Abstract

We document a novel bidding pattern observed in procurement auctions from Japan: winning bids tend to be isolated. We prove that in a general class of models, missing bids robustly indicate non-competitive behavior. In addition, we provide evidence that missing bids coincide tightly with known cartel activity. Finally, we show that missing bids are consistent with efficient collusion in environments where it is difficult for bidders to coordinate on precise bids.

KEYWORDS: missing bids, collusion, isolated winner strategies, cartel enforcement, procurement.
1 Introduction

This paper documents a novel bidding pattern found in multiple datasets describing public procurement auctions held in Japan: the density of bids just above the winning bid is very low. Put differently, winning bids tend to be isolated. We show that these missing bids indicate non-competitive behavior under a general class of asymmetric information models. Indeed, this missing mass of bids makes it a profitable stage-game deviation for bidders to increase their bids. Motivated by these findings, we develop structural tools that allow us to quantify the extent of non-competitive behavior in the data. Finally, we propose an explanation for why this bidding pattern arises, and discuss what it suggests about the challenges of sustaining collusion.

Our data comes from two separate datasets of public procurement auctions taking place in Japan. Our first data-set, already analyzed by Kawai and Nakabayashi (2014), assembles roughly 90,000 national-level auctions for public work projects taking place between 2001 and 2006. Our second dataset, previously studied by Chassang and Ortner (2016), assembles approximately 1,500 city-level auctions for public works projects taking place between 2007 and 2014. In both cases, we are interested in the distribution bidders’ margins of victory/defeat. In other terms, for every (bidder, auction) pair, we are interested in the difference $\Delta$ between the bidder’s own bid and the most competitive bid among this bidder’s opponents, normalized by the reserve price. When $\Delta < 0$, the bidder won the auction. When $\Delta > 0$ the bidder lost. The finding motivating this paper is summarized by Figure 1, which plots the distribution of margins of victory $\Delta$ in the sample of national-level auctions. The distribution follows a truncated bell curve, except that there is a visible gap in the distribution at $\Delta = 0$.

Our primary goal for this paper is to clarify the sense in which this gap is suspicious. For this purpose, we consider a fairly general model of repeated play in first-price procurement auctions. A group of firms repeatedly participates in first-price procurement auctions. Firms’
costs can be serially correlated over time, and we allow for general asymmetric information. We are interested in characterizing the extent to which players’ behavior can be rationalized as competitive, in the sense of being stage-game optimal at the player level.

Our first set of results identifies conditions that any dataset arising from a competitive equilibrium must satisfy. In any competitive equilibrium, firms must not find it profitable to increase their bids. We show that this incentive constraint implies that the elasticity of firms’ counterfactual demand (i.e., the probability of winning an auction at any given bid) must bounded above by -1. This condition is not satisfied in our data: since winning bids are isolated, the elasticity of counterfactual demand is approximately zero for some industrial sectors in our data.

Our second set of results builds on these observations to quantify the extent of non-competitive behavior in the data. We propose a new measure of collusion corresponding to the smallest share of the data that must be excluded, in order to rationalize the remaining
data as competitive. We show that this program is computationally tractable and delineate how different patterns of demand map into restrictions on the set of possibly competitive histories.

Finally, we propose a tentative explanation for missing bids, and why they could plausibly arise as an implication of collusive behavior. This is not entirely obvious because missing bids are not rationalized by standard models of tacit collusion (i.e., Rotemberg and Saloner (1986), Athey and Bagwell (2001, 2008)). In these models, the cartel’s main concern is to incentivize losers not to undercut the winning bid. The behavior of designated winners is stage game optimal. We show that missing bids arise as an optimal response to noise. Keeping the designated winner’s bid isolated ensures that small trembles in play do not cause severe misallocations.

Our paper relates primarily to the literature on cartel detection.\(^1\) Porter and Zona (1993, 1999) show that suspected cartel members use different bidding strategies than non-cartel members. Bajari and Ye (2003) design a test of collusion based on excess correlation across bids. Porter (1983), Ellison (1994) and Chassang and Ortner (2016) build on classic theories of repeated games (i.e., Green and Porter (1984), Rotemberg and Saloner (1986)) to detect collusion. Conley and Decarolis (2016) propose a test to detect collusive bidders competing in average-price auctions. Kawai and Nakabayashi (2014) analyze auctions with re-bidding, and exploit correlation patterns in bids across stages to detect collusion. We provide a new test of collusion that is robust to arbitrary information structures, and that allows us to quantify the extent of collusion in the data.

Our paper also relates to a set of papers studying the internal organization of cartels. Asker (2010) studies stamp auctions, and analyses the effect of a particular collusive scheme on non-cartel bidders and sellers. Pesendorfer (2000) studies the bidding patterns for school milk contracts and compares the collusive schemes used by strong cartels and weak cartels (i.e., cartels that used transfers and cartels that didn’t). Clark and Houde (2013) document

\(^{1}\)See Harrington (2008) for a recent survey of this literature.
the collusive strategies used by the retail gasoline cartel in Quebec. We add to this literature by documenting a novel bidding pattern, and argue that this bidding behavior reflects some of the frictions that cartels face.

The paper is structured as follows. Section 2 describes our data and documents the bidding patterns that motivate our paper. Section 3 introduces our framework. Section 4 presents our main theoretical findings: we show that missing bids are inconsistent with competition, and derive bounds on the maximum share of competitive histories consistent with the data. Section 5 illustrates our approach with data. Section 6 proposes an interpretation of missing bids as a feature of optimal collusive behavior in noisy environments. Proofs are collected in Appendix A unless mentioned otherwise.

2 Motivating Facts

We draw on two sets of data. The first dataset, analyzed in Kawai and Nakabayashi (2014), consists of roughly 90,000 auctions held between 2001 and 2006 by the Ministry of Land, Infrastructure, Transport and Tourism in Japan (the Ministry). The auctions are first-price auctions with secret reserve price, and re-bidding in case there is no successful winner. The auctions involve construction projects, the median winning bid is USD 600K, and the median participation is 10. Our second dataset, analyzed in Chassang and Ortner (2016), consists of roughly 1,500 auctions held between 2007 and 2014 by the city of Tsuchiura in the Ibaraki prefecture. Projects are allocated using a standard first-price auction with public reserve price. The median winning bid is USD 130K, and the median participation is 4. In both cases, the bids of all participants are publicly revealed after the auctions, and reported in our data.

For any given firm, we investigate the distribution of

$$\Delta = \frac{\text{own bid} - \text{most competitive bid}}{\text{reserve price}}.$$

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The value $\Delta$ represents the margin by which a bidder wins or lose an auction. If $\Delta < 0$ the bidder won, if $\Delta > 0$ he won. At $\Delta = -0$, the bidder barely won.

The left panel of Figure 2 plots the distribution of bid differences $\Delta$ for a large firm in the sample of auctions held by the Ministry. The right panel aggregates bid differences over the sample firms in the data. The mass of missing bids around a difference of 0 is starkly visible. This pattern is not limited to a particular firm and remains clearly noticeable when aggregating over all auctions in our sample.

Figure 2 presents plots the distribution of $\Delta$ for auctions held in Tsuchiura. The left panel uses all the bids in the sample. Again, we see a significant mass of missing bids around zero. The right panel shows that the pattern all but disappears when we exclude winning bids from the analysis.

Our objective in this paper is to: 1) formalize why this pattern is suspicious; 2) delineate what it implies about bidding behavior and the competitiveness of auctions in our sample; 3) propose a possible explanation for why this behavior arises as a feature of optimal bidding. To do so we use a model of repeated auctions.

\footnote{Note that the distribution of normalized bid-differences is skewed to the right since the most competitive alternative bid is a minimum over other bidders’ bids.}
3 Framework

We consider a dynamic setting in which, at each period \( t \in \mathbb{N} \), a buyer needs to procure a single project. The auction format is a first-price auction with reserve price \( r \), which we normalize to \( r = 1 \).

In each period \( t \in \mathbb{N} \), a set \( \hat{N}_t \subset N \) of bidders is able to participate in the auction, where \( N \) is the overall set of bidders. We think of this set of participating firms as those eligible to produce in the current period.\(^3\) The sets of eligible bidders can vary over time.

Realized costs of production for eligible bidders \( i \in \hat{N}_t \) are denoted by \( c_t = (c_{i,t})_{i \in \hat{N}_t} \). Each bidder \( i \in \hat{N}_t \) submits a bid \( b_{i,t} \). Profiles of bids are denoted by \( b_t = (b_{i,t})_{i \in \hat{N}_t} \). We let \( b_{-i,t} \equiv (b_{j,t})_{j \neq i} \) denote bids from firms other than firm \( i \), and define \( \land b_{-i,t} \equiv \min_{j \neq i} b_{j,t} \) to be the lowest bid among \( i \)'s opponents at time \( t \). The procurement contract is allocated to the bidder submitting the lowest bid at a price equal to her bid.

In the case of ties, we follow Athey and Bagwell (2001) and let the bidders jointly determine the allocation. This simplifies the analysis but requires some formalism (which can be skipped at moderate cost to understanding). We allow bidders to simultaneously pick numbers \( \gamma_t = (\gamma_{i,t})_{i \in \hat{N}_t} \) with \( \gamma_{i,t} \in [0, 1] \) for all \( i, t \). When lowest bids are tied, the allocation

\(^3\)See Chassang and Ortner (2016) for a treatment of endogenous participation by cartel members.
to a lowest bidder \( i \) is
\[
x_{i,t} = \frac{\gamma_{i,t}}{\sum_{j \in \hat{N}_t \text{ s.t. } b_{j,t} = \min_k b_{k,t}} \gamma_{j,t}}.
\]

Participants discount future payoffs using common discount factor \( \delta < 1 \). Bids are publicly revealed at the end of each period.

**Costs.** We allow for costs that are serially correlated over time, and that may be correlated across firms within each auction. Denoting by \( \langle \cdot, \cdot \rangle \) the usual dot-product we assume that costs take the form
\[
c_{i,t} = \langle \alpha_i, \theta_t \rangle + \varepsilon_{i,t} > 0
\]
where
- parameters \( \alpha_i \in \mathbb{R}^k \) are fixed over time;
- \( \theta_t \in \mathbb{R}^k \) may be unknown to the bidders at the time of bidding, but is revealed to bidders at the end of period \( t \); we assume that \( \theta_t \) follows a Markov chain;
- \( \varepsilon_{i,t} \) is i.i.d. with mean zero conditional on \( \theta_t \).

In period \( t \), bidder \( i \) obtains profits
\[
\pi_{i,t} = x_{i,t} \times (b_{i,t} - c_{i,t}).
\]

Note that costs include both the direct costs of production and the opportunity cost of backlog.

The sets \( \hat{N}_t \) of bidders are independent across time conditional on \( \theta_t \), i.e.
\[
\hat{N}_t | \theta_{t-1}, \hat{N}_{t-1}, \hat{N}_{t-2} \ldots \sim \hat{N}_t | \theta_{t-1}.
\]

**Information.** In each period \( t \), bidder \( i \) gets a signal \( z_{i,t} \) that is conditionally i.i.d. given \((\theta_t, (c_{j,t})_{j \in \hat{N}_t})\). This allows our model to nest many informational environments, including
asymmetric information private value auctions, common value auctions, as well as complete information. Bids $b_t$ are observable at the end of the auction.

**Transfers.** Bidders are able to make positive transfers from one to the other at the end of each period. A transfer from $i$ to $j$ is denoted by $T_{i ightarrow j, t} \geq 0$. Transfers are costly, and we denote by $K\left(\sum_{j \neq i} T_{i \rightarrow j, t}\right)$ the cost to player $i$ of the transfers she makes. We assume that $K$ is positive, increasing and convex. Altogether, flow realized payoffs to player $i$ in period $t$ take the form

$$u_{i, t} = \pi_{i, t} + \sum_{j \neq i} T_{j \rightarrow i, t} - K\left(\sum_{j \neq i} T_{i \rightarrow j, t}\right).$$

**Solution Concepts.** The public history $h_t$ at period $t$ takes the form

$$h_t = (\theta_{s-1}, b_{s-1}, T_{s-1})_{s \leq t},$$

where $T_s$ are the transfers made in period $s$. Our solution concept is perfect public Bayesian equilibrium $(\sigma, \mu)$ (Athey and Bagwell (2008)), with strategies

$$\sigma_i : h_t \mapsto (b_{i,t}(z_{i,t}), (T_{i \rightarrow j,t}(z_{i,t}, b_t))_{j \neq i}),$$

where bids $b_{i,t}(z_{i,t}) \in \Delta([0, r])$ and transfers $(T_{i \rightarrow j,t}(z_{i,t}, b_t))_{j \neq i} \in \Delta(\mathbb{R}^{n-1})$ depend on the public history and on the information available at the time of decision making. We let $\mathcal{H}$ denote the set of all public histories.

We emphasize the class of competitive equilibria, or in this case Markov perfect equilibria (Maskin and Tirole, 2001). In a competitive equilibrium, players condition their play only on payoff relevant parameters.

**Definition 1** (competitive strategy). We say that $(\sigma, \mu)$ is competitive (or Markov perfect) if and only if $\forall i \in N$ and $\forall h_t \in \mathcal{H}$, $\sigma_i(h_t, z_{i,t})$ depends only on $(\theta_{t-1}, z_{i,t})$. 

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We say that a strategy profile \((\sigma, \mu)\) is a competitive equilibrium if it is a perfect public Bayesian equilibrium in competitive strategies.

We note that in a competitive equilibrium, firms must be playing a stage-game Nash equilibrium at every period; that is, firms must play a static best-reply to the actions of their opponents. Generally, an equilibrium may include periods in which (a subset of) firms collude and periods in which firms compete. This leads us to define competitive histories.

**Competitive histories.** Fix a perfect public Bayesian equilibrium \((\sigma, \mu)\). Given a public history \(h_t \in \mathcal{H}\) and firm \(i\)'s private signal \(z_{i,t}\), let \(h_{i,t} = (h_t, z_{i,t})\). Note that, under perfect public Bayesian equilibrium \((\sigma, \mu)\), firm \(i\)'s strategy at time \(t\) depends on \(h_{i,t}\).

**Definition 2** (competitive histories). Fix an equilibrium \((\sigma, \mu)\) and a history \(h_{i,t} = (h_t, z_{i,t})\). We say that \((\sigma, \mu)\) is competitive at \(h_{i,t}\) if play at \(h_{i,t}\) is stage-game optimal for firm \(i\).

### 4 Inference

In this section, we show how to exploit equilibrium conditions at different histories to obtain bounds on the share of competitive histories. The first step is to obtain aggregates of counterfactual demand that can be estimated from data, even though the players’ residual demands can vary with the history.

#### 4.1 Counterfactual demand

Fix a perfect public Bayesian equilibrium \((\sigma, \mu)\). For all public histories \(h_{i,t} = (h_t, z_{i,t})\) and all bids \(b' \in [0, r]\), player \(i\)'s counterfactual demand at \(h_{i,t}\) is

\[
D_i(b'|h_{i,t}) \equiv \text{prob}_{\sigma,\mu}(\land b_{-i,t} > b'|h_{i,t}).
\]
For any finite set of histories $H = \{(h_t, z_{i,t})\} = \{h_{i,t}\}$, and any scalar $\rho \in (-1, \infty)$, define

$$
\overline{D}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} D_i((1 + \rho)b_{i,t}|h_{i,t})
$$

to be the average counterfactual demand for histories in $H$, and

$$
\hat{D}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} 1_{b_{-i,t}>(1+\rho)b_{i,t}}.
$$

**Definition 3.** We say that set $H$ is adapted to the players’ information if and only if the event $h_{i,t} \in H$ is measurable with respect to player $i$’s information at time $t$ prior to bidding.

For instance, the set of auctions for a specific industry with reserve prices above a certain threshold is adapted. In contrast, the set of auctions in which the margin of victory is below a certain level is not.

**Theorem 1.** Consider a sequence of adapted sets $(H_n)_{n\in\mathbb{N}}$ such that $\lim_{n \to \infty} |H_n| = \infty$. Under any perfect public Bayesian equilibrium $(\sigma, \mu)$, with probability 1, $\hat{D}(\rho|H_n) - \overline{D}(\rho|H_n) \to 0$.

In other words, in equilibrium, the sample residual demand conditional on an adapted set of histories converges to the true subjective aggregate conditional demand. This result can be viewed as a weakening of the equilibrium requirement that beliefs be correct. It may fail in settings with sufficiently strong non-common priors.

The ability to legitimately vary the conditioning set $H$ lets us explore the competitiveness of auctions in particular subsettings of interest.

### 4.2 A Test of Non-Competitive Behavior

The pattern of bids illustrated in Figures 1, 2 and 3 is striking. Our first main result shows that its more extreme forms are inconsistent with competitive behavior.
**Proposition 1.** Let \((\sigma, \mu)\) be a competitive equilibrium. Then,

\[
\forall h_i, \quad \frac{\partial \log D_i(b'|h_i)}{\partial \log b'} \bigg|_{b'=b^*_i(h_i)} \leq -1, \quad (2)
\]

\[
\forall H, \quad \frac{\partial \log D(\rho|H)}{\partial \rho} \bigg|_{\rho=0^+} \leq -1. \quad (3)
\]

In other terms, under any non-collusive equilibrium, the elasticity of counterfactual demand must be less than -1 at every history. The data presented in the left panel of Figure 2 contradicts the results in Proposition 1. Note that for every \(i \in N\) and every \(h_i\),

\[
D_i(b'|h_i) = \text{prob}_\sigma(b' - \wedge b_i < 0|h_i) = \text{prob}_\sigma(b' - b_i + \Delta_i < 0|h_i),
\]

where we used \(\Delta_i = \frac{b_i - \wedge b_{-i}}{r} = b_i - \wedge b_i\) (since we normalized \(r = 1\)). Since the density of \(\Delta_i\) at 0 is essentially 0 for some sets of histories in our data, the elasticity of demand is approximately zero as well in these histories.

**Proof.** Consider a competitive equilibrium \((\sigma, \mu)\). Let \(u_i\) denote the flow payoff of player \(i\), and let \(V(h_{i,t}) \equiv \mathbb{E}_{\sigma,\mu} \left( \sum_{s \geq t} \delta^{s-t} u_{i,s} \bigg| h_{i,t} \right)\) denote her discounted expected payoff at history \(h_{i,t} = (h_t, z_{i,t})\).

Let \(b_{i,t} = b\) be the bid that bidder \(i\) places at history \(h_{i,t}\). Since \(b_{i,t} = b\) is an equilibrium bid, it must be that for all bids \(b' > b\),

\[
\mathbb{E}_{\sigma,\mu}\left[ (b - c_{i,t}) 1_{b_{-i,t}>b} + \delta V(h_{i,t+1}) \bigg| h_{i,t}, b_{i,t} = b \right] \\
\geq \mathbb{E}_{\sigma,\mu}\left[ (b' - c_{i,t}) 1_{b_{-i,t}>b'} + \delta V(h_{i,t+1}) \bigg| h_{i,t}, b_{i,t} = b' \right]
\]

Since \((\sigma, \mu)\) is competitive, \(\mathbb{E}_{\sigma,\mu}[V(h_{i,t+1})|h_{i,t}, b_{i,t} = b] = \mathbb{E}_{\sigma,\mu}[V(h_{i,t+1})|h_{i,t}, b_{i,t} = b']\). Hence,
we must have

\[ bD_i(b|h_i,t) - b'D_i(b'|h_i,t) = E_{\sigma,\mu} [b1_{b - i,t > b'} - b'1_{b - i,t > b'} | h_i,t] \]

\[ \geq E_{\sigma,\mu} [c_{i,t}(1_{b - i,t > b} - 1_{b - i,t > b'}) | h_i,t] \geq 0, \quad (4) \]

where the last inequality follows since \( c_{i,t} \geq 0 \). Inequality (4) implies that, for all \( b' > b \),

\[ \frac{\log D_i(b'|h_i) - \log D_i(b|h_i)}{\log b' - \log b} \leq -1. \]

Inequality (2) follows from taking the limit as \( b' \to b \). Inequality (3) follows from summing (4) over histories in \( H \), and performing the same computations.

As the proof highlights, this result exploits the fact that in procurement auctions, zero is a natural lower bound for costs (see inequality (4)). In contrast, for auctions where bidders have a positive value for the good, there is no obvious upper bound to valuations to play that role. One would need to impose an ad hoc upper bound on values to establish similar results.

An implication of Proposition 1 is that, in our data, bidders have a short-term incentive to increase their bids. To keep participants from bidding higher, for every \( \epsilon > 0 \) small, there exists \( \nu > 0 \) and a positive mass of histories \( h_{i,t} = (h_t, z_{i,t}) \) such that,

\[ \delta E_{\sigma,\mu}[V(h_{i,t+1})|h_{i,t}, b_i(h_{i,t})] - \delta E_{\sigma,\mu}[V(h_{i,t+1})|h_{i,t}, b_i(h_{i,t})(1 + \epsilon)] > \nu. \quad (5) \]

In other terms, equilibrium \((\sigma, \mu)\) must give bidders a dynamic incentive not to overcut the winning bid.

Proposition 1 proposes a simple test of whether a dataset \( H \) can be generated by a competitive equilibrium or not. We now refine this test to obtain bounds on the minimum share of non-competitive histories needed to rationalize the data. We begin with a simple
loose bound and then propose a more sophisticated program resulting in tighter bounds.

4.3 A simple bound on the share of competitive histories

Fix a perfect public Bayesian equilibrium \((\sigma, \mu)\) and a finite set of histories \(H\). Let \(H^{\text{comp}} \subset H\) be the set of competitive histories in \(H\), and let \(H^{\text{coll}} = H \setminus H^{\text{comp}}\). Define \(s_{\text{comp}} \equiv \frac{|H^{\text{comp}}|}{|H|}\) to be the fraction of competitive histories in \(H\).

For all histories \(h_{i,t} = (h_t, z_{i,t})\) and all bids \(b' \geq 0\), player \(i\)'s counterfactual revenue at \(h_{i,t}\) is

\[ R_i(b'|h_{i,t}) \equiv b'D_i(b'|h_{i,t}). \]

For any finite set of histories \(H\) and scalar \(\rho \in (-1, \infty)\), define

\[ \overline{R}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} (1 + \rho) b_{i,t} D_i((1 + \rho) b_{i,t}|h_{i,t}) \]

to be the average counterfactual revenue for histories in \(H\). Our next result builds on Proposition 1 to derive a bound on \(s_{\text{comp}}\).

**Proposition 2.** The share \(s_{\text{comp}}\) of competitive auctions is such that

\[ s_{\text{comp}} \leq 1 - \sup_{\rho > 0} \frac{\overline{R}(\rho|H) - \overline{R}(0|H)}{\rho}. \]

**Proof.** For any \(\rho > 0\),

\[ \frac{1}{\rho} [\overline{R}(\rho|H) - \overline{R}(0|H)] = s_{\text{comp}} \frac{1}{\rho} [\overline{R}(\rho|H^{\text{comp}}) - \overline{R}(0|H^{\text{comp}})] \]

\[ + (1 - s_{\text{comp}}) \frac{1}{\rho} [\overline{R}(\rho|H^{\text{coll}}) - \overline{R}(0|H^{\text{coll}})] \]

\[ \leq 1 - s_{\text{comp}}. \]

The last inequality follows from two observations. First, since the elasticity of counterfactual
demand is bounded above by $-1$ for all competitive histories (Proposition 1), it follows that $\bar{R}(\rho|H^{\text{comp}}) - \bar{R}(0|H^{\text{comp}}) \leq 0$. Second,

$$\frac{1}{\rho}[(\rho|H^{\text{coll}}) - \bar{R}(0|H^{\text{coll}})] \leq \frac{1}{\rho}((1 + \rho)\bar{R}(0|H^{\text{coll}}) - R(0|H^{\text{coll}})) = \bar{R}(0|H^{\text{coll}}) \leq r = 1.$$  

In words, if total revenue in histories $H$ increases by more than $\kappa \times \rho$ when bids are uniformly increased by $(1 + \rho)$, the share of competitive auctions in $H$ is bounded above by $1 - \kappa$.

For each $\rho \in (-1, \infty)$, define

$$\hat{R}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} (1 + \rho)b_{i,t}1_{A_{b_{i,t} > (1+\rho)b_{i,t}}}.$$  

Note that $\hat{R}(\rho|H)$ is the sample analog of counterfactual revenue. A result identical to Theorem 1 establishes that $\hat{R}(\rho|H)$ is an unbiased estimate of $\bar{R}(\rho|H)$, whenever set $H$ is adapted. We have the following corollary to Proposition 2.

**Corollary 1.** Fix a set of histories $H$ and a scalar $\rho^* > 0$, and suppose that $s^{\text{comp}} \geq 1 - \kappa$ for some $\kappa > 0$. Then, there exists constants $\alpha > 0$ and $\beta > 0$ such that, with probability at least $1 - \beta \exp(-\alpha|H|)$,

$$\forall \rho \geq \rho^*, \quad \frac{\hat{R}(\rho|H) - \hat{R}(0|H)}{\rho} \leq 2\kappa.$$  

Corollary 1 gives the following statistical test with significance level $1 - \beta \exp(-\alpha|H|)$. Let the null hypothesis be $H_0 = s^{\text{comp}} \geq 1 - \kappa$ for some $\kappa > 0$, and let the alternative hypothesis be $H_1 = s^{\text{comp}} < 1 - \kappa$. We reject the null hypothesis if we can find $\rho \geq \rho^*$ such that $\frac{1}{\rho}[(\rho|H) - \bar{R}(0|H)] > 2\kappa$.  

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4.4 Tight inference

We now seek to establish tight bounds on the set of competitive histories. We expand on Section 4.3 by fully exploiting the empirical content of upwards deviations. In addition we consider the empirical content of downward deviations.

Indeed, although the residual demand in our data is inelastic immediately around winning bids, it is very elastic for large downward deviations. Moderate drops in price (a few percentage points) lead to large increases in the likelihood of winning the contract. This suggests that jointly considering upward and downward deviations will provide a tighter bound on the share of competitive histories than the bound in Proposition 2 and Corollary 1.

For simplicity, we assume that players interact in a private value environment. Besides this, we impose no other restrictions on the information structure.

Take as given an adapted set of histories $H$ and scalars $(\rho_n)_{n=-\pi,...,0,...,\pi}$, with $\rho_n \in (-1, \infty)$ for all $n$, $\rho_0 = 0$ and $\rho_n < \rho_{n'}$ for all $n' > n$. For each history $h_{i,t} \in H$, let $d_{h_{i,t},n} = D_i((1+\rho_n)b_{hi,t}|h_{i,t})$. That is, $(d_{h_{i,t},n})_{n=-\pi,...,\pi}$ is firm $i$’s subjective counterfactual demand at history $h_{i,t}$. For each $n$, define

$$D_n \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} d_{h_{i,t},n} \quad \text{and} \quad \hat{D}_n \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} 1(1+\rho_n)b_{hi,t} < b_{h_{i,t}}.$$ 

Under any perfect Bayesian equilibrium, subjective counterfactual demand at competitive histories must satisfy four types of constraints: feasibility constraints, individual optimality constraints, aggregate consistency constraints, and ad hoc economic plausibility constraints. Formally, for every history $h \in H$ there must exist costs $c_h$ and subjective demands $(d_{h,n})_{n=-\pi,...,\pi}$ satisfying the following conditions

**Feasibility.** Costs and beliefs must be feasible, satisfying

$$\forall h \in H, \ c_h \in [0, b_h]; \quad \forall n, \ d_{h,n} \in [0, 1]; \quad \forall n, n' > n, \ d_{h,n} \geq d_{h,n'}.$$  \(6\)
Individual optimality. Bidding $b_h$ must be optimal, given cost and subjective believes:

$$\forall n, \ (1 + \rho_n)b_h - c_h \leq ((1 + \rho_0)b_h - c_h)d_{h,0} \tag{7}$$

Aggregate consistency. Bidders’ subjective demand must be consistent with aggregate data. Given a tolerance level $T > 0$, aggregate subjective demand at histories $h \in H$ is consistent with the data if and only if

$$\forall n, \ D_n = \frac{1}{|H|} \sum_{h \in H} d_{h,n} \in \left[\hat{D}_n - T, \hat{D}_n + T\right] \tag{8}$$

The aggregate consistency conditions must hold since, by Theorem 1, for all $n$ $\hat{D}_n$ is an unbiased estimator of aggregate counterfactual demand $D_n = \frac{1}{|H|} \sum_{h \in H} d_{h,n}$.

Economic plausibility. In addition to incentive compatibility and aggregate consistency, one may be able to impose plausible ad hoc constraints on the bidder’s economic environment at each history $h$. We focus on two intuitive constraints on the bidder’s costs $c_h$ and interim beliefs $(d_{h,n})$:

$$\frac{b_h}{c_h} \leq 1 + m \tag{9}$$

and

$$\forall n, \ \left|\log \frac{d_{h,n}}{1 - d_{h,n}} - \log \frac{D_n}{1 - D_n}\right| \leq k \tag{10}$$

where $m \in [0, +\infty]$ is a maximum markup, and $k \in [0, +\infty)$ provides an upper bound to the information contained in any signal.\(^4\)

\(^4\)To see why, that that $\log \frac{d_{h,n}}{1 - d_{h,n}} = \log \frac{\text{prob}(Z \mid h)}{\text{prob}(\neg Z \mid h)}$ for $Z$ the event that $\wedge b_{-i} > (1 + \rho_n)b_h$. Hence, $k$ is a bound on the log-likelihood ratio of signals that bidders get. One focal case in which $k = 0$ is that of i.i.d. types.
The following Proposition shows that, if the histories in $H$ are all competitive, then with high probability the conditions above all hold simultaneously.

**Proposition 3.** Consider an economic environment in which conditions (15) and (16) hold. There exists $\alpha > 0$ and $\beta > 0$ such that, for all PBEs $(\sigma, \mu)$ and all adapted sets $H$, whenever $(\sigma, \mu)$ is competitive at histories $h \in H$, then with probability at least $1 - \beta \exp(-\alpha|H|)$, conditions (13), (14) and (8) hold simultaneously.

We define the share of non-competitive histories as the minimum share of histories that must be excluded from the data so that the remaining histories are consistent with competitive play. Formally:

**Definition 4** (share of competitive histories). For any set of histories $H$, we define the maximum share of competitive histories in $H$ as

$$
\hat{s}_{\text{comp}} \equiv \frac{1}{|H|} \max_{p^C = (p^C_h)_{h \in H} \in [0,1]^{|H|}} \sum_{h \in H} p^C_h
$$

such that there exists $((d_{h,n}, c_h)_{h \in H}$ satisfying history-level constraints (13), (14), (15), and modified aggregate constraints and information constraints

$$
\forall n, \quad \frac{1}{\sum_{h \in H} p^C_h} \sum_{h \in H} p^C_h d_{h,n} \in [\hat{D}_n(p^C) - T, \hat{D}_n(p^C) + T] \quad (8')
$$

$$
\forall n, \quad \left| \log \frac{d_{h,n}}{1 - d_{h,n}} - \log \frac{\hat{D}_n(p^C)}{1 - \hat{D}_n(p^C)} \right| \leq k \quad (16')
$$

where

$$
\forall n, \quad \hat{D}_n(p^C) \equiv \frac{1}{\sum_{h \in H} p^C_h} \sum_{h \in H} p^C_h 1_{(1+\rho_n)b_h < \wedge b_{-1,h}}
$$

Note that Program (11) allows us to discard fractions $p_h \in [0,1]$ of each history $h \in H$. As the following result shows, this convexification of the problem implies that $\hat{s}_{\text{comp}}$ is an
upper bound on the true share \( s_{\text{comp}} \) of competitive histories in \( H \).

**Corollary 2.** Consider a public perfect Bayesian equilibrium \((\sigma, \mu)\) and an economic environment in which conditions (15) and (16) hold. Let \( H \) be an adapted set of histories such that a share \( s_{\text{comp}} \in (0, 1] \) is competitive. Then, there exists \( \alpha > 0 \) and \( \beta > 0 \) such that, with probability at least \( 1 - \beta \exp(-\alpha s_{\text{comp}} |H|) \), \( \hat{s}_{\text{comp}} \geq s_{\text{comp}} \).

The proof of Corollary 2 shows that constants \( \alpha \) and \( \beta \) are equal to \( T^2 / 2 \) and \( 2(1 + n + \bar{n}) \), respectively. Corollary 2 can be used to derive the following statistical test. Let \( H_0 = s_{\text{comp}} \geq s \) for some \( s \in (0, 1] \), and let \( H_1 = s_{\text{comp}} < s \). Pick a significance level \( \alpha \), and let \( T \) be the tolerance level such that

\[
a = 1 - \beta \exp(-\alpha s |H|) = 1 - 2(1 + n + \bar{n}) \exp \left( -\alpha \frac{T^2}{2} |H| \right),
\]

where we used \( \alpha = T^2 / 2 \) and \( \beta = 2(1 + n + \bar{n}) \). We then reject the null hypothesis if \( \hat{s}_{\text{comp}} < s \).

**A relaxed program.** A difficulty with Problem (11) is that the optimization variable \( p^C \) belongs to \([0,1]^{|H|}\) and the set of constraints is non-convex, making it computationally intractable. We now propose a convex relaxation that is more amenable to computation.

For each history \( h \in H \), let

\[
(y_{h,n})_{n=-\bar{n},...,-1} \equiv (1(1+\rho_n)b_{h,\land b_{-1,h}})_{n=-\bar{n},...,-1}.
\]

Vector \( y_h \) records the bidding outcomes of each history \( h \), and can take values in \( Y \equiv \{(0,0,...,0), (1,0,...,0), ..., (1,1,...,1)\} \). It turns out that \( ((y_{h,n})_{h \in H} \) is a sufficient statistic of data for Problem (11). As we show now, this allows us to consider solutions \( p^C = (p^C_h)_{h \in H} \)
such that \( p_h^C = q^C(y) \) for all \( h \in H \) with \( y_h = y \). Indeed, note that for any \( p^C = (p_h^C)_{h \in H} \),

\[
(\hat{D}_n(p^C))_{n = -\infty, \ldots, \pi} = \frac{1}{\sum_h p_h^C} \sum_{y \in Y} \left( y \times \sum_{h \mid y_h = y} p_h^C \right)
= \frac{1}{|H|} \sum_{y \in Y} y \times |\{h \mid y_h = y\}| q^C(y)
\]

for \( q^C(y) = \frac{\sum_{h \mid y_h = y} p_h^C}{|\{h \mid y_h = y\}|} \). Looking for solutions of the form \( q : Y \rightarrow [0, 1] \) makes Problem (11) significantly easier in terms of computation.

Let \( Z \) be the set of beliefs \( z_h = (d_{h,n})_{n = -\infty, \ldots, \pi} \) such that there exists a cost \( c_h \in [\frac{1}{1+m} b_h, b_h] \) satisfying (13) and (14). Let

\[
A(p^C) = \Pi_n[\hat{D}_n(p^C) - T, \hat{D}_n(p^C) + T]
\]

denote the set of aggregate constraints. Finally, for any \( D_n \), define

\[
\overline{B}(D_n, k) \equiv \frac{D_n \exp(-k)}{1 + \frac{D_n}{1-D_n} \exp(-k)} \quad \text{and} \quad \underline{B}(D_n, k) \equiv \frac{D_n \exp(k)}{1 + \frac{D_n}{1-D_n} \exp(k)},
\]

and let

\[
I(p^C) = \Pi_n[\underline{B}(\hat{D}_n(p^C), k), \overline{B}(\hat{D}_n(p^C), k)]
\]

denote the information constraints.

For any function \( q : Y \rightarrow [0, 1] \) define \( \tilde{s}_{\text{comp}}(q) \equiv \frac{1}{|H|} \sum_{y \in Y} q(y)|\{h \mid y_h = y\}| \), and let \( p(q) \in [0, 1]^{|H|} \) be such that, for all \( h \in H \), \( p(q)_h = q(y_h) \). For any set \( C \), let \( \text{Conv}(C) \) denote the convex hull of \( C \).

**Proposition 4.** We have that

\[
\hat{s}_{\text{comp}} \leq \max_{q : Y \rightarrow [0, 1]} \{ \tilde{s}_{\text{comp}}(q) \mid \text{Conv}[Z \cap I(p(q))] \cap A(p(q)) \neq \emptyset \}.
\]
5 Case Studies

This Section takes the results of Section 4 to data. We start by using our results to analyze two collusion cases of firms participating in auctions in our national data that were implicated by the Japanese Fair Trade Commission (JFTC). The two cases are: (i) prestressed concrete providers, and (ii) firms installing electric traffic signs. It is worth highlighting that firms in case (ii) admitted that they were violating anti-trust laws soon after the JFTC started investigating them. In contrast, firms in case (i) denied the cases against them and the case went to trial. As it turns out, firms in case (i) continued colluding for some time after the JFTC launched its investigation.

Figure 4 shows the bidding behavior of implicated firms. The left panels plot the distribution of $\Delta$ for prestressed concrete providers, before and after the JFTC started its investigation. Consistent with the fact that firms in this market continued colluding after the investigation, the after-period features missing bids around bid difference $\Delta = 0$, but to a lesser extent than the before-period.

The panels on the right plot the same distributions for firms installing electric traffic signs. Consistent with Proposition 1, the distribution of $\Delta$ has missing mass around zero only during the non-competitive period.

Next, we apply our results of Section 4.4 to these two markets. We proceed as follows. First, we fix two downward deviations: $\rho_{-2} = \rho^- \in (-1, 0)$ and $\rho_{-1} = \lim_{\epsilon \to 0} \epsilon = 0^-$, and one upward deviation $\rho_1 = \rho^+ \in (0, \infty)$. Second, we pick mark-up and information constraint parameters $m \geq 0$ and $k \geq 0$. Third, for each possible null hypothesis $H_0 = s_{\text{comp}} \geq s$ with $s \in (0, 1]$, we pick tolerance level $T(s)$ such that $1 - \beta \exp (-\alpha s |H|) = 0.05$, where $\alpha > 0$ and $\beta > 0$ are the constants in the statement of Corollary 2. Our estimate of the share of competitive histories in the data is the largest $s \in (0, 1]$ such that $\hat{s}_{\text{comp}} \geq s$ given tolerance.

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6 Recall that $\alpha = T^2 / 2$ and $\beta = (1 + n + \pi)$. 

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Figure 4: Distribution of bid-difference $\Delta$. Left-panel: prestressed concrete. Right-panel: traffic signs.

level $T(s)$.

Figure 5 plots our estimate of the share of competitive histories in each of these markets, before and after prosecution. Auctions in both markets became more competitive in the period post-investigation. In the case of traffic signs, our estimates suggest that collusion stopped completely after the investigation. In contrast, our estimates suggest that there was still some collusion after the investigation in the market for prestressed concrete.

Lastly, we look at the auctions run by the city of Tsuchiura. Chassang and Ortner (2016) find evidence consistent with collusion in these auctions. Moreover, they show that the extent of collusion fell after October 2009, when the city changed its procurement auction format and introduced price floors.$^7$

$^7$In October 2009, the city of Tsuchiura switched from a standard first-price auction format to a first-price auction with a minimum price; i.e., an auction in which bids below the minimum price are discarded.
Figure 5: Share of competitive histories. Parameters: $\rho^- = -0.01$, $\rho^+ = 0.001$ and $m = 1.5$.

Figure 6 plots the distribution of bid differences $\Delta$ for these auctions. The left-panel plots the distribution for auctions taking place before October 2009, and the right-panel plots the same distribution for auctions taking place after October 2009. Consistent with Proposition 1 and with the evidence in Chassang and Ortner (2016), auctions before the policy change feature a more pronounced “gap” in the distribution of bid differences around $\Delta = 0$.

Figure 6: Distribution of bid-difference $\Delta$ in Tsuchiura.

Figure 7 plots our estimates of the share of competitive histories for auctions run in
Tsuchiura, before and after the change in the auction format. Our estimates are broadly consistent with the idea that the extent of collusion fell after the city introduced price floors into the auctions.

![Figure 7: Share of competitive histories. Parameters: $\rho^- = -0.01$, $\rho^+ = 0.001$ and $m = 1.5$.](image)

6 Interpreting Missing Bids

This section has two objectives. First, we want to highlight that the bidding behavior we observe in our data is not easily explained by standard models of collusion. Second, we put forward an explanation for the bidding patterns we observe in these two datasets.

**Workhorse model.** We specialize the model in Section 3 as follows. We assume: (i) costs are i.i.d. across firms and across periods, (ii) cost realizations are publicly observed by all firms, and (iii) utility is perfectly transferable.

We denote by $\Sigma$ the set of Subgame Perfect Equilibria of this game. For any $\sigma \in \Sigma$ and any history $h_t$, let

$$V(\sigma, h_t) = \mathbb{E}_\sigma \left[ \sum_{i \in N} \sum_{s \geq 0} \delta^s x_{i,t+s} (b_{i,t+s} - c_i) | h_t \right]$$
denote the total surplus generated by $\sigma$ at history $h_t$. Define
\[ \overline{V} \equiv \max_{\sigma \in \Sigma} V(\sigma, h_0) \]
to be the highest surplus sustainable in equilibrium.

For any cost realization $c = (c_i)_{i \in N}$, we denote by $x^*(c) = (x^*_i(c))_{i \in N}$ the efficient allocation (i.e., the allocation that assigns the contract to the lowest cost bidder and breaks ties randomly). We denote by $b_{(1)}$ and $b_{(2)}$ the lowest and second lowest bids. The following result, which is proved in Chassang and Ortner (2016), characterizes bidding behavior in any equilibrium that attains $\overline{V}$.

**Proposition 5.** Let $\sigma$ be an equilibrium that attains $\overline{V}$. Then:

(i) equilibrium $\sigma$ is stationary on-path, and generates surplus $\overline{V}$ at every history.

(ii) for any cost realization $c = (c_i)_{i \in N}$, the lowest cost bidder wins at bid $b^*(c)$ defined by

\[ b^*(c) \equiv \sup \left\{ b \leq r : \sum_{i \in N} (1 - x^*_i(c)) [b - c_i]^+ \leq \delta \overline{V} \right\}. \]

(iii) there is no money left on the table under equilibrium $\sigma$: $b_{(2)} - b_{(1)} \approx 0$ at all periods.

By Proposition 6, the bidding patterns in our data cannot be rationalized by optimal collusion. In an optimal equilibrium, firms never use strategies under which the winning bidder has a short-run incentive to overcut the winning bid. Indeed, this would mean that firms have to spend continuation surplus to provide incentives to the winner not to bid higher. This creates efficiency losses relative to equilibria in which the winner is given incentives not to overcut by having the second lowest bid right on top of the winning bid.

As a result, bids will be clustered in an optimal collusive equilibria, and the “money left on the table” (i.e., the difference between the winning bid and the second lowest bid) will be negligible. As Figures 1 and 8 show, this is in sharp contrast with the bidding patterns we
observe in our data, under which winning bids are isolated and the money left on the table is significant.

![Figure 8: Distribution of normalized bid difference $b_{(2)} - b_{(1)}$ - national data.](image)

**Missing bids as coordination challenges.** The fact that winning bids are isolated implies that the allocation that this bidding behavior induces is robust to trembles or imprecisions in the communication among cartel members. We now lay out a simple model to illustrate how isolated winning bids may emerge as a response to such imperfections.

Suppose $N = \{1, 2\}$. We continue to assume that cost realizations are publicly observed by all firms, that utility is perfectly transferable, and that costs are drawn i.i.d. across firms and across periods from distribution $F$. Let $\epsilon \sim F_\epsilon$, where $\text{supp} F_\epsilon = [-1, 1]$ and $f_\epsilon(x) \equiv F'_\epsilon(x) \in (\underline{f}, \overline{f})$ for all $x \in [-1, 1]$ and some $0 < \underline{f} < \overline{f}$. We further assume that distribution $F_\epsilon$ is symmetric around zero. Firms choose intended bids $(\hat{b}_1, \hat{b}_2) \in [0, 1 - \gamma]^2$. 

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With probability 1/2 firm 1 trembles and realized bids are \((\hat{b}_1 + \gamma \epsilon, \hat{b}_2)\), and with probability 1/2 firm 2 trembles and realized bids are \((\hat{b}_1, \hat{b}_2 + \gamma \epsilon)\). For simplicity, we assume that distribution \(F\) is such that \(\text{prob}(c_i \leq 1 - \gamma) = 1\).

For any \((\hat{b}_1, \hat{b}_2)\), let

\[ u_i(\hat{b}_1, \hat{b}_2; c_i) = \frac{1}{2} \mathbb{E}\left[1_{b_i + \gamma \epsilon \leq b_j}(b_i + \gamma \epsilon - c_i)\right] + \frac{1}{2} \mathbb{E}\left[1_{b_i \leq b_j + \gamma \epsilon}(b_i - c_i)\right] \]

denote firm \(i\)'s expected payoff from an auction when intended bids are \((\hat{b}_1, \hat{b}_2)\) and firm \(i\)'s cost is \(c_i\). We assume that the distribution \(F_\epsilon\) is such that, for all bids \((\hat{b}_1, \hat{b}_2)\) and all costs \(c_i\), \(\frac{\partial^2 u_i(\hat{b}_1, \hat{b}_2; c_i)}{\partial b_i \partial b_j} \geq 0\). This assumption implies that firm \(i\)'s static best-response is increasing in firm \(j\)'s intended bid \(\hat{b}_j\).

**Definition 5.** We say that an equilibrium is a designated winner equilibrium if, at every on-path history, intended bids \((\hat{b}_1, \hat{b}_2)\) are such that bidder \(i \in \{1, 2\}\) wins the auction with probability 1.

**Proposition 6.** There exists \(\gamma > 0\) such that, if \(\gamma > \bar{\gamma}\), in any designated winner equilibrium the designated winner has a static incentive to increase her bid at every on-path history.

Proposition 7 shows that the bidding profiles we observe in our data can be rationalized by this model with noisy bids. Indeed, in any designated winner equilibrium the winning bidder has a static incentive to increase her bid, provided that the level of noise is non-negligible.

Having a designated winner at each auction is a simple collusive scheme, which cartels commonly use in practice. However, such collusive schemes are in general not efficient within the current model. Indeed, an implication of our next result is that such schemes cannot be

---

\([8] This condition is satisfied whenever, for all \((\hat{b}_1, \hat{b}_2)\) and all \(c_i\), distribution \(F_\epsilon\) satisfies

\[ f_\epsilon\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) + \frac{1}{\gamma} f_\epsilon'\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) \left(\left(\frac{\hat{b}_i + \hat{b}_j}{2}\right) - c_i\right) \geq 0 \].
optimal when the level of noise is non-negligible and cartel members are patient enough. As above, we let $\bar{V}$ denote the highest equilibrium payoff for the cartel.

**Proposition 7.** There exists $\bar{\delta} < 1$ such that, if $\delta \geq \bar{\delta}$, in any equilibrium that attains $\bar{V}$ and at every on-path history, no bidder has a static incentive to increase her bid.

### 7 Conclusion

This paper documents a novel bidding pattern from Japanese procurement auctions: winning bids tend to be isolated. We show that this bidding behavior is a strong marker for collusion, and propose structural methods to estimate the extent of collusion in a given dataset.

Lastly, we show that isolated winning bids can be rationalized by a model with trembles. Indeed, isolated winning bids make the allocation robust, and guarantee that contracts are allocated to the designated winner.

### Appendix

#### A Proofs

**A.1 Proofs of Section 3**

**Proof of Theorem 1.** Let $H$ be a set of histories, and fix $\rho \in (-1, \infty)$. For each history $h_{i,t} = (h_t, z_{i,t}) \in H$, define

\[
\varepsilon_{i,t} \equiv \mathbb{E}_{\sigma, \mu}[1_{\wedge b_{-i,t} > b_{i,t}(1+\rho)|h_{i,t}]} - 1_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}
\]

\[
= \text{prob}_{\sigma, \mu}(\wedge b_{-i,t} > b_{i,t}(1 + \rho)|h_{i,t}) - 1_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}.
\]

Note that $\hat{D}(\rho|H) - D(\rho|H) = \frac{1}{|H|} \sum_{h_{i,t} \in H} \varepsilon_{i,t}$. 

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Note further that, by the law of iterated expectations, for all histories \( h_{j,t-s} \in H \) with \( s \geq 0 \),
\[
E_{\sigma,\mu}[\varepsilon_{i,t} | h_{j,t-s}] = E_{\sigma,\mu}[E_{\sigma,\mu}[1_{b_{i,t-s} > b_{i,t}(1+\rho)} | h_{t}, z_{i,t}] - 1_{b_{i,s} > b_{i,t}(1+\rho)} | h_{t-s}, z_{j,t-s}] = 0. \tag{9}
\]

Number the histories in \( H \) as 1, ..., \(|H|\) such that, for any pair of histories \( k = (h_s, z_{i,s}) \in H \) and \( k' = (h_{s'}, z_{j,s'}) \in H \) with \( k' > k \), \( s' \geq s \). For each history \( k = (h_t, z_{i,t}) \), let \( \varepsilon_k = \varepsilon_{i,t} \), so that
\[
\hat{D}(\rho | H) - D(\rho | H) = \frac{1}{|H|} \sum_{k=1}^{|H|} \varepsilon_k.
\]

Note that, for all \( \hat{k} \leq |H| \), \( S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k \) is a Martingale, with increments \( \varepsilon_k \) whose absolute value is bounded above by 1. By the Azuma-Hoeffding Inequality, for every \( \alpha > 0 \),
\[
\text{prob}(|S_{|H|}| \geq |H|\alpha) \leq 2 \exp\{-\alpha^2|H|/2\}.
\]
Therefore, with probability 1, \( \frac{1}{|H|} S_{|H|} = \hat{D}(\rho | H) - \overline{D}(\rho | H) \) converges to zero as \( |H| \to \infty \). \qed

\section*{A.2 Proofs of Section 4}

**Proof of Corollary 1.** Suppose \( s_{\text{comp}} \geq 1 - \kappa \) for some \( \kappa > 0 \). Then, for all \( \rho > 0 \),
\[
\frac{1}{\rho} [\hat{R}(\rho | H) - \hat{R}(0 | H)] = \frac{1}{\rho} [\overline{R}(\rho | H) - \overline{R}(0 | H) + \hat{R}(\rho | H) - \hat{R}(0 | H) + \hat{R}(0 | H) - \overline{R}(0 | H)]
\]
\[
\leq 1 - s_{\text{comp}} + \frac{1}{\rho} [\overline{R}(\rho | H) - \overline{R}(0 | H) - \hat{R}(0 | H) + \overline{R}(0 | H)]
\]
\[
\leq \kappa + \frac{1}{\rho} [\hat{R}(\rho | H) - \overline{R}(\rho | H) - \hat{R}(0 | H) + \overline{R}(0 | H)],
\]
where the first inequality follows from Proposition 2 and the second follows since \( s_{\text{comp}} \geq 1 - \kappa \).

Next, note that for any scalar \( \rho' \in (-1, \infty) \),
\[
\overline{R}(\rho' | H) - \hat{R}(\rho' | H) = \sum_{h_{i,t} \in H} \varepsilon_{i,t},
\]

\( \text{This holds since, in a perfect public Bayesian equilibrium, bidders’ strategies at any time } t \text{ depend solely on the public history and on their private information at time } t. \)
where
\[ \varepsilon_{i,t} = \mathbb{E}_{\sigma,\mu}[(1 + \rho')b_{i,t}\mathbf{1}_{b_{i,t} > b_{i,t}(1 + \rho')}|h_{i,t}] - (1 + \rho')b_{i,t}\mathbf{1}_{b_{i,t} > b_{i,t}(1 + \rho')}]. \]

By the law of iterated expectations, for all \( h_{j,t-s} \in H \) with \( s \geq 0 \),
\[ \mathbb{E}_{\sigma,\mu}[\varepsilon_{i,t}|h_{j,t-s}] = \mathbb{E}_{\sigma,\mu}[\mathbb{E}_{\sigma,\mu}[(1 + \rho')b_{i,t}\mathbf{1}_{b_{i,t} > b_{i,t}(1 + \rho')}|h_{t,s},z_{i,t}] - (1 + \rho')b_{i,t}\mathbf{1}_{b_{i,t} > b_{i,t}(1 + \rho')}|h_{t-s},z_{j,t-s}] = 0. \]

As in the proof of Theorem 1, number the histories in \( H \) as \( 1, \ldots, |H| \) such that, for any pair of histories \( k = (h_s, z_{i,s}) \in H \) and \( k' = (h_s', z_{j,s'}) \in H \) with \( k' > k \), \( s' \geq s \). For each history \( k = (h_t,z_{i,t}) \), let \( \varepsilon_k = \varepsilon_{i,t} \), so that
\[ \hat{R}(\rho'|H) - \hat{R}(\rho'|H) = \frac{1}{|H|} \sum_{k=1}^{|H|} \varepsilon_k. \]

Note that, for all \( \hat{k} \leq |H| \), \( S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k \) is a Martingale, with increments \( \varepsilon_k \) whose absolute value is bounded above by \( 1 \).\(^{10}\) By the Azuma-Hoeffding Inequality, for all \( \alpha > 0 \),
\[ \text{prob}(|S_{|H|}| \geq |H|\alpha) = \text{prob}(|\hat{R}(\rho'|H) - \hat{R}(\rho'|H)| \geq \alpha) \leq 2 \exp(-\alpha^2|H|/2). \]

Fix any \( \rho \geq \rho^* \). Since this bound holds for all \( \rho' \in (-1, \infty) \), it follows that
\[ \text{prob}(|\hat{R}(\rho|H) - \hat{R}(\rho|H)| \geq \frac{\rho \kappa}{2} \text{ and } |\hat{R}(0|H) - \hat{R}(0|H)| \geq \frac{\rho \kappa}{2}) \leq 4 \exp(-(\rho \kappa)^2|H|/2) \leq 4 \exp(-(\rho^* \kappa)^2|H|/2). \]

Combining this with equation (17), it follows that with probability at least \( 1 - 4 \exp(-(\rho^* \kappa)^2|H|/2) \),
\[ \frac{1}{\rho} [\hat{R}(\rho|H) - \hat{R}(0|H)] \leq 2 \kappa. \]

\(^{10}\)This follows since we normalized reserve price to 1.

**Proof of Proposition 3.** Note first that conditions (13) and (14) must automatically hold at every competitive history \( h \in H \).

Note next that, by the arguments in Theorem 1, for all \( n \), \( \text{prob}(|\hat{D}_n - D_n| \geq T) \leq \)
2 \exp(-T^2|H|/2). \text{ It follows that}

\[ \text{prob}(\forall n, |\hat{D}_n - D_n| \geq T) \leq 2(n + \pi + 1) \exp(-T^2|H|/2). \]

Therefore, with probability at least \(1 - 2(n + \pi + 1) \exp(-T^2|H|/2)\), conditions (13), (14) and (8) hold simultaneously. \qed

\textbf{Proof of Corollary 2.} Let \(H^{\text{comp}} \subset H\) be the set of competitive histories in \(H\), so that \(s_{\text{comp}} = \frac{|H^{\text{comp}}|}{|H|}\). Consider the vector \(p^{\text{comp}} = (p^{\text{comp}}_h)_{h \in H}\) with \(p^{\text{comp}}_h = 1\) for all \(h \in H^{\text{comp}}\) and \(p^{\text{comp}}_h = 0\) otherwise.

Note first that, for all histories \(h \in H^{\text{comp}}\), the firms’ true believes and costs \((d_{h,n}, c_h)\) must satisfy conditions (13), (14), (15) and (16). Hence, for all \(h \in H^{\text{comp}}\), set believes and costs equal to the firms’ true believes and costs. For all \(h \notin H^{\text{comp}}\), pick any beliefs and costs \((d_{h,n}, c_h)\) that satisfy conditions (13), (14), (15) and (16).

For every \(p^C \in [0,1]^{|H|}\) and every \(n\), define

\[ D_n(p^C) \equiv \frac{1}{\sum_{h \in H} p_h} \sum_{h \in H} p^C_h d_{h,n}. \]

Since \(p^{\text{comp}}\) is such that \(p^{\text{comp}}_h = 1\) for all \(h \in H^{\text{comp}}\) and \(p^{\text{comp}}_h = 0\) for all \(h \notin H^{\text{comp}}\), it follows that, for all \(n\),

\[ D_n(p^{\text{comp}}) = \frac{1}{|H^{\text{comp}}|} \sum_{h \in H^{\text{comp}}} d_{h,n}. \]

Similarly, note that for all \(n\),

\[ \hat{D}_n(p^{\text{comp}}) = \frac{1}{|H^{\text{comp}}|} \sum_{h \in H^{\text{comp}}} 1_{(1+p_n)b_h < \wedge b_{-1,h}}. \]
Using the arguments as in the proof of Theorem 1,

\[ \forall n, \quad \text{prob}(|\hat{D}_n(p^{\text{comp}}) - D_n(p^{\text{comp}})| \geq T) \leq 2 \exp(-T^2|H^{\text{comp}}|/2) \]

These inequalities imply that, for \( p^C = p^{\text{comp}} \), conditions (8) hold simultaneously with probability at least \( 1 - 2(n + \pi + 1) \exp(-T^2|H^{\text{comp}}|/2) \). The result follows by noting that \( |H^{\text{comp}}| = s^{\text{comp}}(H)|H| \). ■

Proof of Proposition 4. Let \( p^C \in [0, 1]^C \) be a solution to Problem (11), and let \( ((d_h,n),c_h)_{h \in H} \) be the corresponding beliefs satisfying all the constraints of the problem. For every \( y \in Y \), define \( H(y) \equiv \{ h \in H : y_h = y \} \).

Consider any permutation \( \alpha : H \rightarrow H \) such that, for all \( y \in Y \) and all \( h \in H(y), \alpha(h) \in H(y) \). Let \( \bar{p}^C = (\bar{p}^C_h)_{h \in H} \) be such that, for all \( h \in H, \bar{p}^C_h = p^C_{\alpha(h)} \). Note that \( \bar{p}^C \) is also a solution (11), together with beliefs and costs \( ((\tilde{d}_{h,n}),\tilde{c}_h)_{h \in H} \) such that, for all \( h \in H, ((\tilde{d}_{h,n}),\tilde{c}_h) = ((d_{\alpha(h),n}),c_{\alpha(h)}) \). Moreover, note that, for all \( n, \hat{D}_n(\bar{p}^C) = \hat{D}_n(p^C) \). Hence, beliefs and costs \( ((\tilde{d}_{h,n}),\tilde{c}_h)_{h \in H} \) satisfy the IC constraints, and the aggregate and information constraints given \( \bar{p}^C \).

Since this is true for any such permutation \( \alpha \), it follows that there exists \( \bar{p}^C \in [0, 1]^{|H|} \) and corresponding beliefs and costs \( ((\tilde{d}_{h,n}),\tilde{c}_h)_{h \in H} \), such that

(i) for all \( y \in Y \) and all \( h, h' \in H(y), \bar{p}^C_h = \bar{p}^C_{h'} = p_y \in [0, 1] \),

(ii) \( \frac{1}{|H|} \sum_h \bar{p}^C_h = \frac{1}{|H|} \sum_h p^C_h = \tilde{s}^{\text{comp}}, \)

(iii) \( \frac{1}{\sum_{h \in H} \bar{p}^C_h} \sum_{h \in H} \bar{p}^C_h \times (\tilde{d}_{h,n}) \in A(\bar{p}^C), \)

(iv) for all \( h \in H, (\tilde{d}_{h,n}) \in \text{Conv} \left[ Z \cap I(\bar{p}^C) \right] \).

Let \( q : Y \rightarrow [0, 1] \) be such that \( q(y) = p_y \), so that \( \bar{p}^C = p(q) \). Since \( \sum_{h \in H} \frac{\bar{p}^C_h}{\sum_{h \in H} \bar{p}^C_h} \times (\tilde{d}_{h,n}) \in \text{Conv} \left[ Z \cap I(\bar{p}^C) \right] \),
\[
\hat{s}_{\text{comp}} \leq \max_{q : Y \to [0, 1]} \{ \hat{s}_{\text{comp}}(q) \mid \text{Conv}[Z \cap I(p)] \cap A(p(q)) \neq \emptyset \}.
\]

\section*{A.3 Proofs of Section 6}

\textbf{Proof of Proposition 7.} Fix a designated winner equilibrium, and a history \(h_t\). Without loss of generality, assume that bidder \(i\) is designated winner at history \(h_t\), so that \(\hat{b}_i \leq \hat{b}_j - \gamma\).

If \(\hat{b}_i < \hat{b}_j - \gamma\), bidder \(i\) has a static incentive to increase marginally her intended bid, since doing so doesn’t reduce her probability of winning.

Suppose next that \(\hat{b}_i = \hat{b}_j - \gamma\). Note that for any pair of intended bids \((b_i, b_j)\) with \(|b_i - b_j| \leq \gamma\),

\[
u_i(b_i, b_j, c_i) = \frac{1}{2} \mathbb{E} \left[ 1_{b_i + \gamma \epsilon \leq b_j} (b_i + \gamma \epsilon - c_i) \right] + \frac{1}{2} \mathbb{E} \left[ 1_{b_i \leq b_j + \gamma \epsilon} (b_i - c_i) \right]
\]

\[
\frac{\partial u_i(b_i, b_j, c_i)}{\partial b_i} = \frac{1}{2} \left( 1 - F_\epsilon \left( \frac{b_i - b_j}{\gamma} \right) \right) + \frac{1}{2} F_\epsilon \left( \frac{b_j - b_i}{\gamma} \right) - \frac{1}{2\gamma} \left[ f_\epsilon \left( \frac{b_i - b_j}{\gamma} \right) (b_i - c_i) + f_\epsilon \left( \frac{b_j - b_i}{\gamma} \right) (b_j - c_i) \right]
\]

\[= F_\epsilon \left( \frac{b_j - b_i}{\gamma} \right) - \frac{1}{2\gamma} f_\epsilon \left( \frac{b_j - b_i}{\gamma} \right) \left[ \frac{b_i + b_j}{2} - c_i \right]
\]

where the last equality follows since \(F_\epsilon\) is symmetric around zero. Using \(\hat{b}_i = \hat{b}_j - \gamma\) yields

\[
\frac{\partial u_i(\hat{b}_i, \hat{b}_j, c_i)}{\partial b_i} = 1 - \frac{1}{2\gamma} f_\epsilon(1) \left[ \frac{\hat{b}_i + \hat{b}_j}{2} - c_i \right] \geq 1 - \frac{1}{2\gamma} f_\epsilon(1),
\]

where the last inequality follows since bids are bounded above by the reserve price \(r = 1\), and since costs are bounded below by zero. Thus, there exists \(\tau > 0\) such that, if \(\gamma > \tau\),
\[ \frac{\partial u_i(b_i,b_j,c)}{\partial b_i} > 0, \] so the designated winner has a static incentive to increase her bid. ■

We now move to the proof of Proposition 8. Fix an equilibrium \( \sigma \). For any history \( h_t \) and any cost realization \( c \), we denote by \( \hat{b}(h_t, c) = (\hat{b}_1(h_t, c), \hat{b}_2(h_t, c)) \) the profile of intended bids that firms place under equilibrium \( \sigma \) after history \( h_t \) when costs at time \( t \) are \( c \). We start with a definition and two Lemmas.

**Definition A.1.** We say that an equilibrium \( \sigma \) is stationary if there exists a fixed bidding profile \( \hat{b}^* : [\underline{c}, \overline{c}] \mapsto \mathbb{R}^2 \) such that \( \hat{b}(h_t, c) = \hat{b}^*(c) \) for every on-path history \( h_t \) and every \( c \in [\underline{c}, \overline{c}]^2 \).

**Lemma A.1.** Let \( \sigma \) be an equilibrium that attains \( \bar{V} \). Then, there exists a stationary equilibrium \( \hat{\sigma} \) that also attains \( \bar{V} \).

**Proof.** Let \( \sigma \) be an equilibrium that attains \( \bar{V} \), and let \((V_1, V_2)\) be the firm’s payoff at the start of the game under equilibrium \( \sigma \), with \( V_1 + V_2 = \bar{V} \). Let \( \hat{b}(c) = (\hat{b}_1(c), \hat{b}_2(c)) \) be the intended bids that firms place under \( \sigma \) in the first period, and let \( T(c, b) = (T_1(c, b), T_2(c, b)) \) be the transfers that firms make under equilibrium \( \sigma \) at the end of the first period (which depend on firms’ costs and realized bids). Lastly, let \( V(c, b) = (V_1(c, b), V_2(c, b)) \) be the continuation payoffs that firms get at the second period under equilibrium \( \sigma \) when costs are \( c \) and realized bids are \( b \). Note that \( V_1(c, b) + V_2(c, b) \leq \bar{V} \).

We now show that there exists a stationary equilibrium \( \hat{\sigma} \) in which players bid according to \( \hat{b}(c) \) at every period. At every period, firms exchange transfers \( \hat{T}(c, b) = (\hat{T}_1(c, b), \hat{T}_2(c, b)) \) with \( \hat{T}_i(c, b) + \delta V_i = T_i(c, b) + \delta V_i(c, b) \) for all \( c \), all \( b \) and \( i = 1, 2 \). If firm \( i = 1, 2 \) doesn’t pay her transfer, firms revert to an equilibrium that gives firm \( i \) a payoff of \( V_i \), where \( V_i \) is the lowest possible equilibrium payoff.

Note that, given these transfers, both players have the same incentives as in the first period of the original equilibrium, so bidding according to \( \hat{b}(c) \) remains optimal. Moreover, by construction, both firms are willing to pay transfers \( \hat{T}(c, b) \) at the end of the period.
Indeed, since it was optimal for firms to pay transfers $T_i(c, b)$ under the original equilibrium, for $i = 1, 2$ and for all $c$ and all $b$ it must be that

$$\delta V_i \leq T_i(c, b) + \delta V_i(c, b) = \hat{T}_i(c, b) + \delta V_i.$$ 

Hence, firms find it optimal to pay transfers $\hat{T}$. Lastly, note that

$$\hat{T}_1(c, b) + \hat{T}_2(c, b) = T_1(c, b) + T_2(c, b) + \delta(V_1(c, b) + V_2(c, b) - V_1 - V_2) \leq 0,$$

where we used $V_1(c, b) + V_2(c, b) \leq V = V_1 + V_2$ and $T_1(b) + T_2(b) \leq 0$. Hence, transfers $\hat{T}(c, b)$ are feasible. ■

**Lemma A.2.** Let $\sigma$ be an equilibrium that attains $V$. If $\delta$ is larger than a cutoff $\hat{\delta} < 1$, then there exists an equilibrium $\hat{\sigma}$ that also attains $V$, under which $|\hat{b}_1(h_t, c) - \hat{b}_2(h_t, c)| \leq \gamma$ for all on-path histories $h_t$ and all $c$.

**Proof.** By Lemma A.1, without loss be can take equilibrium $\sigma$ to be stationary. Let $\hat{b}(c)$ be the stationary bidding profile that firms use under $\sigma$, and suppose that there exists $c$ such that $|\hat{b}_1(c) - \hat{b}_2(c)| > \gamma$. Without loss, suppose that $\hat{b}_i(c) < \hat{b}_j(c) - \gamma$.

Let $(V_1, V_2)$ the (stationary) payoffs that players get under equilibrium $\sigma$, with $V_1 + V_2 = V$. Let $\hat{\sigma}$ be a stationary equilibrium that is equal to $\sigma$, except that when costs are $c$ firms bid according to $\bar{b}$ with $\bar{b}_i = \hat{b}_i(c)$ and $\bar{b}_j = \hat{b}_i + \gamma$. Moreover, when costs are $c$, there are no transfers. If player $i$ wins the auction, players continue to play according to the same stationary equilibrium, which yields payoffs $(V_1, V_2)$. If player $j$ wins the auction, players revert to the stage game Nash equilibrium at every future date, obtaining discounted payoffs $(V_{NE}, V_{NE}) < (V_1, V_2)$. Note that, for $k = 1, 2$, $V_k - V_{NE} \to \infty$ as $\delta \to 1$.

Note that the allocation and expected surplus is unchanged when costs are $c$, since bidder $i$ continues to win with probability 1 under this bidding profile. Moreover, since there are no
transfers, the some of the firms’ flow payoffs when costs are \( c \) is weakly larger under this new strategy profile than under the original equilibrium. We now show that there exists \( \delta < 1 \) such that, if \( \delta \geq \delta^i \), both firms find it optimal to bid according to \( \bar{b} \) when costs are \( c \). Hence, this proposed strategy profile is an equilibrium when \( \delta \geq \delta^i \).

Consider firm \( i \). When costs are \( c \), her payoff from bidding \( b > \bar{\bar{b}}_i \) when firm \( j \) bids \( \bar{b}_j \) is

\[
V_i(b) = \frac{1}{2} \mathbb{E}[1_{b+\gamma \epsilon \leq \bar{\bar{b}}_j}(b + \gamma \epsilon - c_i + \delta V_i)] + \frac{1}{2} \left( 1 - F_{\epsilon}\left(\frac{\bar{\bar{b}}_j - b}{\gamma}\right)\right) \delta V^{NE} \\
\quad + \frac{1}{2} \left( 1 - F_{\epsilon}\left(\frac{b - \bar{\bar{b}}_j}{\gamma}\right)\right) (b - c_i + \delta V_i) + \frac{1}{2} F_{\epsilon}\left(\frac{b - \bar{\bar{b}}_j}{\gamma}\right) \delta V^{NE} \\
= \delta V^{NE} + \frac{1}{2} \mathbb{E}[1_{b+\gamma \epsilon \leq \bar{\bar{b}}_j}(b + \gamma \epsilon - c_i + \delta(V_i - V^{NE})] \\
\quad + \frac{1}{2} \left( 1 - F_{\epsilon}\left(\frac{b - \bar{\bar{b}}_j}{\gamma}\right)\right) (b - c_i + \delta(V_i - V^{NE})).
\]

Note that

\[
V'_i(b) = -\frac{1}{2\gamma} \left( f_{\epsilon}\left(\frac{\bar{\bar{b}}_j - b}{\gamma}\right) (\bar{\bar{b}}_j - c_i + \delta(V_i - V^{NE}) \right) + f_{\epsilon}\left(\frac{b - \bar{\bar{b}}_j}{\gamma}\right) (b - c_i + \delta(V_i - V^{NE}) \right) \\
\quad + \frac{1}{2} F_{\epsilon}\left(\frac{\bar{\bar{b}}_j - b}{\gamma}\right) + \frac{1}{2} \left( 1 - F_{\epsilon}\left(\frac{b - \bar{\bar{b}}_j}{\gamma}\right)\right).
\]

Since \( V_i - V^{NE} \to \infty \) as \( \delta \to 1 \), there exists \( \hat{\delta}^i < 1 \) such that \( V'_i(b) < 0 \) for all \( b > \bar{\bar{b}}_i \). Hence, bidding \( \bar{\bar{b}}_i \) is optimal for bidder \( i \) whenever \( \delta \geq \hat{\delta}^i \).

Consider next firm \( j \). When costs are \( c \), her payoff from bidding \( b < \bar{\bar{b}}_j \) when firm \( i \) bids
\[
\bar{b}_i \text{ is }
\]
\[
V_j(b) = \frac{1}{2} \mathbb{E} \left[ 1_{b+\gamma \epsilon \leq \bar{b}_i} (b + \gamma \epsilon - c_j + \delta V^{NE}) \right] + \frac{1}{2} \left( 1 - F_{\epsilon} \left( \frac{\bar{b}_i - b}{\gamma} \right) \right) \delta V_j
\]
\[
\frac{1}{2} \left( 1 - F_{\epsilon} \left( \frac{b - \bar{b}_i}{\gamma} \right) \right) (b - c_j + \delta V^{NE}) + \frac{1}{2} F_{\epsilon} \left( \frac{b - \bar{b}_i}{\gamma} \right) \delta V_j
\]
\[
= \delta V_j + \frac{1}{2} \mathbb{E} \left[ 1_{b+\gamma \epsilon \leq \bar{b}_i} (b + \gamma \epsilon - c_j - \delta(V_j - V^{NE})) \right]
\]
\[
\frac{1}{2} \left( 1 - F_{\epsilon} \left( \frac{b - \bar{b}_i}{\gamma} \right) \right) (b - c_j - \delta(V_j - V^{NE})).
\]

Note that
\[
V'_j(b) = \frac{-1}{2\gamma} \left( f_{\epsilon} \left( \frac{\bar{b}_i - b}{\gamma} \right) (\bar{b}_i - c_j - \delta(V_j - V^{NE})) + f_{\epsilon} \left( \frac{b - \bar{b}_i}{\gamma} \right) (b - c_j - \delta(V_j - V^{NE})) \right)
\]
\[
+ \frac{1}{2} F_{\epsilon} \left( \frac{\bar{b}_i - b}{\gamma} \right) + \frac{1}{2} \left( 1 - F_{\epsilon} \left( \frac{b - \bar{b}_i}{\gamma} \right) \right).
\]

Since \( V_j - V^{NE} \to \infty \) as \( \delta \to 1 \), there exists \( \hat{\delta}^i < 1 \) such that \( V'_j(b) > 0 \) for all \( b < \bar{b}_j \). Hence, bidding \( \bar{b}_j \) is optimal for bidder \( j \) whenever \( \delta \geq \hat{\delta}^j \). Letting \( \hat{\delta} = \max \{ \hat{\delta}^i, \hat{\delta}^j \} \), the two firms find it optimal to bid according to \( \bar{b} \) whenever \( \delta \geq \hat{\delta} \). ■

**Proof of Proposition 8.** Suppose that \( \delta \geq \hat{\delta} \), where \( \hat{\delta} < 1 \) is the cutoff in the statement of Lemma A.2. Let \( \sigma \) be an equilibrium that attains \( V \). By Lemmas A.1 and A.2, we can take \( \sigma \) to be a stationary equilibrium with bidding profile \( \hat{b} \) satisfying \( |\hat{b}_i(c) - \hat{b}_j(c)| \leq \gamma \) for all \( c \).

Towards a contraction, suppose there exists a cost vector \( c \) and a bidder \( i \in \{1, 2\} \) such that, under bidding profile \( \hat{b}(c) \), bidder \( i \) has a static incentive to increase her bid: i.e.,
\[
\frac{\partial u_i(\hat{b}(c), \hat{b}(c), c_i)}{\partial \hat{b}_i(c)} > 0.
\]

Let \( (V_i, V_j) \) be the equilibrium payoffs under \( \sigma \), with \( V_i + V_j = V \). Let \( (T_i(b_i, b_j; c), T_j(b_i, b_j; c)) \) be the transfers that players exchange under equilibrium \( \sigma \) when costs are \( c \) and realized
bids are \((b_i, b_j)\) under equilibrium \(\sigma\).

Consider the following sequence of intended bids \(\{\hat{b}^k\}\) and transfers \(\{\hat{T}^k(\cdot, \cdot)\}\). Given intended bids \(\{\hat{b}^k\}\), transfers are as follows: for each \(k \geq 0\), and for all realized bids \((b_i^*, b_j^*)\), \(\hat{T}^k_j(b_i^*, b_j^*) = \mathbb{E}_{b_i}[T_j(b_i, b_j^*; \cdot) | \hat{b}^k_i]\), and \(\hat{T}^k_i(b_i^*, b_j^*) = -\hat{T}^k_j(b_j^*, b_i^*)\). Note that, for each \(k \geq 0\), \(\hat{T}^k_i\) and \(\hat{T}^k_j\) don’t depend on firm \(i\)’s realized bid \(b_i^*\).

We now specify the sequence of intended bids \(\{\hat{b}^k\}\). Let \(\hat{b}^0_j = \hat{b}_j(c)\) and \(\hat{b}^0_i = \arg\max_b u_i(b, \hat{b}^0_j, c_i)\). Since \(\frac{\partial u_i(b_i, c_i) b_i}{\partial b_i} > 0\) and since \(u_i(b, \hat{b}^0_j, c_i)\) is concave in \(b\),\(^{11}\) it follows that \(\hat{b}^0_i > \hat{b}_i(c)\).

For \(k \geq 1\), \(k\) odd, set \(\hat{b}^k_i = \hat{b}^{k-1}_i\) and \(\hat{b}^k_j = \arg\max_b u_j(\hat{b}^{k-1}_i, b, c_j) + \mathbb{E}[\hat{T}^{k-1}_j(b_j, \hat{b}^{k-1}_j)|\hat{b}_j = b, \hat{b}_i = \hat{b}^{k-1}_i]\). For \(k \geq 1\), \(k\) even, set \(\hat{b}^k_i = \arg\max_b u_i(b, \hat{b}^{k-1}_j, c_i) = \arg\max_b u_i(b, \hat{b}^{k-1}_j, c_i) + \mathbb{E}[\hat{T}^{k-1}_i(b_i, \hat{b}^{k-1}_j)|\hat{b}_i = b, \hat{b}_j = \hat{b}^{k-1}_j]\) (where the second inequality follows since \(\hat{T}^{k-1}_i(b_i, \hat{b}^{k-1}_j)\) is independent of \(i\)’s realized bid \(b_i\)) and \(\hat{b}^k_j = \hat{b}^{k-1}_j\).

Note that the sequence of bids \(\{\hat{b}^k\}\) is increasing, and hence converges to some \(\hat{b}^*\). Indeed, since \(\frac{\partial^2 u_i(\hat{b}_i, \hat{b}_j, c_i)}{\partial \hat{b}_i \partial \hat{b}_j} \geq 0\) and \(\frac{\partial^2 u_i(b_i, \hat{b}_j, c_j)}{\partial \hat{b}_i \partial \hat{b}_j} \geq 0\), standard results in monotone comparative statics (e.g., Vives (2001)) imply that \(\hat{b}^k \geq \hat{b}^{k-1}\) for all \(k \geq 1\). As a result, the sequence of transfers \(\{\hat{T}^k(\cdot, \cdot)\}\) also converges to \(\{\hat{T}^*(\cdot, \cdot)\}\). Moreover, given transfers \(\hat{T}^*(\cdot, \cdot)\) and continuation payoffs \((V_1, V_2)\) (which don’t depend on the realized bids), each firm finds it optimal to bid according to \(\hat{b}^*\) when costs are \(c\) and her opponent also bids according to \(\hat{b}^*\). Note further that if the discount factor is sufficiently large, transfers \(\hat{T}^*_i(\cdot, \cdot)\) can be enforced by punishing a firm who doesn’t pay the specified transfer with a continuation equilibrium in which that firm obtains a payoff of \(V_\cdot\).

\(^{11}\)Indeed, for any \(\hat{b}_i, \hat{b}_j\) with \(|\hat{b}_i - \hat{b}_j| \leq \gamma\),

\[
\frac{\partial^2 u_i(\hat{b}_i, \hat{b}_j, c_i)}{\partial \hat{b}_i^2} = \frac{1}{2\gamma} f_\epsilon\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) - \frac{1}{\gamma} f_\epsilon\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) - \frac{1}{\gamma} f_\epsilon^\prime\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) \left(\frac{\hat{b}_i + \hat{b}_j}{2} - c_i\right) < 0,
\]

where the inequality follows since \(\frac{\partial^2 u_i(\hat{b}_i, \hat{b}_j, c_i)}{\partial \hat{b}_i \partial \hat{b}_j} \geq 0\) implies that

\[
f_\epsilon\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) + \frac{1}{\gamma} f_\epsilon^\prime\left(\frac{\hat{b}_i - \hat{b}_j}{\gamma}\right) \left(\frac{\hat{b}_i + \hat{b}_j}{2} - c_i\right) \geq 0.
\]
The arguments above imply that, whenever $\delta$ is above some cutoff $\delta < 1$, there exists a stationary equilibrium $\hat{\sigma}$, which is identical to the original stationary equilibrium $\sigma$, except that at periods in which firms’ costs are $c$, firms place intended bids $\hat{b}^*$ and exchange transfers $\hat{T}^*(\cdot, \cdot)$.

Finally, we show that the sum of firms’s flow payoffs when costs are $c$ is higher under bidding profile $\{\hat{b}^*\}$ than under the original bidding profile $\{\hat{b}(c)\}$. To see why, note that $u_i(\hat{b}^*_j, \hat{T}^*(\cdot, \cdot), c_j) > u_i(\hat{b}^*_i, \hat{b}^*(c), c_j)$, since $\hat{b}^*_j \geq \hat{b}^*_0$. Similarly, note that $u_j(\hat{b}^*_i, \hat{T}^*(\cdot, \cdot), c_i) > u_j(\hat{T}(\cdot, \cdot), \hat{b}^*(c), c_i)$, since $\hat{b}^*_i \geq \hat{b}^*(c)$. Then, since $\{\hat{b}^k\}$ is an increasing sequence, and since at each step in the iteration firms are best-responding to the previous step, it follows that $\{u_i(\hat{b}^*_i, \hat{b}^k_j, c_i), u_j(\hat{b}^*_j, \hat{b}^k_j, c_j)\}$ is also an increasing sequence. Hence, $u_i(\hat{b}^*_i, \hat{b}^*_j, c) > u_i(\hat{T}(\cdot, \cdot), \hat{b}^*(c), c_i)$ and $u_j(\hat{b}^*_j, \hat{b}^*_j, c) \geq u_j(\hat{T}(\cdot, \cdot), \hat{b}^*(c), c_j)$. But this contradicts the fact that the original equilibrium $\sigma$ attained $V$. Hence, there exists a cutoff $\bar{\delta} < 1$ such that, if $\delta \geq \bar{\delta}$, in any equilibrium that attains $V$ no bidder has a static incentive to increase her bid. ■

References


