PROPERTIES, PROPOSITIONS AND CONDITIONALS

HARTRY FIELD

Abstract. Section 1 discusses properties and propositions, and some of the motivation for an account in which property instantiation and propositional truth behave "naively". Section 2 generalizes a standard Kripke construction for naive properties and propositions, in a language with modal operators but no conditionals. Whereas Kripke uses a 3-valued value space, the generalized account allows for a broad array of value spaces, including the unit interval [0,1]. This is put to use in Section 3, where I add to the language a conditional suitable for restricting quantification. The shift from a value space based on the "mini-space" $\{0, \frac{1}{2}, 1\}$ to one based on the "mini-space" [0,1] leads to more satisfactory results than I was able to achieve in previous work: a vast variety of paradoxical sentences can now be treated very simply. In Section 4 I make a further addition to the language, a conditional modeled on the ordinary English conditional, paying particular attention to how it interacts with the restricted quantifier conditional. This is all done in the [0,1] framework, and two alternatives are considered for how the ordinary conditional is to be handled; one of them results from adding a tweak to a construction by Ross Brady. Section 5 discusses a further alternative, a standard relevance conditional (for the ordinary conditional, perhaps for use with a different quantifier-restricting conditional), but argues that it is not promising. Section 6 discusses the identity conditions of properties and propositions (again in the setting of a value space based on [0,1]); the issue of achieving naivety for coarse-grained properties is seen to be more complicated than some brief remarks in Field 2010 suggested, but a way to get a fair degree of coarse-grainedness is shown.

1. Naivety in a Theory of Properties and Propositions

1.1. Properties and Propositions.

I take it that the main point of talking about propositions and properties[1] is to provide a natural framework for talking about language and the mind. In the language case, this goes as follows:

(EXP): A sentence (as used on a given occasion) is true (at possible or impossible world w) if and only if it expresses (on that occasion) a proposition that is true (at w).
A formula (as used on a given occasion) is true of something o (at world w) if and only if it expresses (on that occasion) a property that is instantiated (at w) by o.

Linguistic truth and satisfaction (the converse of truth-of) are reduced to *propositional* truth and property-instantiation, via the expression relation. In the mental case, e.g. of beliefs, it's similar.

[1]In the "abundant" sense in which there are highly "unnatural" properties like being either hairy or purple or a small ocelet or an African country. A "sparse" conception of properties, e.g. one confined to basic physical properties, might well have another point, but such a conception will not be under discussion.

But what are propositions and properties? And what is it for a proposition to be true (at a possible or impossible world w)? And for a property to be instantiated by an object (at such a world)?

A common procedure is to stipulate at the start what propositions and properties are, characterizing them independently of language. For instance, we might stipulate that propositions are just sets of worlds (perhaps including impossible ones), and that properties are just sets of pairs of worlds and things. On this approach, the theory of propositional truth and property-instantiation is trivial: a proposition is true at a world if and only if the world is a member of it, and something o instantiates a property at a world w if and only if the pair $\langle w, o \rangle$ is a member of that property.

Unfortunately, this leaves all the interesting questions unresolved. Propositions and properties are only of use if we can relate them to our sentences and predicates—that is, if we can sensibly speak of the proposition or property that a given one of our sentences or predicates *expresses (on a given occasion of use).* What we're ultimately interested in is what it is for *whatever proposition is expressed (on a given occasion) by 'Snow is white'* to be true at a world—and analogously for sentences other than 'Snow is white', including potentially paradoxical ones like 'Nothing that Joe is saying is true'. Similarly, what we're ultimately interested in is what it is for *whatever property is expressed (on a given occasion) by 'is a red ball'* to be instantiated by a given thing at a world—and analogously for 'is either a red ball or a property that doesn't instantiate itself'.

Moreover, when it comes to *prima facie* paradoxical properties and propositions, the possible worlds definition of propositions and properties begs questions against many approaches. Perhaps there is *some generalization* of the idea of sets of worlds, or sets of world-object pairs, that could be used instead; but in advance of developing the theory it's hard to know whether this is so, and if it is so then what sort of entities will do the trick.[2]

A better approach to the issues of propositions and their truth, and properties and their instantiation, is to start from the idea that our grasp of the notions of proposition and property is based on locutions like "the proposition *that* ___" and "the property *of being* ___", where in the blanks go sentences and formulas of our language that we already understand. I'll introduce abstraction terms for this: λB for "the proposition that B" where B is a sentence of our language, $\lambda x Px$ for "the property that P" where Px is a formula of our language involving only x free. Then the questions I will start from (before I generalize them below, to allow for parameters) are:

> Under what circumstances is there a proposition λB, and when there is, under what circumstances is it true at a given world?

and

[2]There are also technical issues about the worlds account: taken literally, it assumes that the worlds form a set (which in a standard set theory entails that there is a limitation on the size of worlds), and that within each world the things that instantiate a given property form a set (which in a standard set theory rules out for instance that there be a property that at some world is instantiated by all sets). I don't take these to be integral to the spirit of the proposal; the generalization I'm alluding to will avoid these features, and can do so by offering axioms on propositions and propertes rather than a reductive definition.

> Under what circumstances is there a property $\lambda x P$, and when there is, under what circumstances is it instantiated at a given world by a given object?

('Object' here and in what follows is to be construed broadly: *anything*, including properties and propositions, will count as objects.) It will simplify what follows to imagine that our own language is regimented to exclude ambiguous terms or constructions, indexical elements, and the like; otherwise the ambiguities or indexicalities would be inherited by the λ-terms. Let L be such a regimented version of our language.

I will leave metaphysical questions about the nature of properties and propositions unsettled. But I'm inclined to think that there is nothing to say about properties and propositions beyond the answers to these questions, plus the question of the identity-conditions of properties and propositions to be considered later.

1.2. **Naivety.** The focus of this paper will be on the *naive* theory of properties and propositions, which answers the above questions as follows:

> For each formula $P(x)$ of L with exactly one free variable x,
> (i) there is a corresponding property $\lambda x P(x)$, and
> (ii) $\lambda x P(x)$ is instantiated (at a possible or impossible world w)
> by all and only the objects o such that (at w) $P(o)$.

Similarly:

> For each sentence B of L,
> (i_0) there is a corresponding proposition λB, and
> (ii_0) λB is true (at a world w) if and only if (at w) B.

Such a theory is apparently threatened by paradoxes, e.g. the analog for properties of Russell's paradox for sets; but there is now a large body of literature on ways to keep the naive theory, by weakening the classical logical assumptions used in the derivations of absurdities from it. This paper will suggest some improvements in current approaches to this, with some attention to the assessment of alternatives.

While I won't seriously discuss alternatives to naivety here, I will make one remark regarding the most obvious alternative: a *non-existence theory*, which for paradoxical predicates denies (i), but which accepts that (ii) holds whenever $\lambda x P(x)$ exists and which keeps to classical logic.[3] On such a theory, there can be no such property as $\lambda x(x$ is a red ball $\vee$ x is a property that doesn't instantiate itself). For properties aren't red balls (at least in the actual world, and let's focus on that), so by (ii), if such a property existed it would instantiate itself if and only if it doesn't; and that can't happen in classical logic. Unfortunately, the non-existence of such a property is awkward. For that together with the (EXP) from which we started implies that 'is either a red ball or a property that doesn't instantiate itself' isn't true of red balls. Presumably that isn't acceptable, so presumably the non-existence theorist will reject (EXP).[4] I don't say that's devastating, but I do think

[3] Taking this to include all the usual structural rules, including the transitivity of inference.

[4] The exceptions to (EXP) would be far more pervasive on a non-existence solution to the property-theoretic paradoxes that models its view of property-existence on an iterative theory of classes. For on such a theory there can be no class that contains everything (including all classes); so if properties are conceived similarly, there can be no universal property (property instantiated by everything, whether or not a property). So there can be no property $\lambda x(x = x)$. Also, there can be no property $\lambda x\neg(x$ is an electron). Since presumably both '$x = x$' and '$\neg(x$ is an electron)' are true of protons, we'd have exceptions to (EXP) even for non-paradoxical predicates. (Quine's *New*

that (EXP) is part of our usual conception of "what properties are good for", and a classical theorist is probably best off keeping (i) while restricting (ii) since this leaves (EXP) unthreatened. (A naive theory by definition keeps (i) as well as (ii), so on it there is no threat to (EXP).)

As I hinted, I will want to strengthen the requirements on a naive theory of properties, and correspondingly, a naive theory of propositions: I want to allow for properties and propositions that aren't straightforwardly definable in our language, but only definable from parameters. That is, for properties I will demand that for each formula $P(x; u_1, ..., u_n)$ of our language (with no ambiguous or indexical elements and) with x free and allowing for n additional free variables $u_1, ..., u_n$ for any $n \geq 0$, then for any entities $o_1, ..., o_n$,

(i^+): there is a corresponding property $\lambda x P(x; o_1, ..., o_n)$, and

(ii^+): $\lambda x P(x; o_1, ..., o_n)$ is instantiated (at a world w) by all and only those entities o such that (at w) $P(o; o_1, ..., o_n)$.

(The analogous expansion (i_0^+) and (ii_0^+) is to be made in the naive theory of propositions.) I stress that there are no restrictions on the "parameters"$o_1, ..., o_n$: in particular, they might themselves be properties or propositions.

One advantage of allowing for parameters—and in particular, parameters that might themselves be properties or propositions—is that it immediately gives us compositional laws. Suppose we want the law

(C): For any properties P and Q there is a property P *and* Q that is instantiated (at any world) by all and only the things that (at that world) instantiate them both.

Before the expansion to include parameters, (i) and (ii) would have allowed us to derive specific instances of (C), such as

> There is a property of *being red and a ball* that is instantiated (at any w) by all and only the things that (at w) instantiate both the property of *being red* and the property of *being a ball*.

But that is insufficient for two reasons: (a) it isn't a general law, just a collection of instances; and (b) it doesn't even cover all the relevant instances, e.g. it doesn't cover any properties that don't happen to be expressable in our language. But with the expansion, everything is fine: apply (i^+) and (ii^+) to the predicate $R(x; u_1, u_2)$ "x instantiates u_1 and x instantiates u_2", and then restrict the universal generalizations over o_1 and o_2 in (i^+) and (ii^+) to properties, including ones not definable in our language even from parameters (if there are such properties).

And in fact there may not even be any properties that aren't definable in our language *from parameters*, when those parameters can include properties. That's because (i^+) implies that for each property P, there is a property $P^* =_{df} \lambda x(x$ instantiates $P)$; and by (ii^+), it is instantiated at each world by precisely those things that instantiate P. Provided we take necessary co-instantiation as sufficient for property-identity,[5] P^* just is P, and so P is in a trivial sense definable in the language from itself as parameter. This is obviously only of limited interest: if there

---

Foundations, though generally regarded as an unattractive set theory, would serve considerably better than iterative set theory as a model for a property theory on which to base semantics; but the problems in the main text hold for it too.)

[5]In the end, I won't quite do this. However, when P is a property that can't be instantiated by properties or propositions, the proposals in Section 6 will take P^* to just be P.

is a property of being ferschlugginer that is otherwise undefinable in our language, this gives no way of introducing it into our language in a way that lets us understand it. Still, it does allow the theory to apply to such properties even though we don't understand them.

It will simplify many of the formulations below to introduce the ideas of a *parameterized 1-formula* and a *parameterized sentence*. The latter is simply a pair of a formula and an assignment of objects (in a broad sense that includes properties and propositions) to *all* its free variables; the former is a pair of a formula and an assignment of objects to *all but one of* its free variables. Then (i^+) and (ii^+) are subsumed under (i) and (ii) provided that we understand $P(x)$ to be a *parameterized* 1-formula; and analogously for (i_0^+) and (ii_0^+) and parameterized sentences. For future use, I'll also let a *fully parameterized abstraction term* be a pair of an abstraction term together with an assignment of objects to all its free variables.

How fine-grained are properties and propositions to be? That is, what is the relation that $P(x)$ must bear to $Q(x)$ (where P and Q may contain parameters) for $\lambda xP(x)$ to equal λxQx, and what is the relation that B must bear to C for λB to equal λC? I'm inclined to think that there is no one "right answer" to this question, that it is basically a matter for stipulation (though with some limitations on which stipulations are coherent).

That said, there is much to be said for developing a theory that is quite coarse grained, something very much in the spirit of the worlds account with which I started. It can't be exactly that; and I won't be able to deal with the matter until Section 6, when the basic theory is in place. Until then, I will simply leave the notion of property-identity out of the theory.

Two final remarks before we get going:

First, the theory to be offered here extends also to relations (viewed non-extensionally); indeed, relations might be thought of as multi-place properties and propositions as 0-place properties (whereas properties in the sense originally intended were 1-place). If we do so, it is possible to develop the naive theories of all three in a common way. But there is a slight awkwardness about the format of the common theory: if one wants a theory that deals with relations with arbitrarily many places, the natural way to do it is by an instantiation predicate with arbitrarily many places; but predicates of varying adicity present certain complications. The awkwardness certainly isn't insurmountable, but for simplicity I will leave relations out of my presentation of the basic theory. This is not much of a loss, since if we have arbitrary k-tuples available, we can slightly artificially think of a k-place relation as a property of k-tuples.

Second, though the official focus of this paper is on the non-linguistic (properties and propositions), the main problems that must be overcome in achieving naivety are the same in the linguistic case as in the non-linguistic. (The one exception is that the problems about property-identity don't arise in the linguistic case.) The reader who wishes can easily modify the discussion in Sections 2-5 to avoid all talk of properties and propositions and to instead focus on language.

2. Generalizing Kripke's Construction

I'm going to begin the formal discussion with a generalization of a familiar tool, from Kripke 1975. (It's more familiar in the context of sentential truth than of property-instantiation or propositional truth, but it's well-known that it can be applied to those as well.) The main generalization is that Kripke developed the tool in the context where the valuation space is the Kleene algebra: the set $\{0, \frac{1}{2}, 1\}$, ordered in the usual way, with the obvious involution operation (taking value v to $1-v$) to handle negation.[6] Visser 1984 generalized the value space for the construction in one way, to the context of the Dunn 4-algebra: the set {0, b, c, 1}, partially ordered in such a way that $\forall v(0 \leq v \leq 1)$ and that b is incomparable to c, with the involution operation that flips 1 with 0 and that takes each of b and c to itself. I'm going to generalize the algebra in a different way (which could be further generalized to incorporate Visser's, though I won't bother to do so). This generalization will turn out to be very useful when it comes to conditionals.

Let a *Kripke algebra* be a complete deMorgan algebra in which (unlike with the Dunn algebra) there is an element $\frac{1}{2}$ that is a fixed point of negation and is comparable to every element. More fully: it has the form $\langle V, \leq, 1, \frac{1}{2}, 0, \natural\rangle$ where[7]

(i): V is a set with at least three distinct elements 0, $\frac{1}{2}$ and 1
(ii): $\leq$ is a partial ordering on V, with $0 \leq \frac{1}{2} \leq 1$
(iii): every subset of V has a least upper bound in V and a greatest lower bound in V
(iv): $(\forall v \in V)(0 \leq v \leq 1)$
(v): $\natural$ is an operation on V such that$(\forall u, v \in V)$ $(\natural u \leq \natural v$ iff $v \leq u)$
(vi): $(\forall v \in V)(\natural\natural v = v)$
(vii): $\natural(\frac{1}{2}) = \frac{1}{2}$
(viii): $(\forall v \in V)(v \leq \frac{1}{2} \vee \frac{1}{2} \leq v)$

(The last condition together with the partial ordering entails that $\frac{1}{2}$ is the only fixed point of the involution (or "negation") $\natural$.)

One example of a Kripke algebra that will be useful to bear in mind (though special in that the order is total) has V the unit interval [0,1], $\leq$ the usual order on it, and $\natural = 1 - v$. This is the example that I will put to use for conditionals, starting in Section 3.

2.1. **Background Framework.** Let L_0 be a first order modal language (for simplicity I'll assume that it has no primitive function symbols), and L_0^+ the result of expanding it in the obvious way to include (i) an abstraction operator λ for forming terms for properties and propositions, together with (ii) a 1-place predicate 'Property', another 1-place predicate 'Proposition', and a 2-place predicate ξ. (This is short of the full L we'll later use, which includes also two binary operators '$\rightarrow$' and '$\triangleright$' on formulas.) For any formula A, λA is to be a singular term whose free variables are just those of A; relative to any assignment of objects to all the variables of A, it denotes a proposition. For any formula A and any variable x, λxA is to be a singular term whose free variables are just those *other than* x that are free in A; relative to any assignment of objects to those other variables, it denotes a property. (We could if we like restrict to the case where x is free in A, it won't matter. But if we don't, then even if x isn't free in A, I'll regard λxA as denoting a property

[6]Kripke also considered supervaluationist alternatives, where the valuation space is a Boolean algebra, but this isn't suited to naive theories.

[7]There's a bit of redundancy here, which I've kept for the sake of clarity.

rather than a proposition; for instance, $\lambda x \forall y(Number(y))$ will denote the property of being in a world where everything is a number.)[8]

What does 'ξ' mean? The parameterized formula '$o_1 \, \xi \, o_2$' is to be regarded as false whenever o_2 isn't a proposition or property. When o_2 is a property, '$o_1 \, \xi \, o_2$' can be read as 'o_1 instantiates o_2'. When o_2 is a proposition, '$o_1 \, \xi \, o_2$' can be read as 'o_2 is true'; in this case, o_1 won't matter. So 'ξ' embodies truth and instantiation together: $True(y)$ is to be an abbreviation of "y is a proposition and $\forall x(x \, \xi \, y)$" (though we could replace the $\forall$ with an $\exists$ without affecting anything); and $Instantiates(x, y)$ as an abbreviation of "y is a property and $x \, \xi \, y$".

Let M_0 be any V-valued modal model for L_0. More precisely, M_0 will consist of

(i): a non-empty set W of worlds
(ii): for each $w \in W$, a subset W_w of W (the set of worlds "accessible from w")
(iii): for each $w \in W$, a set $|M_0|_w$ of objects (the domain of w). I'll let $|M_0|$ be $\bigcup_{w \in W} |M_0|_w$, and refer to this as the domain of the modal model.
(iv): for each name of L_0, a member of $|M_0|$
(v): for each k-place predicate p of L_0, and each $w \in W$, a function p_w from $|M_0|^k$ to V. ("The V-valued extension of p at w, in M_0".)

(For simplicity, I'm allowing worlds to overlap, so that one doesn't need a counterpart relation for cross-world identifications. If one likes one could impose "actualist" restrictions, such as that if neither b nor c is in $|M_0|_w$ then $p_w(o_1, ..., o_{i-1}, b, o_{i+1}, ..., o_k) = p_w(o_1, ..., o_{i-1}, c, o_{i+1}, ..., o_k)$; but I won't bother.) For the definition of validity (see Section 2.3), I also include

(vi): a nonempty subset N of W: the set of "normal" worlds.

But N could just be W. For simplicity in the construction to follow, I'll assume that nothing in $|M_0|$ is a term or formula of L_0^+, or a set built in part out of such terms or formulas. There will be no loss of generality, since if we have a modal model that doesn't meet this condition we can replace it by an isomorphic one that does.

We want a modal model M_0^+ (which I'll often just call M) with a domain that extends $|M_0|$ to include representatives of properties and propositions, but which looks like M_0 on the domain of the latter. (I emphasize that this is only a model: it makes no claim to include representatives of all properties in its domain.) For simplicity I'll do the construction in such a way that any proposition or property in the domain of the expanded model is in the domain of every world of the model; so that even if o isn't in a given world, a property defined using o as a parameter is. This won't play any essential role in the theory, it just makes for simplicity.

The expansion of the domain of M_0 goes in successive stages: in each stage, we construct representatives of properties and propositions from formulas, allowing parameters both from the ground model and from prior stages. (The stratification into n-formulas isn't used; this is a different stratification.) More fully: let

---

[8]The formation rules are given in cumulative levels. The *0-terms* are just the names and variables of L_0. From these we use the usual formation rules for the connectives to construct *0-formulas*, which will be just the formulas with no abstraction terms. The *1-terms* will be the 0-terms together with abstraction terms formed from 0-formulas. (They can have free variables, to be filled by parameters.) From these we construct the *1-formulas*, and then *2-terms*, which are the 0-terms together with abstraction terms formed from 1-formulas. And so on. The terms and formulas are whatever appears at some finite level.

$PROPOS[X]=$ {parameterized sentences whose parameters are in X}, and

$PROPTY[X]=$ {parameterized 1-formulas whose parameters are in X}.

For any natural number j, let

$PROPOS_j = PROPOS[|M_0| \cup \bigcup\{PROPOS_k \cup PROPTY_k : k < j\}]$, and

$PROPTY_j = PROPTY[|M_0| \cup \bigcup\{PROPOS_k \cup PROPTY_k : k < j\}]$.

Let $PROPOS_\omega$ be the union of the $PROPOS_j$, and similarly for $PROPTY_\omega$. (These depend on the starting model M_0: we shouldn't expect the model to contain a representation of a property of being ferschlugginer, if that is intuitively an "object level" property and there is no corresponding formula in the ground language. But this is no problem, we're only constructing a model, not making a claim about reality.) The members of $PROPOS_\omega$ and $PROPTY_\omega$ are *representations of* propositions and properties: we should think of the representation of properties and propositions as many-one, but we're not yet in a position to specify the equivalence relation on representations that make them representations of the same property or proposition. In Section 6 I'll consider ways of defining property identity in the language, with a formula R whose extension in the model is the desired equivalence relation.

Until further notice, the domain of M_0^+ is to be $|M_0| \cup PROPOS_\omega \cup PROPTY_\omega$. Once a definition of property identity is in place, we can contract the model. And (following the simplifying stipulation of two paragraphs back) each $|M_0^+|_w$ is also just $|M_0|_w \cup PROPOS_\omega \cup PROPTY_\omega$.

The next step in the specification of M_0^+ is the treatment of denotation. I'll assign denotation to fully parameterized terms of L_0^+. Since the only terms are L_0-names, abstraction terms, and variables, these are just the L_0-names plus parameterized closed abstraction terms plus "parameterized variables". The latter are in effect proper names; so we in effect have a name for every object in the domain $|M_0^+|$. In M_0^+, each name of L_0 will denote what it denotes in M_0; and each parameterized closed abstraction term will denote a member of $PROPOS_\omega$ or $PROPTY_\omega$.

To complete the specification of the modal model M, we need to assign a value in V to every parameterized sentence at each world. That's where Kripke comes in.

2.2. The Kripke Construction Generalized to Kripke Algebras.

As generalized to arbitrary Kripke algebras, Kripke's construction for the modal language under consideration is as follows. As a preliminary:

Let an *OPW triple* ("Object, Property/proposition, World", where again "objects" include properties and propositions) be a member of $|M_0^+| \times (|M_0^+| - |M_0|) \times W$.

Let a *valuation for* ξ be any function I that assigns a member of V to each OPW triple.

Relative to such an I, we evaluate parameterized sentences at worlds by generalized Kleene rules:

- For any k-place predicate p of L_0 and fully parameterized terms $t_1, ..., t_k$: (i) if for each i, the denotation o_i of t_i is in the ground model $|M_0|$, then

$|p(t_1, ..., t_k)|_{I,w}$ is the same as in M_0 (i.e. it's the value that p_w assigns to $\langle o_1, ..., o_k \rangle$);
(ii) if for at least one i the denotation o_i of t_i is a property or proposition, then for each w, $|p(t_1, ..., t_k)|_{I,w} = 0$.

- $|Property(t)|_{I,w}$ is 1 if the object denoted by t is (a representative of) a property; 0 otherwise. Analogously for $|Proposition(t)|_{I,w}$.
- *If* t_2 denotes a member of $|M_0|$ (i.e. isn't a property or proposition), then $|t_1 \, \xi \, t_2|_{I,w}$ is 0
- Otherwise $|t_1 \, \xi \, t_2|_{I,w}$ is $I(o_1, o_2, w)$, where o_1 and o_2 are the objects denoted by parameterized terms t_1 and t_2.
- $|\neg A|_{I,w} = \natural(|A|_{I,w})$
- $|A \wedge B|_{I,w} = glb\{|A|_{I,w}, |B|_{I,w}\}$
- $|A \vee B|_{I,w} = lub\{|A|_{I,w}, |B|_{I,w}\}$
- $|\forall x A|_{I,w} = glb\{|A(o/x)|_{I,w} : o \; in \; |M_0^+|\}$, where $A(o/x)$ is the parameterized formula just like A except with o assigned to the free variable x
- $|\exists x A|_{I,w} = lub\{|A(o/x)|_{I,w} : o \; in \; |M_0^+|\}$
- $|\Box A|_{I,w} = glb\{|A|_{I,w^*} : w^* \in W_w\}$
- $|\Diamond A|_{I,w} = lub\{|A|_{I,w^*} : w^* \in W_w\}$.

(The clauses for quantifiers and modal operators depend on the completeness assumption (iii) for Kripke algebras.)

Note that if L_0 contains an identity predicate, the rule for atomic formulas dictates that in L_0^+ it is not treated as a full identity predicate but as identity restricted to the ground model. Identity among properties and propositions is not yet defined.

Now for the crucial definition, whose interest depends on clauses (vii) and (viii) in the definition of Kripke algebras. If I, J are valuations for ξ, let $I \leq_K J$ mean:

For all OPW triples $\langle o_1, o_2, w \rangle$, either $\frac{1}{2} \leq I(o_1, o_2, w) \leq J(o_1, o_2, w)$, or else $J(o_1, o_2, w) \leq I(o_1, o_2, w) \leq \frac{1}{2}$.

The importance of this is that it allows us to generalize Kripke's key Lemma to arbitrary Kripke algebras:

Kripke Monotonicity Lemma (Generalized Form): if I and J are valuations for ξ with $I \leq_K J$, then for every parameterized sentence B and world w, either $\frac{1}{2} \leq |B|_{I,w} \leq |B|_{J,w}$ or else $|B|_{J,w} \leq |B|_{I,w} \leq \frac{1}{2}$.

Proof: By induction on complexity of B.

- When B atomic, then the only case where the value depends on I is when B has form $t_1 \, \xi \, t_2$ for parameterized terms, and t_2 denotes a property or proposition. And in that case, $|B|_{I,w}$ is just $I(den(t_1), den(t_2), w)$. Similarly for J. And so the assumption that $I \leq_K J$ yields the desired conclusion that either $\frac{1}{2} \leq |B|_{I,w} \leq |B|_{J,w}$ or else $|B|_{J,w} \leq |B|_{I,w} \leq \frac{1}{2}$.
- Negation: If $\frac{1}{2} \leq |B|_{I,w} \leq |B|_{J,w}$ then by order-reversing nature of $\natural$, $(|B|_{J,w})^{\#} \leq (|B|_{I,w})^{\#} \leq \frac{1}{2}$, i.e. $|\neg B|_{J,w} \leq |\neg B|_{I,w} \leq \frac{1}{2}$. Similarly for the other case.
- Conjunction: Suppose B and C both satisfy the claim.
Case 1: $\frac{1}{2} \leq |B|_{I,w} \leq |B|_{J,w}$ and $\frac{1}{2} \leq |C|_{I,w} \leq |C|_{J,w}$. Then $\frac{1}{2} \leq |B \wedge C|_{I,w} \leq |B \wedge C|_{J,w}$.
Case 2: $|B|_{J,w} \leq |B|_{I,w} \leq \frac{1}{2}$ and $|C|_{J,w} \leq |C|_{I,w} \leq \frac{1}{2}$. Then $|B \wedge C|_{J,w} \leq |B \wedge C|_{I,w} \leq \frac{1}{2}$.

Case 3: $|B|_{J,w} \leq |B|_{I,w} \leq \frac{1}{2}$ and $\frac{1}{2} \leq |C|_{I,w} \leq |C|_{J,w}$. Then $|B \wedge C|_{J,w} = |B|_{J,w}$ and $|B \wedge C|_{I,w} = |B|_{I,w}$, so result holds.
The other case is analogous to Case 3.

- Universal quantification and $\Box$ are similar. For instance, if $B(o)$ obeys the assumption for all o, then
(i) if all the $|B(o)|_{I,w}$ are at least $\frac{1}{2}$ then for all o $\frac{1}{2} \leq |B(o)|_{I,w} \leq |B(o)|_{J,w}$ and hence $\frac{1}{2} \leq |\forall x Bx|_{I,w} \leq |\forall x Bx|_{J,w}$;
(ii) otherwise, $|\forall x Bx|_{I,w}$ and $|\forall x Bx|_{J,w}$ are the glbs of the $|B(o)|_{I,w}$ and $|B(o)|_{J,w}$ for which $|B(o)|_{I,w} < \frac{1}{2}$. For these, $|B(o)|_{J,w} \leq |B(o)|_{I,w} \leq \frac{1}{2}$, and so $\frac{1}{2} \leq |\forall x Bx|_{J,w} \leq |\forall x Bx|_{I,w}$.
- Disjunction and existential quantification and $\Diamond$ are similar. ■

Given any valuation I for ξ, we define its "Kripke jump" $K(I)$ as follows:

> $K(I)$ is the valuation for ξ that, for each o in the full $|M_0^+|$ and each parameterized sentence or parameterized 1-formula A and each world w, assigns to $\langle o, \lambda z A(z), w\rangle$ or $\langle o, \lambda A, w\rangle$ the value $|A(o)|_{I,w}$.

(Obviously $K(I)$ makes all propositions "instantiation invariant": if $o\,\xi\, c$ where c represents a proposition, then for any other o^*, $o^*\,\xi\, c$.)

The following is then just a restatement of the Monotonicity Lemma:

Kripke Monotonicity Corollary: if I and J are valuations for ξ with $I \leq_K J$, then $K(I) \leq_K K(J)$.

Moreover, the completeness requirement (iii) on Kripke algebras allows us to define a valuation for ξ corresponding to any sequence of valuations for ξ, say as follows:

> $LIM\{I_\rho : \rho < \tau\}$ is the valuation for ξ that assigns to any OPW triple $\langle o_1, o_2, w\rangle$ the following:
> $lub\{I_\rho(o_1, o_2, w) : \rho < \tau\}$, if $(\exists\rho < \tau\ )(I_\rho(o_1, o_2, w) > \frac{1}{2})$
> $glb\{I_\rho(o_1, o_2, w) : \rho < \tau\}$, if $(\forall\rho < \tau)(I_\rho(o_1, o_2, w) \leq \frac{1}{2})$.

This isn't of much interest in general, but it is when the sequence $\{I_\rho : \rho < \tau\}$ is an $\leq_K$*-chain* of valuations for ξ; that is, a sequence of valuations for ξ such that for any ρ and σ such that $\rho < \sigma < \tau$, $I_\rho \leq_K I_\sigma$. In that case we have:

Observation on Chains: If $\{I_\rho : \rho < \tau\}$ is an $\leq_K$-chain, then for any $\sigma < \tau$, $I_\sigma \leq_K LIM\{I_\rho : \rho < \tau\}$.

Proof: Suppose that $\{I_\rho : \rho < \tau\}$ is an $\leq_K$-chain and $\sigma < \tau$; and consider any $\langle o_1, o_2, w\rangle$. We need (i) that if $I_\sigma(\langle o_1, o_2, w\rangle) > \frac{1}{2}$ then $LIM\{I_\rho : \rho < \tau\})(\langle o_1, o_2, w\rangle) \geq I_\sigma(\langle o_1, o_2, w\rangle)$, and (ii) that if $I_\sigma(\langle o_1, o_2, w\rangle) < \frac{1}{2}$ then $LIM\{I_\rho : \rho < \tau\})(\langle o_1, o_2, w\rangle) \leq I_\sigma(\langle o_1, o_2, w\rangle)$. (i) is immediate from the definition of LIM, independent of the chain assumption. (ii) holds because $I_\sigma(\langle o_1, o_2, w\rangle) < \frac{1}{2}$ together with the chain condition implies that for all $\rho < \tau$, $I_\rho(\langle o_1, o_2, w\rangle) \leq \frac{1}{2}$; and so $LIM\{I_\rho : \rho < \tau\})(\langle o_1, o_2, w\rangle)$ is the glb of the $I_\rho(\langle o_1, o_2, w\rangle)$ and hence $\leq I_\sigma(\langle o_1, o_2, w\rangle)$. (Of course the asymmetry in the proof is just due to the asymmetry in the definition, an asymmetry that only makes a difference in the uninteresting case of non-chains.) ■

We can now easily prove the existence of Fixed Points, and in particular a minimal one. For the latter, let I_0 be the trivial valuation for ξ that assigns $\frac{1}{2}$ to every triple in its domain; whenever I_ρ has been defined, define $I_{\rho+1}$ as $K(I_\rho)$; and when I_ρ has been defined for $\rho < \lambda$ when λ is a limit ordinal, let I_λ be $LIM\{I_\rho : \rho < \lambda\}$. By the Monotonicity Corollary and the Observation on Chains, it is clear that whenever $\rho < \sigma$, $I_\rho \leq_K I_\sigma$. So there must be an ordinal τ of cardinality no greater than that of the set of all valuations for ξ such that $I_{\tau+1} = I_\tau$. This is the minimal fixed point. (If at stage 0 we had started out not with the trivial valuation, but with another I^* such that $I^* \leq_K K(I^*)$, we would reach a fixed point by the same argument, the minimal valuation *that extends* I^*.) Restating (in a slightly loose notation using "parameterized formulas" to avoid talk of functions assigning objects to variables):

Kripke Fixed Point Theorem (Generalized to Arbitrary Kripke Algebras): There are valuations I for ξ (including one that is minimal in the ordering $\leq_K$) such that for each $o, o_1, ..., o_k$ in the full $|M_0^+|$ and each formula A that has exactly k variables other than x free (and may or may not have x free), $|o\,\xi\,\lambda x A(x, o_1, ..., o_k)|_{I,w} = |A(o, o_1, ..., o_k))|_{I,w}$,
and for each without x free,
$|o\,\xi\,\lambda A(o_1, ..., o_k)|_{I,w} = |A(o_1, ..., o_k))|_{I,w}$.

(The left hand side is what I_τ assigns to $\langle o, \lambda x A(x, o_1, ..., o_k), w\rangle$, the right hand side is what $I_{\tau+1}$ assigns it, and $I_{\tau+1} = I_\tau$.)

By using such fixed points, then, we are guaranteed (ii^+) and (ii_0^+) of the naive theory of properties and propositions. (Recall the definitions of 'instantiates' and 'True' in terms of ξ.) (i^+) and (i_0^+) were built into the construction from the start.

This generalization of the Kripke construction on $\{0, \frac{1}{2}, 1\}$ to arbitrary Kripke algebras isn't particularly surprising, but it is useful.[9] A minor illustration (a more substantial one will be given in Section 3): Suppose we think (as many people do) that vague language is best evaluated with values in the unit interval [0,1] (and not confined to 0, $\frac{1}{2}$ and 1), with the rules for conjunction, disjunction and the quantifiers given in terms of greatest lower bound and least upper bound and where $|\neg A|$ is $1 - |A|$. Then it's natural to hope that we can add propositions and properties to a language with vague terms in a naive way, without disrupting the values of the ground level sentences. Since [0,1] with this negation is a Kripke algebra, the generalized Kripke construction gives just what we need.

2.3. Choices for Validity in the Conditional-free Context.

The construction I've been developing can be used as a model-theoretic semantics for many different logics, for there are many different choices for how to use it to define validity. In the ones I'll be interested in, validity is defined in terms of the values of parameterized formulas *at all normal worlds of all models.*

One kind of choice is what constraints if any one puts on the modal structure. I'll eventually impose the reflexivity condition for normal worlds: that at least for normal worlds, $w \in W_w$. That's what's needed for the validity of $\Box A \models A$. Also, to simplify a discussion later on I will assume that if there are non-normal worlds then for each of them, there is a normal world from which it is accessible. The point of this is to make the claim $\models \Box B$ require that B is true at *every* world of every model, even the non-normal ones.

But a more fundamental choice, with the kind of multi-valued semantics considered here, is how exactly the values enter into the account of validity. This greatly affects the logic that the semantics validates. For instance, if we take an inference

[9] Further generalizations are possible: e.g. for any Kripke algebra V, we can add a value incomparable to all the values in V other than 1 or 0, and let the negation involution $\natural$ take it to itself; this generalizes the way Visser went from the Kleene 3-algebra to the Dunn 4-algebra. Also, the Fixed Point Theorem as stated above automatically extends to products and subproducts of Kripke algebras (which typically will not themselves be Kripke algebras due to condition (viii)).

to be valid if in all models and all normal worlds in them, if the premises have value 1, so does the conclusion, then we'll get a "paracomplete" logic in which excluded middle isn't valid. If we take an inference to be valid if in all models and normal worlds in them where the conclusion has value 0, so does at least one of the premises, then we'll get a "paraconsistent" logic in which disjunctive syllogism (the inference from $A \vee B$ and $\neg A$ to B) isn't valid. (Similarly if we replace '0' by 'less than $\frac{1}{2}$'.) If we take an inference to be valid if in all models and normal worlds in them, the value of the conclusion is at least the greatest lower bound of the values of the premises, we have both failures: the logic is both paracomplete and paraconsistent. Nothing said so far is a reason for going one way or another. And indeed, that is an issue on which I will mostly remain neutral in this paper, though there will be some remarks in Section 5 tending to favor the paracomplete approach.

3. Restricted Quantifier Conditionals: Revising the Revision Approach.

Much of the recent work on naive theories of truth and property-instantiation has been devoted to conditionals. As first noted in Beall *et al* 2006 and Chapter 18 of Priest 2006, an adequate theory needs to deal with at least two kinds of conditionals: one (which I'll symbolize as $\rightarrow$) for defining restricted universal quantification from unrestricted,[10] another (which I'll symbolize as $\triangleright$) to symbolize the kind of conditional we use in everyday sentences like

> He's under the delusion that if he runs for President, he'll win. But he doesn't want the job, so he won't run.

In a classical setting, $\triangleright$ is obviously not the material $\supset$, whereas the $\rightarrow$ is. In a non-classical setting of the sort required for naive properties and truth, even the $\rightarrow$ can't obey the usual classical definition of $\supset$ (that is, $\neg A \vee B$), if the $\rightarrow$ is to obey reasonable laws (modus ponens and $A \rightarrow A$). But presumably it needs to reduce to $\supset$ when the antecedent and consequent behave classically. Whereas $\triangleright$ had better *not* reduce to $\supset$ in such classical contexts.

There are various approaches to handling each of the two conditionals (and to handling how they interact). One approach that can be used for each is a revision construction. In the past I've developed this approach in the context of a valuation space based on $\{0, \frac{1}{2}, 1\}$ (resulting in a much bigger valuation space that is a product or subproduct of $\{0, \frac{1}{2}, 1\}$, that is, a space of functions from some set X to $\{0, \frac{1}{2}, 1\}$). But it can be adapted for other Kleene algebras in place of $\{0, \frac{1}{2}, 1\}$, with results that have a rather different feel. In the rest of this section I'll use [0,1] rather than $\{0, \frac{1}{2}, 1\}$ in this role of "mini-space" (so that the resulting valuation space is a space of functions from some set X to [0,1]).[11]

---

[10]Instead of defining "All A are B" as $\forall x(Ax \rightarrow Bx)$ for some appropriate $\rightarrow$, we could take a binary restricted quantifier as primitive. But that wouldn't effect the logical issues to be discussed. For we could then use primitive restricted quantification to define a conditional $\rightarrow$: $A \rightarrow B$ would mean that all v such that A are such that B, where v is a variable not free in either A or B; and with this defined $\rightarrow$, $\forall x(Ax \rightarrow Bx)$ will be equivalent to the original "All A are B", assuming only very uncontroversial laws.

[11]There are other approaches that have been used with the 3-valued mini-space. The earliest is due to Brady, and I will make use of an improved version of it in Section 4.3.2, in connection with the $\triangleright$ conditional. For reasons to be mentioned there, I don't think the Brady approach has

In this section I'll illustrate this for the restricted quantifier conditional $\rightarrow$, saving the $\triangleright$ for the next section.

There is an initial worry about how to make the adaptation from $\{0, \frac{1}{2}, 1\}$ to [0,1].

The basic idea of the revision construction (in a language that adds $\rightarrow$ to the L_0^+ of Section 2) is that we start out with an assignment h of values in [0,1] to parameterized conditional sentences at worlds (that is, to pairs of conditional formulas and functions assigning objects to their free variables, at worlds). We require of h that it be "transparent", in the sense that if C_1 and C_2 are conditional formulas such that one results from another by substitutions of formulas of form "$o \,\xi\, B(x, o_1, ..., o_k)$" for the corresponding "$B(o, o_1, ..., o_k)$" or vice versa, then for any world and any common assignment of objects to the variables of C_1 and C_2, h assigns the same value to the parameterizations of C_1 and C_2 at that world. Given such a transparent valuation h of (parameterized) conditionals, we first do a generalized Kripkean construction as in the previous section, but in the language with $\rightarrow$, holding the values of parameterized conditionals fixed at the values given by h throughout the construction. We choose a fixed point: say the minimal fixed point. For any sentence A, let $|A|_{h,w}$ be the Kripke fixed point value that A gets at world w on the construction starting from h. It is clear that because h is transparent, the Kripke construction guarantees that the assignment $|A|_{h,w}$ that it generates is transparent in the corresponding sense. The key is then to introduce a revision rule, that takes any h to a new one $R(h)$, based on the values $|A|_{h,w}$, which will be transparent given that h is. (The rule for $\rightarrow$ operates world by world: that is, the value that $R(h)$ assigns to a conditional at a given w depends only on the values that h assigns its antecedent and consequent at that same w. The situation will be different when we come to $\triangleright$.)

More specifically, the revision rule I've come to advocate for $\rightarrow$ in the case of the space $\{0, \frac{1}{2}, 1\}$ is this:[12]

---

much relevance to restricted quantification, and Brady doesn't either (judging from Beall *et al* 2006, of which he was a co-author). [But see note 33.]

Another alternative to revision semantics is the "higher order fixed point" approach of Field 2014: higher order in that the fixed point isn't an individual hypothesis but a set of final hypotheses together with their ordering. This approach can be used for either conditional. The Field 2014 version uses valuations in a space of form $\{0, \frac{1}{2}, 1\}^X$ though in this case X will not be a set of ordinals; it too can be generalized to $[0,1]^X$. This approach has deep structural similarities to the revision-theoretic approach. Indeed, as Harvey Lederman remarked to me, it selects the unique "minimal" fixed point in a higher order construction in which the revision constructions with different starting hypotheses produce the "maximal" fixed points. (Perhaps the terms 'maximal' and 'minimal' should be reversed: the set of hypotheses that survive in the revision-theoretic systems that I've called 'maximal' are proper subsets of those in the "minimal" fixed point. Nonetheless, the one I've called 'minimal' is the one generated most easily, and it produces the fewest sentences with extreme values 0 and 1.)

The higher order fixed point approach is more complicated than the revision, and especially once one shifts to the [0,1]-valued versions I'm not sure that the extra complications have much payoff. (In the $\{0, \frac{1}{2}, 1\}$ case, Standefer 2015 has pointed to some oddities in the revision approach that don't apply to the higher order fixed point approach; but the oddities are avoided in the [0,1]-valued revision approach as well.) I won't discuss the higher order approach further.

[12]In Field 2008 I primarily used a different 0 clause (though I mentioned the one in the text in Section 17.5): the 0 clause there (generalized to the current modal framework) was "0 iff $|A|_{h,w} > |B|_{h,w}$". The one in the text (also employed in Field 2014 and Field 2016) is substantially

$$[R(h)](A \to B, w) \text{ is } \begin{cases} 1 & \text{if } |A|_{h,w} \leq |B|_{h,w} \\ 0 & \text{if } |A|_{h,w} = 1 \text{ and } |B|_{h,w} = 0 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

The natural generalization of this to [0,1] is

$$[R(h), w](A \to B) \text{ is } \begin{cases} 1 & \text{if } |A|_{h,w} \leq |B|_{h,w} \\ 1 - (|A|_{h,w} - |B|_{h,w}) & \text{otherwise.} \end{cases}$$

But the problem is that if, as is natural, we start the construction with a function h_0 that assigns to every conditional one of the values in $\{0, \frac{1}{2}, 1\}$ at each world, then this rule for $R(h)$ will also assign only values in $\{0, \frac{1}{2}, 1\}$; the extra richness in the space [0,1] won't be exploited.

The best way around this, I think, is to insist on "slow corrections" in the revision process. More fully, $R(h)$ as given above looks like it gives natural values based on the old h; but it produces big jumps, sometimes from 1 to 0 or 0 to 1, which we might have to reverse later. Rather than make the big jump at once, let's have a rule that averages the $R(h)$ value with the h value:

$$[R^*(h), w](A \to B) \text{ is } \begin{cases} \frac{1}{2}[h(A \to B, w) + 1] & \text{if } |A|_{h,w} \leq |B|_{h,w} \\ \frac{1}{2}[h(A \to B, w) + 1 - (|A|_{h,w} - |B|_{h,w})] & \text{otherwise.} \end{cases}$$

That's the "slow correction" process.

Let us now use $R^*(h)$ to construct a revision sequence. Let h_0 be any transparent valuation: say the one that gives every conditional the value $\frac{1}{2}$ at every world, though I'll suggest what I regard as a better alternative in note 16. (For most purposes the details of h_0 won't matter much, as long as it is transparent: these details are very largely washed away as the revision construction proceeds, though there are a few special sentences for which it matters.) Once h_μ has been constructed, let $h_{\mu+1}$ be $R^*(h_\mu)$.

What about limits? It turns out that a great many A and B are such that for each world w, the sequence $\{|A \to B|_{h_n,w} : n < \omega\}$ approaches a particular point r_w as limit; and in that case we presumably want to take that r_w as the value that h_ω assigns the conditional $A \to B$ at w. More generally when there is convergence prior to any limit ordinal λ, that should determine the value at λ. But what about when there's no convergence? One might explore taking $h_\lambda(A \to B, w)$ to be the average of the liminf and limsup of $\{h_\mu(A \to B, w) : \mu < \lambda\}$. But I prefer a limit rule where it is to be the liminf when that is at least $\frac{1}{2}$, the limsup when that is at most $\frac{1}{2}$, and $\frac{1}{2}$ in all other cases, i.e. when the liminf is less than $\frac{1}{2}$ and the limsup more than $\frac{1}{2}$. ("The value is as close to $\frac{1}{2}$ as it can sensibly be.") Either of these rules generalizes the limit rule I've previously used in the case of $\{0, \frac{1}{2}, 1\}$.[13]

---

better in several respects. In any case, the basic point next to be made doesn't depend on the difference.

(The 1 clause as given makes $\to$ contraposable, which I think reasonable. In the 3-valued case, there is also a natural non-contraposable $\to$ to consider, where the 1-clause is altered to: "1 iff either $|A|_{h,w} < 1$ or $|B|_{h,w} = 1$". But this doesn't generalize as neatly to the continuum-valued case.)

[13] A *prima facie* advantage of the averaging rule over the one I prefer is that it would lead outside the space $\{0, \frac{1}{2}, 1\}$ even without "slow corrections" in the revision rule. But it doesn't lead outside $\{0, \frac{1}{2}, 1\}$ as much as is desirable: later in this section I note that the resolution of "ordinary" paradoxes on the present semantics reduces to the Łukasiewicz resolution, and this attractive feature of the semantics depends on its use of slow corrections. (It doesn't depend on

Whichever of these limit rules we use, the general theory of revision sequences (Gupta and Belnap 1993) tells us that there are some "hypotheses" that occur arbitrarily late in the sequence; call these *recurring hypotheses*. (That is, h_κ is recurring iff for any ς, there is an $\eta > \varsigma$ for which $h_\eta = h_\kappa$.) Indeed, there are ordinals μ such that for any $\kappa \geq \mu$, h_κ is recurring; call such μ *final*. And among these final ordinals, there are ones of particular interest, the *reflection ordinals*: these are the final limit ordinals Δ such that for any $\mu < \Delta$, and any final η, there is a $\kappa \in [\mu, \Delta)$ such that $h_\kappa = h_\eta$. And it's easy to see that for any two reflection ordinals, the values of all parameterized conditionals and hence of all parameterized sentences are the same.[14] More generally, on the preferred limit rule, it's easy to see that if Δ is a reflection ordinal and μ any other final ordinal, then:

(*): For any parameterized conditional $A \to B$ and world w, either $\frac{1}{2} \leq h_\Delta(A \to B, w) \leq h_\mu(A \to B, w)$ or $h_\mu(A \to B, w) \leq h_\Delta(A \to B, w) \leq \frac{1}{2}$.

And one can then, by an induction on complexity, extend this to the values of non-conditional parameterized sentences:

(FT): For any parameterized sentence A and world w, either $\frac{1}{2} \leq |A|_{\Delta,w} \leq |A|_{\mu,w}$ or $|A|_{\mu,w} \leq |A|_{\Delta,w} \leq \frac{1}{2}$.

The proof is a straightforward generalization of the one I've given elsewhere (e.g. Field 2008) for the special case of $\{0,\ \frac{1}{2},\ 1\}$: by induction on the stages σ of the Kripke construction, with a subinduction on the complexity of parameterized sentences A, one proves that (deleting the world parameter for readability) if $|A|_{\Delta,\sigma} < \frac{1}{2}$ then for all final α $|A|_\alpha \leq |A|_{\Delta,\sigma}$, and if $|A|_{\Delta,\sigma} > \frac{1}{2}$ then for all final α $|A|_\alpha \geq |A|_{\Delta,\sigma}$.[15]

A special case of (*) is that if a conditional gets value 1 at a reflection ordinal, it gets value 1 for every final ordinal; and similarly for 0. And a special case of (FT) is that that's so for every parameterized sentence of the language. Thus (FT) is a generalized version of what in the 3-valued case I've called the "Fundamental Theorem". (These special cases of (*) and of (FT) would hold also on the alternative limit rule where we average the liminf and limsup, and it may be that only the special cases are crucial. But in what follows I'll use the preferred rule.)[16]

---

the choice between the two limit rules, but if we want the reduction, it removes the *prima facie* advantage of the averaging rule.)

[14]The term 'reflection ordinal' is sometimes used more broadly, for any ordinal κ for which h_κ is the same as h_Δ where Δ is as described here. The difference won't matter much, though confusion might result if it were not realized that there are infinitely many reflection ordinals in the latter sense between any two reflection ordinals in the former.

[15]There are only two places where the move to the space [0,1] might be thought to matter to the proof: for the clause for 'True' at limit ordinals in the main induction, and in the quantifier clause in the subinductions. But in neither case is there a problem.

(1) Suppose for instance that $|True(t)|_{\Delta,\lambda} > \frac{1}{2}$, where t denotes C. Then by the Kripke rules, it must be that for each r in the open interval $(\frac{1}{2}, r)$, there is a $\tau < \lambda$ such that $|C|_{\Delta,\tau}$ is at least r. So by induction hypothesis, for each final α and each such r, $|C|_\alpha \geq r$. That can only be so if $|C|_\alpha \geq |True(t)|_{\Delta,\lambda}$; which by transparency means that $|True(t)|_\alpha \geq |True(t)|_{\Delta,\lambda}$.

(2) Suppose for instance that $|\forall x Bx|_{\Delta,\sigma} < \frac{1}{2}$. (The case where $|\forall x Bx|_{\Delta,\sigma} > \frac{1}{2}$ is slightly easier.) Then the set $\Sigma = \{o : |B(o)|_{\Delta,\sigma} < \frac{1}{2}\} \neq \varnothing$, and $|\forall x Bx|_{\Delta,\sigma}$ is $glb\{|B(o)|_{\Delta,\sigma} : o \in \Sigma\}$. And $|\forall x Bx|_\alpha \leq glb\{|B(o)|_{\Delta,\sigma} : o \in \Sigma\}$. By subinduction hypothesis, $|B(o)|_\alpha \leq |B(o)|_{\Delta,\sigma}$ for all $o \in \Sigma$, so $|\forall x Bx|_\alpha \leq |\forall x Bx|_{\Delta,\sigma}$.

[16]This is all independent of the choice of transparent starting valuation h_0, and as I've said, that choice has only a minor effect on the results. But for the record, I now prefer to first do the construction in a general way, independent of the transparent starting valuations h_0. For

The upshot is that the value in $[0,1]$ of a parameterized sentence A at a reflection ordinal Δ tells us a lot about how A behaves in the model, but the full story requires how it behaves in a semi-open interval $[\Delta, \Delta+\Pi)$ between two reflection ordinals. (In such an interval, every recurring hypothesis shows up. It's best to think of it as the closed interval $[\Delta, \Delta+\Pi]$ but with endpoints identified to form a circle.) So the obvious value space is the space $[0,1]^\Pi$ of functions from Π to $[0,1]$, where Π is an ordinal that when added (on the right) to a reflection ordinal yields a bigger reflection ordinal. I'll let $||A||_w$ be the value in $[0,1]^\Pi$ of A at w. For later reference, I'll use the notation $\mathbf{r}$ (where $r \in [0,1]$) for the constant function that assigns the value r to every predecessor of Π. So for a sentence to have value $\mathbf{r}$ (at a world) is for it to have the same value r (at that world) *at all final ordinals*; it can have different values for earlier ordinals, but those earlier values get washed out.

How much of an improvement do we get in this construction by using [0,1] instead of $\{0, \frac{1}{2}, 1\}$ as the mini-space? In one sense, not much: the most general laws that don't hold with $\{0, \frac{1}{2}, 1\}$, such as the permutation axiom

$$[A \to (B \to C)] \to [B \to (A \to C)]$$

(or its weaker rule form), tend not to hold with [0,1] either. However, with [0,1] we have to go to much greater lengths to get exceptions: for "ordinary" paradoxical sentences, the new construction yields far more satisfactory results, and axioms such as permutation will hold for them.

That's because

(i) for "ordinary" paradoxical sentences, their value in the new construction is the same for all h_μ where μ is final (often even, for all h_μ when $\mu \geq \omega$); that is, their value is one of the constant functions $\mathbf{r}$; and

(ii) for such sentences, the semantics reduces to Łukasiewicz semantics on [0,1].

Łukasiewicz semantics on [0,1] uses the evaluation rules for $\neg$, $\wedge$, $\vee$ and the quantifiers and modal operators that were employed in Section 2, together with the rule

$$|A \to B|_{h,w} \text{ is } \begin{cases} 1 & \text{if } |A|_{h,w} \leq |B|_{h,w} \\ 1 - (|A|_{h,w} - |B|_{h,w}) & \text{otherwise.} \end{cases}$$

The "jumpy correction" revision rule R was obviously modeled on this: it's essentially this except with $|A \to B|_{R(h),w}$ instead of $|A \to B|_{h,w}$ as its left hand side, so that what the jumpy revision rule semantics says of $R(h)$, Łukasiewicz semantics says of h itself. The "slow correction" revision rule R^* is still a further step from Łukasiewicz semantics. However, it's easy to see that if A and B are sentences each of whose values is a constant function (say $\mathbf{a}$ and $\mathbf{b}$), then neither step makes a difference: the value of $A \to B$ will itself be the constant function $\mathbf{c}$, where the value c is determined from the values a and b by the Łukasiewicz rule.

---

each one, and each parameterized conditional $A \to B$ and world w, we get a reflection value $|A \to B|_{\Delta,w,h_0}$; let the set of these for a given $A \to B$ and w but varying the starting hypothesis be $REFL(A \to B,w)$. Then my preferred starting valuation assigns to $A \to B$ at w the greatest lower bound of $REFL(A \to B,w)$ if that's at least $\frac{1}{2}$, the least upper bound if that's no more than $\frac{1}{2}$, and $\frac{1}{2}$ in other cases. This choice gives more natural values to some sentences: e.g. to "conditional truth-teller" sentences whose antecedent is itself conditional, as on p. 319 of Yablo 2003. But its overall impact on the theory is small.

Łukasiewicz semantics doesn't include a specification for the generalized instantiation predicate ξ (or for a truth predicate). But it's well-known that for the quantifer-free sublanguage (supplemented with a means to achieve self-reference and referential loops), a predicate that behaves naively in this sublanguage can be added.[17] Even outside this sublanguage, a great many paradoxical sentences can be consistently evaluated in Łukasiewicz semantics (often in a unique way). And it's natural to conjecture that for those sentences, the revision process given above will lead to the constant function corresponding to one of those values.[18] I won't attempt here to make this conjecture precise or to prove it, but there are countless examples to illustrate it. For instance, consider a Curry-like sentence K_2 that is equivalent to $True(\lambda K_2) \to [True(\lambda K_2) \to \bot]$.[19] Applying the Łukasiewicz rules without assuming naive truth, we get

$$|K_2|_{h,w} = \begin{cases} 1 & \text{if } |True(\lambda K_2)|_{h,w} \leq \frac{1}{2} \\ 2(1 - |True(\lambda K_2)|_{h,w}) & \text{if } |True(\lambda K_2)|_{h,w} > \frac{1}{2}. \end{cases}$$

We see that in Łukasiewicz semantics, at each world there is a unique value of K_2 where the naivety requirement that $|True(\lambda K_2)|_w = |K_2|_w$ is met, and it is the same at each world: it is $\frac{2}{3}$. (There is no world-dependence since K_2 is a "non-contingent" Curry-like sentence.) And in the modified revision semantics, it isn't hard to show (independent of what one assumes the starting valuation h_0 to be) that for all $\mu \geq \omega$, h_μ assigns value $\frac{2}{3}$ to K_2 at each world.[20] That value of course wasn't available on the revision construction using only $\{0, \frac{1}{2}, 1\}$. (There, the values repeat indefinitely in groups of three, each group consisting of $\frac{1}{2}$ followed by 1 followed by $\frac{1}{2}$.)[21]

Well, you might say, why not just use the Łukasiewicz semantics, it's simpler? The answer is basically that there's no way to expand the Łukasiewicz logic to include '$True$' (or more generally, 'ξ') that will yield a naive theory once quantifiers are included.[22] Here's one well known example (from Restall 1992). Define a

[17]This is a consequence of the Brouwer Fixed Point Theorem on spaces of form $[0,1]^X$: see Field 2008, pp 97-9.

[18]A more general (and more tentative) conjecture is that if a sentence can be handled in Łukasiewicz semantics by assigning it more than one value, then the constant function corresponding to any one of those values could emerge from the revision procedure either by varying the starting valuation for conditionals or choosing non-minimal Kripkean fixed points.

[19]$\bot$ is an absurd proposition. Recall that $True(x)$ is short for "$Proposition(x) \land \forall y(y\,\xi\,x)$".

[20]Letting s_μ be $h_\mu(K_2) - \frac{2}{3}$ (I drop the world parameter since it doesn't matter in the example), it's easy to see that for all μ, $s_{\mu+3} = -s_\mu/8$; so each sequence $s_{\lambda+n}$ rapidly converges to 0, so the value at all limits is $\frac{2}{3}$. And once it reaches the first limit, at ω, it never moves from there. This analysis is independent of the somewhat arbitrary assumptions made about the values that h_0 gives to conditionals.

[21]These average to $\frac{2}{3}$, so in this case the value in the new system is the average (defined as a limit) of the values in the old. While this is so for many sentences, it is not so for all. Consider a sentence W equivalent to $True(\lambda W) \to \neg True(\lambda W)$. This also gets value $\frac{2}{3}$ in Łukasiewicz semantics and the corresponding constant function in the present semantics; but in the revision construction using only $\{0, \frac{1}{2}, 1\}$, the value is a function that alternates between 0 and 1 at successors and has value $\frac{1}{2}$ at limits. W is also an illustration of the point made in note 13, that the averaging rule for limits, without slow corrections, would yield different (and presumably less satisfying) results than we get with slow corrections.

[22]More accurately, naive truth theory in Łukasiewicz logic is ω-inconsistent, in the semantic sense: arithmetically standard models of the ground language can't be consistently expanded to include naive truth in the logic. (See Restall 1992.) If we build into the naivety condition that the

function $F(n,x)$ from natural numbers and properties to properties, with $F(0,x)$ being $\lambda x(\perp)$ and for each n, $F(n+1,x)$ being $\lambda x[x\,\xi\,x \rightarrow x\,\xi\,F(n,x)]$. By an easy induction we get that for any n, and any x in the domain and world w, $|x\,\xi\,F(n,x)|_w$ is $min\{1,\, n \cdot (1 - |x\,\xi\,x|_w)\}$; so it's 1 iff $|x\,\xi\,x|_w \leq 1 - \frac{1}{n}$. Let $G(x)$ be $\lambda x[\exists n(x\,\xi\,F(n,x))]$; it follows from the previous (together with an arithmetic standardness assumption: see previous footnote) that for any x and w, $|G(x)|_w$ is 1 if $|x\,\xi\,x|_w < 1$, and 0 if $|x\,\xi\,x|_w = 1$. Now let R be $\lambda z G(z)$. The above requires that $|G(R)|_w$ is 1 iff $|R\,\xi\,R|_w < 1$, hence $|G(R)|_w \neq |R\,\xi\,R|_w$; but naivety requires that $|R\,\xi\,R|_w = |G(R)|_w$, so naivety can't be achieved in Łukasiewicz semantics.

How is this handled on the "slow correcting" revision-theoretic semantics? It's not hard to show that the value of $R\,\xi\,R$ at each world at an ordinal goes in ω^2-length cycles. Each one starts with the ω-sequence $\langle \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \rangle$, followed by ω more ω-sequences $\langle 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \rangle$.[23] (The discontinuity at limits is possible because $R\,\xi\,R$ isn't equivalent to a conditional, but to something like an infinite disjunction of conditionals; the "disjuncts" $R\,\xi\,F(n,R)$ are continuous at limits.)[24]

In summary: there is no general Łukasiewicz specification for truth or property-instantiation; Łukasiewicz semantics only tells us the value that *certain* sentences would have to have in order that naivety "hold locally" for them.[25] If a sentence is such that we can get naivety to hold locally in Łukasiewicz semantics, then the modified revision semantics will assign it a constant function. So the modified revision semantics is, in a sense, "Łukasiewicz done better": it is a coherent proposal for how to deal with truth and property instantiation, that yields essentially what Łukasiewicz semantics yields where that works, but expands the value space to handle cases where Łukasiewicz fails.[26] And to repeat, in a context where these "extraordinary" paradoxical sentences (all of which involve quantification essentially) are excluded, then in that context Permutation and all the other axioms and rules of Łukasiewicz logic can be used.[27]

---

induction rule extends to formulas that contain 'True' and that suitable composition principles hold, then naivety is flat out inconsistent in Łukasiewicz logic: see Hajek, Paris and Sheperson 2000.

[23]If the starting valuation assigns 1 to every conditional then the initial ω^2-cycle starts with an ω-sequence of 1's, but the rest of that ω^2-cycle and all later ones are the same as if all conditionals were given starting value $\frac{1}{2}$.

[24]If λ is a limit ordinal and m a fixed finite number, then as n increases, $|R\,\xi\,F(n,R)|_{\lambda+m}$ increases from 0 (at $n=0$) to a maximum value; that value is $1 - \frac{1}{2^m}$ (reached at $n = m$) whenever λ isn't a multiple of ω^2, and $1 - \frac{1}{2^{m+1}}$ (reached at $n = m+1$) when it is a multiple. So this is also the value of $|R\,\xi\,R|_{\lambda+m}$. And since this approaches 1 as m increases, it's clear that for fixed n, $|R\,\xi\,F(n,R)|_{\lambda+m}$ approaches 0 as m increaches. So by continuity for conditionals, $|R\,\xi\,F(n,R)|_{\lambda+\omega}$ is 0 for each n, which is why $|R\,\xi\,R|_{\lambda+\omega}$ is 0. At limits that are multiples of ω^2 the limsup of the values $|R\,\xi\,F(n,R)|_\alpha$ for fixed $n > 0$ as α approach the limit is at least $\frac{1}{2}$ (indeed at least $1 - \frac{1}{2^n}$, since setting $m = n$ above, we see that that's the value at any predecessor of form $\chi + \omega + n$). And the liminf is 0, so the value at multiples of ω^2 is $\frac{1}{2}$.

[25]I.e., hold for those sentences and other closely related ones. I won't bother to make this precise.

[26]Rossi forthcoming proposes a less unified way of generalizing Łukasiewicz semantics: he takes the value space to be the union of [0,1] with a set L of sets of equations. Also, he takes the valuation space to be partial: even after enriching to sets of equations as values, not every sentence gets a value.

[27]The Permutation rule lacks obviousness in a naive theory, since no such theory can have Importation ($A \rightarrow (B \rightarrow C) \models A \wedge B \rightarrow C$), whereas the obvious argument for Permutation depends on Importation and its converse. But deductively, Permutation is of immense convenience,

4. "Ordinary" conditionals in naive theories without $\rightarrow$

4.1. "Ordinary" conditionals without 'ξ' or '$\rightarrow$': 3-valued case.

Let's go back to conditionals like

If I run for President I'll win,

which need to be understood in terms of a conditional $\triangleright$ other than $\supset$ or $\rightarrow$ (assuming we're not happy to regard them as true). A Łukasiewicz-like semantics would be totally inappropriate for $\triangleright$. In the first place, the Łukasiewicz conditional is contraposable, but ordinary conditionals aren't: from the likely fact that if Sarah Palin runs for President she won't win, it doesn't follow that if she wins she won't have run. More generally, the Łukasiewicz conditional reduces to $\supset$ when the antecedent and consequent take on classical values, and we don't want that for $\triangleright$.

The most developed proposal for how $\triangleright$ might work, in contexts that *don't* involve paradoxical sentences, is in terms of a variably strict conditional in the general ballpark of Stalnaker 1968. This involves adding a "neighborhood structure" to the space W of worlds, perhaps given by a ternary relation $x \leq_w y$ meaning intuitively that the change from w to x is no bigger than the change from w to y. For each w, the binary relation $x \leq_w y$ is required to be transitive, and reflexive on its field (i.e. $\forall x \forall y(x \leq_w y \vee y \leq_w x \supset x \leq_w x)$; that field is thus $\{x : x \leq_w x\}$, and it is natural to identify this with the set W_w of worlds accessible from w used in the semantics for $\Box$. A simple version of the semantics in the 2-valued case is:

(VS$_{simple}$): $|A \triangleright B|_w$ is

$$\begin{cases} 1 & \text{if either } \neg(\exists y \in W_w)(|A|_y = 1) \text{ or } (\exists y \in W_w)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 1)] \\ 0 & \text{otherwise.} \end{cases}$$

But that is the appropriate form only if you assume, with Stalnaker, that for any w, and any x and y in W_w, either $x \leq_w y$ or $y \leq_w x$. If we don't make that Connectivity assumption then the appropriate version in the 2-valued case (see Burgess 1981 and Lewis 1981) is

(VS$_{general}$): $|A \triangleright B|_w$ is

$$\begin{cases} 1 & \text{if } (\forall x \in W_w)[|A|_x = 1 \supset (\exists y \leq_w x)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 1)]] \\ 0 & \text{otherwise.} \end{cases}$$

This reduces to (VS$_{simple}$) when the Connectivity assumption is made.

In a 2-valued context, the definition of validity will be that an inference is valid if it preserves value 1 at all normal worlds. (This is my preferred definition in multi-valued contexts too, though I will be neutral about the "value 1" part as much as possible.) I will now strengthen the first of the two structural assumptions about modal models made in Section 2.3: instead of just that for all normal worlds w, $w \in W_w$, I assume also that for all normal worlds w and all $x \in W_w$, $w \leq_w x$. ("Weak centering at normal worlds".) This guarantees that modus ponens for $\triangleright$ is valid.

Even prior to introducing ξ into the language, there might be a motivation for moving to a semantics with at least three values. For if there are worlds arbitrarily close to w where $|A|$ and $|B|$ are both 1, and other worlds arbitrarily close to w where $|A|$ is 1 and $|B|$ is 0, it is rather natural to think that $A \triangleright B$ should have

---

as anyone who works through Sections C and D of Schechter 2005 will see. So I think there is significant value in extending its range.

value $\frac{1}{2}$ at w, rather than the value 0 delivered by either version of (VS). So it seems natural to keep the 1 clause of (VS) (except perhaps for the decision that "vacuous conditionals", where $\neg(\exists y \in W_w)(|A|_y = 1)$, are to have value 1), but tighten the 0 clause and give value $\frac{1}{2}$ to the remaining cases. In particular, I'd suggest

(MVS$_{simple}$): $|A \triangleright B|_w$ is

$$\begin{cases} 1 & \text{if } (\exists y \in W_w)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 1)] \\ 0 & \text{if } (\exists y \in W_w)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 0)] \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for when Connectivity is assumed, or without that assumption

(MVS$_{general}$): $|A \triangleright B|_w$ is

$$\begin{cases} 1 & \text{if } (\exists y \in W_w)(|A|_y = 1) \text{ and} \\ & \quad (\forall x \in W_w)[|A|_x = 1 \supset (\exists y \leq_w x)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 1)]] \\ 0 & \text{if } (\exists y \in W_w)(|A|_y = 1) \text{ and} \\ & \quad (\forall x \in W_w)[|A|_x = 1 \supset (\exists y \leq_w x)[|A|_y = 1 \wedge (\forall z \leq_w y)(|A|_z = 1 \supset |B|_z = 0)]] \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

(MVS$_{general}$) reduces to (MVS$_{simple}$) when Connectivity is assumed. The decision to let vacuous conditionals have value $\frac{1}{2}$ is a matter of convenience that we should feel free to reconsider later.

It's going to be important to introduce $\triangleright$ into a language with '$\rightarrow$' as well as 'ξ', but I'll save that for Section 4.4. Without the '$\rightarrow$', adding 'ξ' (or 'True') requires adding the value $\frac{1}{2}$ if we didn't have it already, but doesn't create a strong reason for using [0,1]:[28] that's because the variably strict semantics (MVS) used for $\triangleright$ is different in character from the Łukasiewicz semantics. Nonetheless, since we will ultimately want values in [0,1], it's useful to see how they might work independent of '$\rightarrow$' and even of 'ξ'. I'll treat that next, and get back to the paradoxes only in Section 4.3.

4.2. **"Ordinary" conditionals without 'ξ' or '$\rightarrow$': [0,1]-valued case.** Imagine that (perhaps for treating vagueness) we start from a model where arbitrary values in [0,1] can be assigned to atomic sentences at worlds. How (independent of any issues about 'True' and 'ξ' and '$\rightarrow$') are we to evaluate conditionals whose antecedents and consequents are [0,1]-valued? There are several slightly different ways to do it, but they have a common theme. I'll select one that seems natural to me. For simplicity of formulation, I'll introduce the notions of lub* and glb*, which for nonempty sets are least upper bound and greatest lower bound, but where the lub* and glb* of the empty set are both $\frac{1}{2}$.

For simplicity I'll concentrate on the case where the modal model obeys Connectivity (though I will cover the case where Connectivity is not assumed in passing, starting with the next footnote). For any world w and any $y \in W_w$ and any (parameterized) sentences A and B, let

$$Lowerlim_{w,A,y}(B) =_{df} glb^*\{|B|_z : z \leq_w y \wedge |A|_z = 1\}.$$

[28]There might be some reason to invoke [0,1] even without 'ξ': one might want to assign to each world w a measure on W_w, and evaluate $A \triangleright B$ using it together with the ordering $\leq_w$, with the idea being that the proportion of nearby A-worlds that have a give value for B is important. But that would further complicate the discussion, so I will not pursue it.

(In an obvious terminology, $Lowerlim_{w,A,y}(B)$ is the glb* of $|B|_z$ for "A-worlds z in the w-neighborhood generated from y".) Note that if y_1 is an A-world and $y_1 \leq_w y_2$ then for any B, $Lowerlim_{w,A,y_1}(B) \geq Lowerlim_{w,A,y_2}(B)$. For any w and A and B, let

$$Liminf_{w,A}(B) =_{df} lub^*\{Lowerlim_{w,A,y}(B) : y \in W_w \wedge |A|_y = 1\}$$

Roughly, it's the largest number r such that there's a w-neighborhood that contains A-worlds and where at all A-worlds in it, $|B|$ is at least r.[29] Then a *partial* generalization of both (VS_{simple}) and (MVS_{simple}) is

(CV-special): $|A \triangleright B|_w = 1$ iff $Liminf_{w,A}(B) = 1$.[30]

This is only a partial account: it only tells us when a conditional gets value 1. For a full account, we have to decide whether it is (VS) or (MVS) that we want to generalize to the $[0,1]$ case.

If it's (VS) that we want to generalize, we set $|A \triangleright B|_w = Liminf_{w,A}(B)$ (or if Connectivity is not assumed, $LOWERVAL_{w,A}(B)$: see note 30). For later reference, I'll use the label (CV-B) for this proposal, or rather, for this proposal modified to give value 1 to vacuous conditionals. However, I'm inclined to think it more desirable to generalize (MVS). In that case, we introduce notions dual to the previous ones:

$$Upperlim_{w,A,y}(B) =_{df} lub^*\{|B|_z : z \leq_w y \wedge |A|_z = 1\}, \text{ and}$$
$$Limsup_{w,A}(B) =_{df} glb^*\{Upperlim_{w,A,y}(B) : y \in W_w \wedge |A|_y = 1\}.$$

Then my proposed generalization of (MVS) when Connectivity is assumed is

(CV): $|A \triangleright B|_w$ is

$$\begin{cases} Liminf_{w,A}(B) & \text{when that is at least } \frac{1}{2}; \\ Limsup_{w,A}(B) & \text{when that is at most } \frac{1}{2}; \\ \frac{1}{2} & \text{when } Liminf_{w,A}(B) < \frac{1}{2} < Limsup_{w,A}(B). \end{cases}$$

A feature that I find attractive is that the value of $\neg(A \triangleright B)$ is the same as that of $A \triangleright \neg B$. (If we used value 1 instead of $\frac{1}{2}$ for the vacuous case, we'd need an exception to this feature for vacuous A.) I don't insist on the details of rule (CV), but it will serve as a good illustration for an account of ground-level conditionals that a naive theory should extend.

Everything is the same when Connectivity is dropped except that we use the $LOWERVAL_{w,A}$ of note 30, and its obvious dual $UPPERVAL_{w,A}$, instead of $Liminf_{w,A}$ and $Limsup_{w,A}$.

This, to repeat, is a natural generalization of variably strict semantics for '$\triangleright$', prior to adding 'ξ' or '$\rightarrow$'. I now turn to what happens when we add 'ξ', saving until Section 4.4 what happens when we add 'ξ' and '$\rightarrow$' together.

---

[29]More accurately: it's the largest number r such that for any $\epsilon > 0$, there's a w-neighborhood that contains A-worlds and where at all A-worlds in it, $|B|$ is at least $r - \epsilon$.

[30]If we don't assume that the $\leq_w$ relation obeys Connectivity, then in the [0,1] framework, we first add an argument x to the notion of $Liminf$, and define a notion $LOWERVAL$ from that:

$$Liminf_{w,A,x}(B) =_{df} lub^*\{Lowerlim_{w,A,y}(B) : y \leq_w x \wedge |A|_y = 1\}.$$
$$LOWERVAL_{w,A}(B) =_{df} glb^*\{Liminf_{w,A,x}(B) : x \in W_w \wedge |A|_x = 1\}.$$

(When Connectivity is assumed, $LOWERVAL_{w,A}$ just is $Liminf_{w,A}$.) The analog of (CV-special) is then

($\mathbf{CV}_{gen}$-special): $|A \triangleright B|_w = 1$ iff $LOWERVAL_{w,A}(B) = 1$.

4.3. **"Ordinary" conditionals with 'ξ' but not '$\rightarrow$': two approaches.** How should we handle '$\triangleright$' in the presence of 'ξ' when we don't have to worry about '$\rightarrow$'? I'll discuss two approaches: a revision-theoretic approach, and an approach that generalizes the fixed point construction of Brady 1983. (As mentioned in note 11, there's also a "higher order fixed point" approach, but it is more complicated than the revision and doesn't appear to have any compelling advantages.) The revision-theoretic approach doesn't quite keep to the letter of (CV), but gives something in the ballpark, and reduces to (CV) for propositions without 'ξ'. For Brady it's similar (though the Brady approach is more natural in a context where (CV-B) rather than (CV) is the target). Unfortunately, the most straightforward version of the Brady-like approach leads to some rather undesirable results: for instance, if $\top$ is a vacuous conditional-free proposition such as $\lambda\forall x(x = x)$, and $\perp$ is its negation, then $\neg(\top \triangleright \perp)$ comes out valid but$\top \triangleright \neg(\top \triangleright \perp)$ doesn't; indeed the negation of the latter comes out valid. But I'll discuss a modified version that seems to avoid the undesirable results. I will not decide among the approaches, and there could well be alternatives preferable to both.

4.3.1. *The revision approach.* On the revision approach we can be brief, because the situation is much like the revision approach for $\rightarrow$, simplified in that there is no need for slow corrections. The main difference is that here the revision procedure doesn't operate world by world, but instead operates on the assignment of values to all worlds at once.

Suppose we're given a [0,1]-valued modal model (including neighborhood structure) for the language with '$\triangleright$' but not '$\rightarrow$' or 'ξ', and with no property or proposition abstracts. Now add property and proposition abstracts, and 'ξ'. Then if j is a "hypothesis" assigning values in [0,1] to all parameterized $\triangleright$-conditionals in the language at each world, the Kripkean procedure of Section 2 will generate a fixed-point value $|A|_{j,w}$ based on j, for every parameterized sentence A of the language and every world w. And then we can use this to come up with a revised valuation $S(j)$ for $\triangleright$-conditionals, in analogy with (CV) (or the generalized version that avoids the Connectivity assumption). That is, in the version that assumes Connectivity, it will be

(REV): $S(j)(A \triangleright B, w)$ is

$$\begin{cases} Liminf_{w,A,j}(B) & \text{when that is at least } \frac{1}{2}; \\ Limsup_{w,A,j}(B) & \text{when that is at most } \frac{1}{2}; \\ \frac{1}{2} & \text{when } Liminf_{w,A,j}(B) < \frac{1}{2} < Limsup_{w,A,j}(B). \end{cases}$$

(The extra subscript j on Liminf and Limsup is for the hypothesis that assigns values to A and B at each world.) Then starting with a transparent initial valuation of conditionals at worlds, we use this rule to give valuations at successor ordinals. At limit ordinals we proceed as with $\rightarrow$: use the liminf of the values at prior ordinals when that's at least $\frac{1}{2}$, the limsup when it's at most $\frac{1}{2}$, and $\frac{1}{2}$ in other cases.

Here too, the general theory of revision sequences applies, to yield a non-empty set FIN^* of ordinals after which only recurring hypotheses occur.[31] And as before,

[31]The * is simply to emphasize that the set of final ordinals in the $\triangleright$-construction needn't be the same as in the $\rightarrow$-construction. For the same reason I'll use different Greek letters for the reflection ordinals of the $\triangleright$-construction.

we single out the "reflection ordinals" for this construction as distinguished members. Every recurring hypothesis appears between any two reflection ordinals, and the value of a conditional at a reflection ordinal Ω is

(#): $j_\Omega(A \triangleright B, w)$ is the greatest lower bound of $\{j(A \triangleright B, w) : j \text{ is recurrent}\}$, when that is at least $\frac{1}{2}$; the least upper bound when that is no more than $\frac{1}{2}$; and $\frac{1}{2}$ in all other cases.

Since valuations at successors of reflection ordinals are recurrent, this together with the revision rule gives us "one direction of" (CV): if $|A \triangleright B|_{w,\Omega} > \frac{1}{2}$ then $Liminf_{w,A,\Omega}(B) \geq |A \triangleright B|_{w,\Omega}$, and if $|A \triangleright B|_{w,\Omega} < \frac{1}{2}$ then $Liminf_{w,A,\Omega}(B) \leq |A \triangleright B|_{w,\Omega}$. (So for instance if $|A \triangleright B|_{w,\Omega} = r > \frac{1}{2}$ then for any $\epsilon > 0$, there is a y_ϵ in W_w such that $(\forall z \leq_w y_\epsilon)(|A|_{z,\Omega} = 1 \supset |B|_{z,\Omega} \geq r - \epsilon)]$, and dually when $|A \triangleright B|_{w,\Omega} = r < \frac{1}{2}$.) The reverse inequalities of (CV) do not in general hold at reflection ordinals (that is, we can have $Liminf_{w,A,\Omega}(B) > |A \triangleright B|_{w,\Omega} > \frac{1}{2}$) and $Liminf_{w,A,\Omega}(B) < |A \triangleright B|_{w,\Omega} < \frac{1}{2}$), because of the quantification over all recurrent valuations; but they do of course hold for conditionals that get the same value at every ordinal in FIN^*, which includes all conditionals with no occurrences of 'ξ'.

When there is no '$\rightarrow$' in the language, we can generalize (#) from conditionals to arbitrary sentences: $|A|_{\Omega,w}$ is the greatest lower bound of the $|A|_{\alpha,w}$ for α in FIN^*, when that's at least $\frac{1}{2}$; the least upper bound when that is no more than $\frac{1}{2}$; and $\frac{1}{2}$ in all other cases. This is an analog of the "Fundamental Theorem" above, but this time for $\triangleright$-sentences. However, the result does not hold in full generality with '$\rightarrow$' in the language: it still holds for $\triangleright$-conditionals, but the inductive argument extending it to other sentences in the language is blocked, and there are sentences where $\triangleright$ is inside the scope of an $\rightarrow$ where the result fails. I don't think this is a serious problem for the revision approach: whereas the Fundamental Theorem for $\rightarrow$-sentences is important in establishing some important laws, as we'll see in Section 4.5, there is no obvious such need for a corresponding result for $\triangleright$-sentences. Nonetheless, the lack of such a theorem makes the theory less tidy, and might be seen as an advantage of the "revised general Brady construction" that I consider next.

Note that in the revision construction we've left the modal model structure completely unchanged. What is changed is the valuation space: it is now of form $[0, 1]^\Psi$, where Ψ is the set of values from one reflection ordinal to the next (including at least one of them). We take an inference from Γ to B to be valid if for every modal model M, and every normal world in W_M (which may be every world in W_M, depending on the modal model), if all members of Γ have the constant function $\mathbf{1}$ as values, so does the conclusion.

4.3.2. *The Brady-like and revised Brady-like fixed point approaches.* Ross Brady introduced a procedure for naive conditionals, for 3-valued models in which W consisted of only a single world so that the "variably strict" element is lost. But it easily generalizes to multi-world 3-valued models W, and indeed to multi-world continuum-valued models. I'll now state the idea in a way that incorporates both generalizations; several paragraphs from now I'll add a substantial tweak. In the Brady construction with or without the tweak, it's best to think of vacuous conditionals as having value 1, and probably to build upon the asymmetric rule (CV-B) of Section 4.2 rather than the symmetric (CV). But for present purposes we need not fuss about these details.

The simplest way to view the generalized Brady construction is to leave the original modal model structure unaltered, but to alter the revision rule S; instead, let $S^B(j)(A \triangleright B, w) = min\{S(j)(A \triangleright B, w), j(A \triangleright B, w)\}$ where S is much as in the revision approach (though probably modeled on (CV-B) instead of (CV)). What I want to focus on is the minimization used in S^B. This modification guarantees that $S^B(j)(A \triangleright B, w) \leq j(A \triangleright B, w)$, for each $A \triangleright B$ and w; so that this "revision rule" is monotonic. At limit ordinals, we set $j_\lambda(A \triangleright B, w) = glb\{j_\alpha(A \triangleright B, w) : \alpha < \lambda\}$. As a result, the entire construction reaches a fixed point. We take validity to be preservation of value 1 at the fixed point, at all normal worlds in all models.

I haven't specified the question of the starting valuation j_0. Brady's approach was to take this as assigning the value 1 to every conditional in the one world in his model. The obvious generalization of to the multi-world context is to let it assign value 1 to every conditional at every world. But with or without this generalization to the multi-world context, this has very odd results (and would even in the original 3-valued setting). For instance, at the single world of Brady's actual approach, while $\top$ and $\neg(\top \triangleright \bot)$ quite properly get fixed point values 1, $\top \triangleright \neg(\top \triangleright \bot)$ gets fixed point value 0. That's because at the initial stage, $\neg(\top \triangleright \bot)$ gets the wrong value 0, and though this is corrected at the next stage, its effects survive: it makes $\top \triangleright \neg(\top \triangleright \bot)$ get value 0 at that next stage, and once a conditional gets value 0 it can never recover. The oddity carries over to the multi-world generalization: indeed, now $\top \triangleright \neg(\top \triangleright \bot)$ gets fixed point value 0 *at every world*.

This feature of the Brady approach is due to the starting valuation. Can we find a better starting valuation that doesn't have this feature? For a long time I didn't think one could do so without introducing ideas foreign to his approach, but it now occurs to me that there is a way; it involves adapting the suggestion for a starting valuation for '$\rightarrow$' that I made in note 16. Let j be an arbitrary transparent valuation of (parameterized) conditionals at worlds; then a Brady construction that takes j as its starting valuation yields as its fixed point a new valuation $Reg(j)$ of conditionals at worlds (with $[Reg(j)](A \triangleright B, w) \leq j(A \triangleright B, w)$). Now, for any set V of transparent valuations, let $j^{\#(V)}(A \triangleright B, w)$ be the least upper bound of the $[Reg(j)](A \triangleright B, w)$ for all j in V. Then a simple version of the tweak is to take as starting valuation $j^{\#(V_0)}$, where V_0 is the set of transparent valuations. That's enough to avoid at least the most obvious problems of the construction without the tweak. (A natural further improvement is to use instead $j^{\#(V_{fp})}$, where V_{fp} is constructed by a natural fixed point procedure given in the attached footnote.)[32] Call the use of a starting valuation based on a $j^{\#}$ (whether applied to V_0 or to V_{fp}) a *revised general Brady construction*. ("General" reflecting both the use of a multi-world starting point with variably strict semantics and the use of [0,1]. The *unrevised* general Brady construction is the one starting from the valuation that assigns value 1 to every conditional at every world.)

$A \triangleright B$ won't be value functional in the values in [0,1] that $Reg(j^{\#})$ assigns at worlds: to compute its value you need the values of A and B at worlds supplied earlier in the fixed point construction from $j^{\#}$. But the construction does make

[32]For each α let $V_{\alpha+1}$ be the set of valuations j where for each $A \triangleright B$ and each w, $j(A \triangleright B, w) \leq j^{\#(V_\alpha)}(A \triangleright B, w)$; repeated application (taking intersections at limits) gives rise to a decreasing sequence of V_α, non-empty at each stage (as can be seen from the fact that it includes the valuation that assigns every conditional value 0 at all worlds, though of course it also includes much better ones).

it value functional in the function space $[0,1]^X$, where X is the set of the fixed point ordinal and its predecessors. So this kind of product space turns up on the Brady-based construction as well as on the revision. As with the unrevised Brady, validity is taken to be the preservation of value 1 at the fixed point ordinal at all normal worlds.

I haven't investigated this construction closely, but it looks to me as if it and the revision construction do about equally well in the respects I most care about. They will not deliver exactly the same laws, but as we'll see in Section 4.5, each when combined in the right way with $\rightarrow$ leads to the most obviously desirable laws about how restricted quantification and $\triangleright$ interact; and I'm not in a position now to systematically evaluate the laws on which they differ. On simplicity of use there are tradeoffs: often the values in revised Brady are heavily dependent on the starting valuation $j^{\#}$, and calculating that adds a layer of complexity; on the other hand, in revised Brady certain conditionals can be easily seen to have value 0 at the fixed point whatever their starting values. (An informed decision on simplicity/convenience as well as on laws could only be made after we've added $\rightarrow$ to the language.)

If the revised Brady might be good as an alternative to the revision account for $\triangleright$, might it also be good as an alternative to the revision account for '$\rightarrow$'? No. For it's clear that an adequate account of a restricted quantifier conditional '$\rightarrow$' requires the Weakening Rule

$\rightarrow$-WEAKENING: $B \models A \rightarrow B$;

otherwise, the obviously-desirable inference from "Everything is B" to "All A are B" wouldn't come out valid. And this law fails on even the revised Brady for $\rightarrow$, though less obviously so than for the unrevised. The inference from $\neg W$ to $W \rightarrow \neg W$, where W is equivalent to $True(\langle W \rangle) \rightarrow True(\langle \neg W \rangle)$, is no longer a counterexample: $j^{\#}$ (in either version) in fact assigns it the same value $\frac{2}{3}$ to W that it gets with Łukasiewicz, and it retains this value all the way through to the fixed point. But a counterexample can be obtained from the Restall sentence R considered earlier, which (in very sloppy notation) is equivalent to $\exists n(R \rightarrow^n \bot)$. It isn't hard to see that $j^{\#}$ (in either version) assigns to $R \rightarrow^n \bot$ the value $\frac{n}{n+1}$, and that at the α^{th} stage its value is 0 when $\alpha \geq n$ and $\frac{n-\alpha}{n-\alpha+1}$ otherwise. The value of R at any stage is the least upper bound of the values of these conditionals, so it's 1 at all finite stages, and 0 from stage ω on. From this it's clear that $\top \rightarrow \neg R$ gets value 0 at least from stage 1. Consequently, while $\neg R$ gets value 1 at the fixed point, $\top \rightarrow \neg R$ gets value 0, in violation of Weakening.[33]

An analogous sentence using $\triangleright$ for $\rightarrow$ would of course show the invalidity of a Weakening rule for $\triangleright$, but that's no problem since for $\triangleright$ we don't expect Weakening to hold. (We might expect $\Box B \models A \triangleright B$, but I don't regard its failure to hold unrestrictedly on the revised Brady as obviously crippling.)

[33][Added at the last minute.] One might consider changing the notion of validity, to preserving the property of having value 1 *throughout the fixed point construction.* On the untweaked Brady construction that would validate Weakening, but would be obviously unsatisfactory: no negation of a conditional would be valid. With the new starting valuation that objection doesn't apply, but it is also unobvious that Weakening holds. ($\neg R$ is no longer a counterexample, but I've been unable either to prove there are no others or to find any.) If Weakening is valid on this definition, and if the recapture of Łukasiewicz values that we get for sentence W extends to all sentences in the quantifier-free sublanguage, then this rival approach to $\rightarrow$ is worth seriously considering.

4.4. $\triangleright$ **and** $\rightarrow$ **together.** If we want to employ $\triangleright$-conditionals in a setting with naive truth or property-instantiation, we need to deal with sentences that contain $\rightarrow$ and $\triangleright$ together. Indeed, many obvious laws of conditionals have $\rightarrow$ embedded inside $\triangleright$: e.g. $[\forall xBx \wedge \forall x(Bx \rightarrow Cx] \triangleright \forall xCx$, i.e. "If everything is B, and all B are C, then everything is C". (There are fewer obvious laws that essentially require embedding $\triangleright$ inside $\rightarrow$, but there are some: for instance, $True(\lambda(A \triangleright B)) \leftrightarrow [True(\lambda A) \triangleright True(\lambda B)]$.)

We have the pieces, but need to combine them, and do so in a way that will yield desirable laws. The right way to do so is asymmetric between $\triangleright$ and $\rightarrow$ (as it would pretty much have to be if we decide on the Brady option for $\triangleright$, since it's hard to see how to combine that symmetrically with a very different kind of account for $\rightarrow$).

Suppose we're given a modal model for the ground language, taken now to include the neighborhood structure given by the relative closeness ordering in worlds. We want to use it to assign values to all parameterized sentences in the language with 'ξ', '$\rightarrow$' and '$\triangleright$' (plus abstraction terms, 'Property' and 'Proposition').

In either case, the overall procedure I propose involves a multi-stage *macro-construction*, focused on $\triangleright$; each stage of which is a *mid-level construction*, focused on $\rightarrow$; each stage of which is a (generalized) Kripkean *micro-construction*, focused on 'ξ'.

In the microconstructions, we hold fixed both a [0,1]-valuation j of parameterized $\triangleright$-conditionals at each world and a [0,1]-valuation h of parameterized $\rightarrow$-conditionals at each world, and use the generalized Kripke fixed point construction in Section 2 to get a value in [0,1] for every parameterized sentence of the language at every world relative to j and h.

In the mid-level constructions, we hold fixed a [0,1]-valuation j of parameterized $\triangleright$-conditionals at each world, and use the revision procedure of Section 3 to get a value $||A||_{j,w}$ for every sentence at each world. These values $||A||_{j,w}$ are in a space $[0,1]^{\Pi_j}$ where Π_j is the difference between two reflection ordinals of the $\rightarrow$-construction based on j.[34] As we've seen, for typical sentences A including finitely iterated Curry sentences, the value $||A||_{j,w}$ for any j and w will be a constant function on Π_j: that was the advantage of using [0,1] instead of $\{0, \frac{1}{2}, 1\}$. But for weird enough sentences, it won't be. In that case, though, the value at the reflection ordinal has a privileged status, which will be exploited in the macro-construction. (The values that sentences take on at other final ordinals of the midlevel construction won't be directly used in the macro-construction.)

In the macro-construction, we vary the [0,1]-valued valuation j of parameterized $\triangleright$-conditionals at worlds, either by the revision procedure described in 4.3.1 or the revised Brady procedure described in 4.3.2. The only difference between here and the previous section is that the $\triangleright$-conditionals that are evaluated in the stages of the macro-construction may now contain $\rightarrow$, but this is no problem since the mid-level reflection values of all sentences including ones with $\rightarrow$ get values at prior stages of the macro-construction.

So the overall architecture of the two constructions is the same. In both cases we can think of the overall valuation space as rather like a fiber bundle. The "base space" Z is a segment of the ordinals, with a distinguished member z_0 (the reflection ordinal or fixed point ordinal of the macro-construction, as the case may be).

[34]If we like, we can get a common space $[0,1]^{\Pi}$ that works for all $\triangleright$-valuations j. (Π need only be a common right-multiple of each of the Π_j. A big enough initial ordinal will certainly do.)

To each member of Z is attached a circular "fiber", obtained from $[\Delta_j, \Delta_j + \Pi_j]$ by identifying endpoints, which is attached to the base space at its distinguished member Δ_j. In the evaluation of sentences, values in [0,1] are assigned to $\triangleright$-conditionals at worlds primarily at the base points (though the value is used at all points of the fiber attached to the base point); whereas values of $\rightarrow$-conditionals at worlds are assigned primarily to points of the various fibers. Moreover, the rules for evaluating sentences on the fibers are the same in both constructions: they're given by the mid-level procedure. The only difference is on the valuation rules in the base space.

It might well be thought that the "fiber bundle" structure that these constructions have in common is implausibly complicated. But I know of no other way of adequately accommodating the two kinds of conditionals in a naive theory. (I'll briefly consider another approach in Section 5.)

4.5. **Some laws.** All these constructions validate some important laws, including the following:

(1): $[\forall x(Ax \rightarrow Bx) \wedge Ay] \triangleright By$ "If all A are B, and y is A, then y is B"
(1c): $[\forall x(Cx \rightarrow Dx) \wedge \neg Dy] \triangleright \neg Cy$
"If all C are D, and y is not D, then y is not C"
(2): $\forall x Bx \triangleright \forall x(Ax \rightarrow Bx)$ "If everything is B, then all A are B"
(2c): $\forall x \neg Cx \triangleright \forall x(Cx \rightarrow Dx)$ "If nothing is C, then all C are D".
(3): $\forall x(Ax \rightarrow Bx) \wedge \forall x(Bx \rightarrow Cx) \triangleright \forall x(Ax \rightarrow Cx)$
"If all A are B and all B are C then all A are C"
(4): $\forall x(Ax \rightarrow Bx) \wedge \forall x(Ax \rightarrow Cx) \triangleright \forall x(Ax \rightarrow Bx \wedge Cx)$
"If all A are B and all A are C then all A are both B and C"
(5): $\neg\forall x(Ax \rightarrow Bx) \triangleright \exists x(Ax \wedge \neg Bx)$
"If not all A are B, then something is both A and not B"
(5*): $\neg\exists x(Ax \wedge \neg Bx) \triangleright \forall x(Ax \rightarrow Bx)$
"If nothing is both A and not B, then all A are B"
(6): $\exists x(Ax \wedge \neg Bx) \triangleright \neg\forall x(Ax \rightarrow Bx)$
"If something is both A and not B, then not all A are B".

(1c) and (2c) follow from (1) and (2) respectively on the supposition that the $\rightarrow$ contraposes, as the $\rightarrow$ in this paper does; but they are worth stating separately for those who would like a non-contraposable $\rightarrow$. Also (5*) needs to be stated separately from (5), since the ordinary conditional $\triangleright$ definitely does not contrapose. (5*) is just a more general form of (2).

These are very simple laws: all of them are schemas of form $X \triangleright Y$, where neither X nor Y involve $\triangleright$ essentially though they do involve $\rightarrow$. That makes them very easy to verify.

For in order that $X \triangleright Y$ be valid, it is required only that for every modal model for the ground language and every member j of the base space Z (that is, the set of recurrent macro-valuations, on the revision macro-construction; and the set of stages along the way to the fixed point, on the Brady) and every normal world:

> if $|X|_{j,w,\Delta_j}$ is 1 then so is $|Y|_{j,w,\Delta_j}$ (where Δ_j is a reflection ordinal of the fiber attached to j).

Using the Fundamental Theorem for fibers, this is in turn equivalent to:

(*): for all normal w and all $j \in Z$, if $|X|_{j,w,\alpha} = 1$ for all final α in the $\rightarrow$-construction for j,w, then $|Y|_{j,w,\Delta_j} = 1$.

In the case of each of the laws $X \rhd Y$ above, (*) would hold even without the restriction to normal w, or the restriction to $j \in Z$. In other words, the validity is guaranteed by only the fiber construction together with basic structural features of the macro-construction that are common to its different versions.

For instance, in the case of law 1, what we need (dropping the j and w subscripts from the notation, since they are irrelevant to the argument) is that if $(\forall\alpha \in FIN)[|\forall x(Ax \to Bx) \wedge Ao|_\alpha = 1]$ then $|Bo|_\Delta = 1$, for any object o in the domain. But the assumption requires that $|Ao|_\Delta$ be 1 and also that for any final α in the $\to$-construction, $|Ao|_\alpha \leq |Bo|_\alpha$; and since Δ is itself one of those final α, $|Bo|_\Delta = 1$ as desired.

And in the case of 5*, what we need is that if $(\forall\alpha \in FIN)[|\neg\exists x(Ax \wedge \neg Bx)|_\alpha = 1]$ then $|\forall x(Ax \to Bx)|_\Delta = 1$. But the assumption requires that for any o in the domain, $|\neg Ao \vee Bo|_\Delta = 1$, which requires that either $|Ao|_\Delta = 0$ or $|Bo|_\Delta = 1$, which by the Fundamental Theorem requires that either $(\forall\alpha \in FIN)(|Ao|_\alpha = 0)$ or $(\forall\alpha \in FIN)(|Bo|_\alpha = 1)$. And that guarantees $|\forall x(Ax \to Bx)|_\Delta = 1$, as desired.

The other laws I've listed are verified similarly.

5. Standard "Relevant" Conditionals.

Much of the technical literature on the paradoxes, especially that in the dialetheic tradition, is focused on relevant conditionals, especially those in the vicinity of the System **B** of Priest 2008. (In some of the literature this system is weakened to exclude even the rule form of contraposition. In some cases (with or without the modification about contraposition) it is strengthened to include excluded middle. There may be other variations as well.)

It is not entirely clear to me the motivation behind this focus. Such relevant conditionals are ill-suited both for restricted quantification and for the ordinary English conditional that we find in such sentences as

(1): If I get a reservation at that restaurant, I'll eat dinner there tonight.

One of the most obvious features of such sentences is that they do not obey the rule of antecedent strengthening

(AS): If A then $C \models$ If A and B then C;

for (1) clearly doesn't imply

(2): If I get a reservation at that restaurant and die immediately after doing so, I'll eat dinner there tonight.

But (AS) is built into System **B** and the logics in its vicinity that the technical literature above employs.

It is no defense to say that if a logic is compatible (in the sense of Post-consistency) with naivety in a logic with (AS), then it is compatible with naivety in the weaker logic with (AS) dropped. For what we want is more than the Post-consistency of the naivety claims alone, we want that *adding naivety to whatever acceptable assumptions we start with* is Post-consistent. And since assumptions violating (AS) are acceptable, we need to demonstrate that naivety is compatible with such violations. Showing that naivety holds *in some models of a logic that includes (AS)* can't be of any help.

It is equally clear that relevant conditionals are of no help for restricted quantification. Indeed, it seems pretty clear that for that, we need a logic whose $\to$ reduces to $\supset$ when the antecedent and consequent are classical; and the relevant conditionals were designed precisely *not* to have that feature. That aside, even many who think relevant conditionals important have conceded that they can't be used for restricted quantification because we need at least the rule form of Law 2 above, viz.

Everything is $B \models$ All A are B.

If $\rightarrow$ is the restricted quantifier conditional, the conclusion is $\forall x(Ax \rightarrow Bx)$, and so the law clearly requires $D \models C \rightarrow D$; and that law is not valid for relevant conditionals. For this reason, the authors of Beall *et al* 2006 have advocated using both a relevant conditional for purposes other than restricted quantification and an "irrelevant" one for restricted quantification. I disagree about the details of their "irrelevant" one (theirs doesn't reduce to $\supset$ in classical contexts), but they are certainly right that no relevant conditional will do for restricted quantification.

The Beall *et al* paper was a great advance: indeed my own focus on the use of two distinct conditional operators, one for restricted quantification and one for more ordinary conditionals, was inspired by it. But for the reasons just given, I don't think that either of the two particular conditional operators they use is what is needed for their respective purposes.

Further evidence for this, were it needed, is that the laws that one gets with their conditionals are very far from what we need for restricted quantification. I concede that there might be some dispute as to exactly which laws we need. Dialetheists *must* dispute my list: for instance, even *the rule forms of* (1) and (5*) together entail

Everything is either B or is not A, c is $A \models c$ is B;

and taking B to be $x = x \wedge \bot$ and A to be $x = x \wedge \lambda$ where λ is a dialetheia, this yields

$\neg\lambda, \lambda \models \bot$,

which no dialetheist can accept. Beall *et al* do accept the rule form of (1), and so reject even the rule form of (5*). It's hard for me to find an independent rationale for rejecting it, short of attributing to the restricted quantifier a modal element which I don't think it has; but I recognize that I'm unlikely to convince the committed dialetheist that there is a problem here.[35]

But dialetheism aside, the Beall *et al* system also doesn't include the full (1), or (3), (5) or (6). (The failure to get the full (1) is because they assume an intimate relation between the two conditionals: they assume that $A \rhd B \models A \rightarrow B$ (using $\rhd$ for their relevant conditional and $\rightarrow$ for their restricted quantifier conditional), which would mean that (1) would require "Pseudo Modus Ponens" for $\rightarrow$ (i.e. $[(A \rightarrow B) \wedge A] \rightarrow B$); as is well-known, that conflicts with genuine modus ponens in any naive theory.) There's a lot more that could be said about this, but one moral seems to be that relevant conditionals are just the wrong tool for naive theories.[36]

And whatever one thinks of the treatment of $\rhd$ in these theories, the treatment of the restricted quantifier conditional is highly problematic: though these theorists

[35]The weak consequence (2_c) of (5*) is also incompatible with the rule form of (1). Their restricted quantifier conditional is non-contraposable, so they can accept (2).

[36]This is perhaps a slight overstatement, in that I've suggested a possible use for the Brady construction, which is connected to relevance. But I've suggested that it has no role for the $\rightarrow$-conditional, and the main departure of $\rhd$ from a conditional that reduces to $\supset$ in classical contexts isn't due to the Brady construction but to the variably strict semantics.

recognize that it is not a relevant conditional, they treat it as intimately related to one, in a way that prevents its collapse to $\supset$ in classical contexts.

6. Property identity

An important issue about naive property theory, raised originally in Restall 2010, is how to include within it satisfactory identity conditions for properties.

6.1. **A negative result.**

Restall assumed that a satisfactory account of property-identity should satisfy the condition that if $P(x) \models Q(x)$ and $Q(x) \models P(x)$ then $\models \lambda x P(x) = \lambda x Q(x)$, and showed that if so, then no satisfactory account is possible in a theory like mine. But as I argued in Field 2010, that condition on property identity is unreasonable: if μ is a Liar sentence, and $P(x)$ is "$\mu \wedge x$ is a cockroach" and $Q(x)$ is "$\mu \wedge x$ is a kangaroo", we shouldn't expect $\lambda x P(x)$ to be the same as $\lambda x Q(x)$.

A more plausible criterion, for the language without $\Box$ or $\triangleright$ that was there under consideration, is this:

(?): If $\models \forall x(P(x) \leftrightarrow Q(x))$ then $\models \lambda x P(x) = \lambda x Q(x)$.

And Restall's impossibility proof doesn't work for this loosened criterion. Something I said in the above paper implies that (?) can be achieved for the language there under consideration, but the argument-sketch I gave for that claim was seriously flawed. Indeed, Harvey Lederman pointed out to me that on the specific version of the $\rightarrow$ there under consideration (which involved a jumpy correction rule and a starting valuation that assigned every conditional value $\frac{1}{2}$), (?) must fail for any reasonable notion of identity when $P(x)$ is $\bot$ and $Q(x)$ is $\top \rightarrow \bot$.[37]

That argument would be blocked by the choice of initial valuation recommended in note 16, or alternatively by the use of slow corrections together with a starting valuation that assigns all conditionals value 1. But any hope this might generate would be misplaced.

For Tore Fjetland Øgaard gave a proof that (?) leads to triviality in any system that includes the $\rightarrow$-Weakening rule plus minimal other laws; it's reported in Section 10 of Field, Lederman and Øgaard forthcoming (as is the earlier Lederman result). Our discussion there might suggest that Øgaard's proof undermines only the idea that *coextensiveness* suffices for identifying abstracts, but in fact it undermines even the idea that *validity of* coextensiveness is sufficient.

---

[37]If identity is to behave at all reasonably, then for any formula $S(y)$, (?) implies

(?-S): If $\models \forall x(P(x) \leftrightarrow Q(x))$ and $\models S(\lambda x Q(x))$, then $\models S(\lambda x P(x))$.

Using the P and Q in the text, the first antecedent holds, so this becomes

(?*-S): If $\models S(q)$, then $\models S(p)$, where p is $\lambda x(\bot)$ and q is $\lambda x(\top \rightarrow \bot)$.

Let $S(y)$ be $\forall u(y \,\xi\, u \leftrightarrow q \,\xi\, u)$. Then $\models S(q)$. But not $\models S(p)$, at least with fast corrections and a starting valuation that values all conditionals the same. To see this, we proceed in three steps. First we argue that if $\alpha > 0$ then $|S(p)|_\alpha \leq |\top \rightarrow S(p)|_\alpha$. (That's because by the definition of S, $|S(p)|_\alpha \leq |p \,\xi\, \lambda z S(z) \leftrightarrow q \,\xi\, \lambda z S(z)|_\alpha$, and by naivety the right hand side equals $|S(p) \leftrightarrow S(q)|_\alpha$, and for $\alpha > 0$ $|S(q)|_\alpha = 1$.) Second, we use this to argue that on the fast-correction construction, if $S(p)$ has value less than 1 when α is 1 then it can't have a higher value for larger α. Finally we observe that on the assumed starting valuation, $S(p)$ does have a value less than 1 when α is 1: for letting e be $\lambda z[\neg \exists w(w \,\xi\, z)]$, $|p \,\xi\, e|_0 = |\neg\bot|_0 = 1$ and $|q \,\xi\, e|_0 = |\neg(\top \rightarrow \bot)|_0 < 1$ and so $|S(p)|_1 \leq |p \,\xi\, e \leftrightarrow q \,\xi\, e|_\alpha < 1$. The second of the three steps fails with slow corrections when the starting value for conditionals is 1, and the third fails for any starting valuation where $\top \rightarrow \bot$ is given value 0.

The failure of (?) might be unsurprising in the presence of non-normal worlds, since its antecedent only requires that $\forall x(P(x) \leftrightarrow Q(x))$ have value **1** at *normal* worlds, and it might well be thought that failure of coextensivity at non-normal worlds precludes property identity. That could be handled by putting a '$\Box$' or a '$\top\triangleright$' before the '$\forall$', given the structural assumption that if there are non-normal worlds then each is accessible from a normal world. Will this or some similar modality $\Box^*$ (perhaps defined using '$\top \rightarrow$' as well as '$\Box$' and '$\top\triangleright$') solve the problem?[38] That is, is there some modality obeying minimally reasonable laws for which we can define property identity so as to get

($?_w$): If $\models \Box^*\forall x(P(x) \leftrightarrow Q(x))$ then $\models \lambda x P(x) = \lambda x Q(x)$?

No: Øgaard's proof generalizes to rule that out. Though the generalization is totally routine, I include a proof in a footnote since I think our original presentation was hard to survey.[39]

6.2. **A positive result.** Despite the Øgaard proof, there is room for a great deal of coarse-graining. Precisely how coarse-grained to go is somewhat arbitrary: we can pick any formula $R(x, y)$ that satisfies the following conditions.[40]

(I): $\models \forall x, y[R(x, y) \rightarrow (Property(x) \wedge Property(y)) \vee (Proposition(x) \wedge Proposition(y)) \vee x =_o y]$ (where $=_o$ is the ground level identity predicate)

(II): $\models \forall x, y[R(x, y) \vee \neg R(y, x)]$

(III): $\models \forall x R(x, x)$

---

[38]In fact the extra operators couldn't help: while (e.g.) $\top \triangleright (\top \rightarrow B)$ doesn't follow from $\Box B$, still the validity of the former follows from the validity of the latter in the logics we've discussed.

[39] Define $b \sim c$ ("congruence") to mean $\forall z(b \,\xi\, z \leftrightarrow c \,\xi\, z)$; so by transparency, we have $\models b \sim c \rightarrow (A(b) \leftrightarrow A(c))$, for any A. In any of these logics that in turn implies $A(b) \models b \sim c \rightarrow (\top \leftrightarrow A(c))$. (This depends on $\rightarrow$-Weakening.) Then for any reasonable modality $\Box^*$ we get the "modal quasi-substitutivity lemma" $\Box^* A(b) \models \Box^*[b \sim c \rightarrow (\top \rightarrow A(c))]$.

Then let $\emptyset$ be $\lambda x(\bot)$, and if p is an instantiation-invariant property, let $F(p)$ ("the quasi-complement of p") be $\lambda x(p \sim \emptyset)$. Trivially, $\emptyset \,\xi\, F(\emptyset)$; so making use of the instance of quasi-substitutivity where $A(p)$ is $\emptyset \,\xi\, p$, b is $F(\emptyset)$ and c is $\emptyset$, we easily prove

(1) $\models \Box^*[F(\emptyset) \sim \emptyset \rightarrow \bot]$.

Consider the "Hinnion property" $H =_{df} \lambda y[\lambda x(y \,\xi\, y) \sim \emptyset]$. Let κ be $H \,\xi\, H$, and let B be $\lambda x[\kappa]$. Making use of (1) together with the instance of quasi-substitutivity where $A(p)$ is $p \sim F(p)$, b is B and c is $\emptyset$, we easily prove

(2) $\Box^*(B \sim F(B)) \models \Box^*(B \sim \emptyset \rightarrow \bot)$.

We easily prove

(3) $\models \Box^*(\kappa \leftrightarrow (B \sim \emptyset))$;

in effect, that B is necessarily coextensive with $F(B)$. And from (2) and (3) we easily prove

(4) $\Box^*(B \sim F(B)) \models \Box^*(\kappa \leftrightarrow \bot)$,

whose conclusion says in effect that B is necessarily coextensive with $\emptyset$.

So far, the proof has made no use of ($?_w$). I'll assume that property identity is "rigid" in the sense that if properties p and q are identical then $\Box^*\forall r(p\xi r \leftrightarrow q\xi r)$ (that is, $\Box^*(p \sim q)$). So ($?_w$) entails

(?*) If $\models \Box^*\forall x(P(x) \leftrightarrow Q(x))$ then $\models \Box^*[\lambda x P(x) \sim \lambda x Q(x)]$.

Applying that first to (3), we get the premise of (4); hence we get

(5) $\models \Box^*(\kappa \leftrightarrow \bot)$.

And applying (?*) again to that, we get

(6) $\models \Box^*(B \sim \emptyset)$.

But (6) and (3) yield $\models \Box^*\kappa$, which with (5) yields $\Box^*\bot$ and hence $\bot$.

[40]It may be possible to liberalize this by dropping Condition (II), weakening (V) to

(V_{weak}): $R(x, y) \models \forall z[R(y, z) \rightarrow R(x, z)]$,

and finding a suitable weakening of (VI). But the use of non-bivalent property identity raises issues that would take us too far afield.

(IV): $\models \forall x[R(x,y) \to R(y,x)]$
(V): $\models \forall x,y,z[R(x,y) \wedge R(y,z) \to R(x,z)]$
(VI): If b and c are properties or propositions such that $\models R(b,c)$, and j is a $\triangleright$-valuation and v is an $\to$-valuation and *the pair $\langle j,v\rangle$ occurs at some Kripke fixed point in the overall construction*, then for all objects o in the domain and all worlds w, $|o\,\xi\,b|_{j,v,w} = |o\,\xi\,c|_{j,v,w}$.

Many such R are easily definable in the language (even the fragment without 'True'), as long as the ground language is adequate to syntax, or to set theory.

An obvious proposal is to take R to mean provable equivalence in some suitable logic; for instance, $R[\lambda uB(u), \lambda vC(v)]$ would be of form $\vdash_{log} \forall v(B(v) \leftrightarrow C(v))$. This will automatically validate the first five conditions. The "suitable logic" is naturally taken to include quantified S_3 ("symmetric Kleene logic"),[41] and also to include the naivety conditions plus a selection of laws involving $\to$ and $\triangleright$. The latter laws need to be chosen in a way that is compatible with condition (VI), but that allows for quite a bit. For instance, (VI) isn't violated by building in the equivalence of $\lambda u(B(u) \to C(u))$ to $\lambda u(B(u) \to C(u) \wedge C(u))$ or to $\lambda u(B(u) \to u\,\xi\,\lambda zC(z))$. It also isn't violated by building in the equivalence of $\lambda uB(u)$ to $\lambda u(B(u) \wedge (C(u) \to C(u))$, or the equivalence of $\lambda u(B(u) \to B(u))$ to $\lambda u(B(u) \triangleright B(u))$, provided that we start the revision process by assigning these conditionals value 1 rather than say $\frac{1}{2}$. (So this is one place where my choice of $\frac{1}{2}$ as initial value for conditionals was sub-optimal. The choice of 1 for all conditionals, though less natural, would have been marginally better and have no obvious downside; the valuation suggested in note 16 would be better still.)

I claim that any R meeting (I)-(VI) meets the formal conditions on identity: in particular, the requirement of substitutivity of identity. To show this, I use the "Micro-Extensionality Theorem" from Field *et al* forthcoming (which perhaps was implicit in some much earlier papers by Ross Brady). Or rather, I use a generalization of this theorem, not only to a modal setting but more substantially, to the setting of the Kripke algebra [0,1]. (In fact it generalizes to arbitrary Kripke-algebras, but there will be no need for that.) Let v and j be transparent valuation functions for $\to$-conditionals and $\triangleright$-conditionals respectively. Let b and c be any two closed property abstracts. Call v *$\langle b,c\rangle$-congruent* if for any parameterized formulas $P(u)$ and $Q(u)$ and any world w, $v(P(b) \to Q(b), w) = v(P(c) \to Q(c), w)$; and analogously for j, using $\triangleright$ instead of $\to$. (And call a pair $\langle j,v\rangle$ *$\langle b,c\rangle$-congruent* if both its members are.) Call a pair $\langle j,v\rangle$ *$\langle b,c\rangle$-good* if for any object o in the domain and world w, $|o\,\xi\,b|_{j,v,w} = |o\,\xi\,c|_{j,v,w}$, where these are the Kripke fixed point values. Call a pair $\langle j,v\rangle$ *strongly $\langle b,c\rangle$-congruent* if for any parameterized 1-formula $A(z)$ in the domain, $|A(b)|_{j,v,w} = |A(c)|_{j,v,w}$. Then we have

Generalized Micro-Extensionality Theorem: For any closed property abstracts b and c, and any pair $\langle j,v\rangle$ of transparent valuations for the two kinds of conditionals: if $\langle j,v\rangle$ is $\langle b,c\rangle$-congruent and $\langle b,c\rangle$-good then it is *strongly* $\langle b,c\rangle$-congruent.

Proof: Suppose $\langle j,v\rangle$ is not $\langle b,c\rangle$-congruent. Then there is at least one parameterized 1-formula $A(x)$ and world w such that $|A(b)|_{j,v,w} \neq |A(c)|_{j,v,w}$. Call any such pair of $A(x)$ and w a *counterexample*. For any counterexample $\langle A(x),w\rangle$, either

[41]This is the 3-valued logic with Kleene evaluation rules in which for an inference to be valid, its conclusion must in every model have value at least that of the minimum of the values of its premises.

(i): $|A(b)|_{j,v,w} < |A(c)|_{j,v,w} \leq \frac{1}{2}$
(ii): $|A(c)|_{j,v,w} < |A(b)|_{j,v,w} \leq \frac{1}{2}$
(iii): $|A(b)|_{j,v,w} > |A(c)|_{j,v,w} \geq \frac{1}{2}$, or
(iv): $|A(c)|_{j,v,w} > |A(c)|_{j,v,w} \geq \frac{1}{2}$.

That is, for any counterexample $\langle A(x), w \rangle$, there are ordinals σ such that in the Kripke microconstruction, either

(i_σ): $|A(b)|_{j,v,w,\sigma} < |A(c)|_{j,v,w} \leq \frac{1}{2}$;
(ii_σ): $|A(c)|_{j,v,w,\sigma} < |A(b)|_{j,v,w} \leq \frac{1}{2}$;
(iii_σ): $|A(b)|_{j,v,w,\sigma} > |A(c)|_{j,v,w} \geq \frac{1}{2}$;
(iv_σ): $|A(c)|_{j,v,w,\sigma} > |A(b)|_{j,v,w} \geq \frac{1}{2}$.

So for any $A(x)$ that is the first component of a counterexample, there is a smallest σ such that for some world w, one of (i_σ)-(iv_σ) holds. Call this the *height* of $A(x)$. (All and only those $A(x)$ that are the first components of counterexamples are assigned heights.)

Assuming there are counterexamples, there are ones whose first component has lowest height; let δ be the lowest height at which there are counterexamples.

Lemma: No parameterized 1-formula of form $t(x)\,\xi\,x$ can have height δ.

Proof of Lemma: If $t(x)\,\xi\,x$ had height δ, then one of the cases (i_δ)-(iv_δ) would apply. Relabelling if necessary, we can stick to cases (i_δ) and (iii_δ); and the proofs for them are similar, so let's just focus on (i_δ). That is, we suppose

$$|t(b)\,\xi\,b|_{j,v,w,\delta} < |t(c)\,\xi\,c|_{j,v,w} \leq \tfrac{1}{2}.$$

δ can't be 0, since $|t(b)\,\xi\,b|_{j,v,w,0} = \frac{1}{2}$, and it can't be a limit since if $|t(b)\,\xi\,b|_{j,v,w,\sigma} \geq |t(c)\,\xi\,c|_{j,v,w}$ for all σ less than a limit λ then the same holds for λ. So it would have to be of form $\tau+1$. But b and c are of form $\lambda x B(x)$ and $\lambda w C(w)$ where $B(x)$ and $C(w)$ are parameterized 1-formulas, so $|t(b)\,\xi\,b|_{j,v,w,\delta} = |B(t(b))|_{j,v,w,\tau}$; and since by the Kripke fixed point condition $|t(c)\,\xi\,c)|_{j,v,w} = |C(t(c))|_{j,v,w}$, we have

$$|B(t(b))|_{j,v,w,\tau} < |C(t(c))|_{j,v,w} \leq \tfrac{1}{2}.$$

But since $\tau < \delta$, $B(t(x))$ is not part of a counterexample. So the left side equals $|B(t(c))|_{j,v,w\tau}$, and so

$$|B(t(c))|_{j,v,w,\tau} < |C(t(c))|_{j,v,w} \leq \tfrac{1}{2}.$$

But then monotonicity yields the following relation among the fixed point values:

$$|B(t(c))|_{j,v,w} < |C(t(c))|_{j,v,w} \leq \tfrac{1}{2}.$$

So by the fixed point condition,

$$|t(c)\,\xi\,b|_{j,v,w} < |t(c)\,\xi\,c|_{j,v,w} \leq \tfrac{1}{2}.$$

But this violates the $\langle b, c\rangle$-goodness of $\langle j, v\rangle$. ∎

Continuing the proof of the theorem, we generalize the Lemma: we show that no parameterized 1-formula *of any form* can have height δ. This is by induction on complexity. Any atomic parameterized 1-formula of the language, say with x the free variable if there is one, is either

(A) an atomic formula of the ground language
(B) of form $Property(t(x))$ or $Proposition(t(x))$
(C) of form $t(x)\,\xi\,N$ where N is a name in the ground language
(D) of form $t(x)\,\xi\,\lambda y D(x,y)$ or $t(x)\,\xi\,\lambda D(x)$
(E) of form $t(x)\,\xi\,x$.

(We allow that the parameterized terms and formulas not contain free the variables shown.) Clearly no atomic formulas of form (A)-(C) can be (components of) counterexamples of any height. By the Lemma, none of form (E) can be one of height δ. So it remains only to show that none of form (D) can have height δ. Again we need to divide up into four cases, but a typical one would be that for some v,

$$|t(b)\, \xi\, \lambda y D(b,y)|_{j,v,w,\delta} < |t(c)\, \xi\, \lambda y D(c,y)|_{j,v,w} \le \tfrac{1}{2}.$$

But again, δ can't be $\frac{1}{2}$ or a limit, so letting τ be its immediate predecessor, we have

$$|D(b,t(b))|_{j,v,w,\tau} < |D(c,t(c))|_{j,v,w} \le \tfrac{1}{2},$$

and so we have a counterexample of height less than δ, which is a contradiction.

We've shown that no atomic parameterized formula can be (a component of) a counterexample. And it is routine to extend this to non-atomic, which we do by induction on complexity. The clauses for the ordinary connectives and quantifiers are routine, and for the $\rightarrow$ and $\triangleright$ clauses we use (for the only time) the assumption that j and v are $\langle b,c\rangle$-congruent. ■ ■

Now let's apply the theorem. Condition (VI) says that if b and c are R-equivalent then every $\langle j,v\rangle$ pair that occurs (as a Kripkean fixed point) anywhere in the overall construction is $\langle b,c\rangle$-good. So by the theorem, any such $\langle j,v\rangle$ that is $\langle b,c\rangle$-congruent is strongly $\langle b,c\rangle$-congruent. So defining *R-congruence* as $\langle b,c\rangle$-congruence for all R-equivalent $\langle b,c\rangle$, and similarly for *strong R-congruence*, we have:

(#): Any pair of R-congruent valuations that occurs as a Kripke fixed point anywhere in the overall construction is strongly R-congruent.

And using (#), we can easily show by going through the construction that *every pair of valuations in the construction is strongly R-congruent.*[42]

Given this, it is a routine matter to "contract the model by R-equivalence": to replace the model by a reduced model where R-equivalent abstracts are identified. The value of every parameterized sentence is unaffected by the contraction, so naivety is preserved.

It might seem desirable to weaken (VI) by strengthening the assumption that $\langle j,v\rangle$ occur somewhere in the construction to the assumption that it be a recurring pair, in the sense that v is recurring in the mid-level construction for j and j is either recurring in the revision-theoretic macro-construction or a stage along the way to the fixed point in the Brady. But the positive result couldn't hold for (I)-(V) plus this weakened version of (VI): Øgaard's proof rules it out. The argument for the positive result would fail since we could then prove only a weaker form of (#):

($\#_w$): Any pair of R-congruent valuations that occurs as a *recurring* Kripke fixed point in the overall construction is strongly R-congruent.

---

[42]Pick any R-congruent j. The mid-level construction from j starts out from a valuation $v_{j,0}$ that assigns the same value to every $\rightarrow$-conditional, so it is trivially R-congruent; so by (#), $\langle j, v_{j,0}\rangle$ is strongly R-congruent. By the rules for the construction, this guarantees the R-congruence of the next member $v_{j,1}$ of the mid-level construction from j. Continuing in this way, we establish

($): Every $\rightarrow$-valuation $v_{j,\alpha}$ that occurs in the mid-level construction from a R-congruent j is such that $\langle j, v_{j,\alpha}\rangle$ is strongly R-congruent.

Proceed analogously in the macro-construction: the starting j_0 assigns the same value to all $\triangleright$-conditionals, so is trivially R-congruent, so ($) applies to it; and the rules of the macro-construction then guarantee that the next j_1 is R-congruent, and so on throughout the macro-construction.

And that doesn't suffice for the inductive proof of

(%): Every recurring pair of valuations in the construction is strongly R-congruent.

For in that proof (sketched in note 42), it was essential to start the induction from a pair $\langle j, v\rangle$ whose components are obviously R-congruent.

The upshot is that laws like $(\top \to \bot) \to \bot$, though in some sense valid, don't have the kind of "uniform validity" that is required for predicates coextensive by virtue of them to be sensibly regarded as expressing the same property. I don't see that that should be terribly upsetting.

7. Conclusion

I began by discussing the advantages of a naive theory of properties and propositions, and the paper has looked at several issues for such a theory, including (i) how it treats restricted quantification (and a conditional $\to$ used to define it), (ii) how it treats an ordinary conditional $\triangleright$, (iii) how it makes these interact so as to achieve the laws of restricted quantification we might expect, and (iv) what kinds of identity conditions for properties it permits.

A novelty in my treatment of (i) is the use of a continuum valued framework which generalizes Łukasiewicz continuum-valued semantics, and allows paradoxes that are treated there to be treated in essentially the same way while also providing a natural generalization that handles the paradoxes that Łukasiewicz semantics can't handle.

Under (ii) I argued that several methods do well: once one has decided how to treat an ordinary conditional without consideration of the paradoxes, any of a number of methods can be used to generalize it to handle the paradoxes peculiar to that conditional. This includes a simple method of Ross Brady's, provided it is given a novel tweak.

Under (iii) I argued that reasonable restricted quantifier laws don't depend too much on the details of the ordinary conditional used in stating the laws, they are largely settled by the laws of the $\to$ together with the "fiber bundle architecture" by which the treatment of the two conditionals is combined.

Under (iv) I adapted work in a joint paper with Harvey Lederman and Tore Fjetland Øgaard to show that there are some limitations on how coarse-grained one can take properties to be in a naive theory, but also to show that those limitations aren't that severe and that there are easy ways to attain fairly coarse-grained properties.

References

[1] Beall, JC, Ross Brady, Allen Hazen, Graham Priest and Greg Restall 2006. "Relevant Restricted Quantification." *Journal of Philosophical Logic* 54: 87-103.

[2] Brady, Ross 1983. "The Simple Consistency of a Set Theory Based on the Logic CSQ". *Notre Dame Journal of Formal Logic* 24: 431-49.

[3] Burgess, John 1981. "Quick Completeness Proofs for Some Logics of Conditionals". *Notre Dame Journal of Formal Logic* 22: 76-84.

[4] Field, Hartry 2008. *Saving Truth from Paradox.*

[5] ———— 2010. "Reply to Greg Restall". *Philosophical Studies* 147: 460-67.

[6] ———— 2014. "Naive Truth and Restricted Quantification". *Review of Symbolic Logic* 7: 147-91.

[7] ———— 2016. "Indicative Conditionals, Restricted Quantification and Naive Truth". *Review of Symbolic Logic* 9: 181-208.

[8] Field, Hartry, Harvey Lederman and Tore Fjetland Øgaard forthcoming. "Prospects for a Naive Theory of Classes". *Notre Dame Journal of Formal Logic.*
[9] Gupta, Anil, and Nuel Belnap 1993. *The Revision Theory of Truth.* MIT Press.
[10] Hajek, Petr, Jeff Paris and John Sheperdson 2000. "The Liar Paradox and Fuzzy Logic", *Journal of Symbolic Logic* 65: 339-346.
[11] Kripke, Saul 1975. "Outline of a Theory of Truth." *Journal of Philosophy* 72: 690-716.
[12] Lewis, David 1981. "Ordering Semantics and Premise Semantics for Counterfactuals". *Journal of Philosophical Logic* 10: 217-34.
[13] Priest, Graham 2006. *In Contradiction*, second edition. Oxford University Press.
[14] ——————— 2008. *Introduction to Nonclassical Logic*, second edition. Cambridge University Press.
[15] Restall, Greg 1992. "Arithmetic and Truth in Łukasiewicz's Infinitely Valued Logic". *Logique et Analyse* 139-40: 303-12.
[16] ——————- 2010. "What are we to accept, and what are we to reject, in saving truth from paradox?" *Philosophical Studies* 147: 433-43.
[17] Rossi, Lorenzo forthcoming. "Graphs, Truth, and Conditionals." *Review of Symbolic Logic.*
[18] Schechter, Eric 2005. *Classical and Nonclassical Logics.* Princeton University Press.
[19] Stalnaker, Robert 1968. "A Theory of Conditionals." In N. Rescher, ed., *Studies in Logical Theory*, Blackwell Publishers.
[20] Standefer, Shawn 2015. "On Artifacts and Truth-preservation". *Australasian Journal of Logic* 12(3): 135-58.
[21] Visser, Albert 1984. "Four-Valued Semantics and the Liar". *Journal of Philosophical Logic* 13: 181-212.
[22] Yablo, Stephen 2003. "New Grounds for Naive Truth Theory". In JC Beall, ed., Liars and Heaps, *Oxford University Press.*