{3}{2} \frac{d\Omega}{d\nu_3} \]
\[ \nu = \frac{1}{3} \frac{d\Omega}{d\nu_1} \]
\[ = \frac{3}{2} \frac{d\Omega}{d\nu_1} + \frac{3}{2} \frac{d\Omega}{d\nu_2} + \frac{3}{2} \frac{d\Omega}{d\nu_3} \]
\[ = \frac{3}{2} \frac{d\Omega}{d\nu_1} + \frac{3}{2} \frac{d\Omega}{d\nu_2} + \frac{3}{2} \frac{d\Omega}{d\nu_3} \]
4. \[ E_\xi = \frac{1}{c^2} \sum_{e=1}^n \left( \frac{q e \xi_i}{c^2} - m \right) = \frac{1}{c^2} \int \left( \sum_{\xi} q e \xi_i \right) \left( \frac{1}{c^2} \right) d^3 \xi_i = \frac{1}{c^2} \int \left( \sum_{\xi} q e \xi_i \right) \left( \frac{1}{c^2} \right) d^3 \xi_i = - \frac{1}{c^2} N \sum_{\xi} \left( \frac{q e \xi_i}{c^2} - m \right) \]

\[ N = \frac{2}{c^2} \int \left( \frac{1}{c^2} \right) \left( \frac{q e \xi_i}{c^2} - m \right) d^3 \xi_i = \frac{1}{c^2} \left( \sum_{\xi} \left( \frac{q e \xi_i}{c^2} - m \right) \right) \]

5. \[
\frac{\partial}{\partial t} \left( \frac{1}{c^2} \xi_i \right) + \nu \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) = \frac{1}{m} \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \right) \left( \frac{1}{m} \right) \xi_i + \frac{1}{m} \int \left( \sum_{\xi} q e \xi_i \right) \left( \frac{1}{c^2} \right) d^3 \xi_i + \frac{1}{m} \sum_{\xi} \left( \frac{q e \xi_i}{c^2} - m \right) \]

\[ d^3 \nu : \]

\[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \xi_i \right) + \nu \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) = 0 \]

\[ \nu d^3 \nu : \]

\[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \xi_i \right) + \nu \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) = \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) \]

\[ + \frac{1}{m} \int \left( \sum_{\xi} q e \xi_i \right) \left( \frac{1}{c^2} \right) d^3 \xi_i + \frac{1}{m} \sum_{\xi} \left( \frac{q e \xi_i}{c^2} - m \right) \]

But \[ \nu u + \gamma = \left( \nu u + \gamma \right) + \left( \nu - u \right) \left( \nu - u \right) + \gamma \]

where \( u = \nu + \gamma \)

\[ = \frac{1}{m} \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) + \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) \]

and \( \left( \nu - u \right) \left( \nu - u \right) + \gamma = 0 \)

\[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \xi_i \right) + \nu \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) = \frac{\partial}{\partial \xi_i} \left( \frac{1}{c^2} \xi_i \right) \]

\[ + \frac{1}{m} \int \left( \sum_{\xi} q e \xi_i \right) \left( \frac{1}{c^2} \right) d^3 \xi_i \]