Some Recent Trends of Nonlinear Partial Differential Equations

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It is a great honor for me to be awarded the Silver Professorship and to have this opportunity to tell you about some of the central aspects of the research field in which I have been involved. I was told that the readers of this article will be mostly non-mathematicians, including some non-scientists. This is very difficult for me; all of my experience as a speaker or writer has been addressing specialist in a particular research field—and so this will most likely be a dismal attempt. Though I will try to be as non-technical as possible, I will have to use some mathematical language and notation and I apologize in advance (also to specialists who happen to read this article for whom the matters explained here will be rather familiar).

My work is mostly concerned with partial differential equations (PDEs). A partial differential equation for an unknown function is a simply a relationship between the unknown and its various partial derivatives with respect to several variables. Like algebraic equations, such a PDE could be linear, quadratic, or genuinely nonlinear in the unknown function. Naturally, nonlinear PDEs are often much harder to understand than the linear ones. So in this research field, we all have problems...... few will actually admit theirs to be linear.

Before proceeding any further, I wish to note that, there is in truth no central core theory to the subject of nonlinear PDEs, nor can there be. The sources are so numerous and the applications so many that topic is vast and extensively developed. The whole subject seems to be a confederation of diverse subareas, each a study different phenomena for different PDEs by utterly different methods. It is the “nonlinearity” that makes each equation and each problem unique, and one should never expect an all-inclusive theory. On the other hand, one certainly does not want to treat each problem individually and have a collection of unrelated facts and techniques. Therefore, in a technical sense, one always seeks new principles, ideas, and methods that may be applied to at least a class of such problems and PDEs.

Looking back, the study of PDEs started in the 18th century with the work of Euler, d’Alembert, Lagrange, Laplace, and others. It was used as a central tool in descriptions of mechanics of continua and, more generally, as the principle mode of analytical study of models in the physical sciences. Indeed, the one-dimensional linear wave equation was introduced and studied by d’Alembert around 1752 to model vibrating strings.
Laplace first formulated his equation in his work on gravitational fields in 1780, and
the heat equation appeared a bit later in the work of Fourier (1810) on heat conduction.
These three simple linear, second-order equations lead to a major classification of PDEs
into the so-called hyperbolic, elliptic and parabolic PDEs, and most of the theory of
linear PDEs is built around these three fundamental examples.
In fact, science and technology, engineering, and industry are always the constant
source of motivation and inspiration for the development of the theory of PDEs. Historically there are many such examples and they are presently still of great importance and
interest.

- the famous Euler and Navier-Stokes equations of compressible and incompressible
  fluid flow;
- the Maxwell, Maxwell-Dirac, and Born-Infeld equations for electromagnetism;
- the Korteweg de Vries equations as a model for solitary waves;
- the Schrödinger equations for quantum mechanics;
- the Fokker-Planck equations and Boltzmann equations for kinetic models of gas
  and fluids; and
- the Einstein equations for general relativity.

The list continues and is by no means exclusive. These fundamental equations of
physical sciences have occupied the most important positions in the developments of
the subject. It is not an exaggeration that most physical phenomena are modeled
and described by PDEs and that models that use PDEs are certainly not limited to
physical sciences alone. This brief overview of history of the subject has overwhelmingly
characterized one of the basic roles of PDEs in the general sciences. One certainly
should expect that such a role will continue for a quite some time. One may also
believe that the trends in general sciences and technology will dictate the developments
and trends of research in the PDEs subject. This observation leads me to choose two
special topics that I shall discuss in slightly more detail below.

On the other hand, there is a remarkable dual role played by PDEs. Beginning
in the middle of the 19th century, particularly with the work of Riemann (on the ge-
ometrization of complex function theory and the Dirichlet principle), PDEs also became
an essential tool for studying other branches of mathematics. This duality of viewpoint
has been central to the study of PDEs throughout the 19th and 20th centuries. On the
one hand, as we explained above, one has intimate connections to models in sciences;
on the other hand, there are potential applications (which have often turned out to
be quite revolutionary) of PDEs as instruments in developments of other branches of
mathematics. The work of Poincaré, studies on the minimal surfaces equation and the
Monge-Ampère equation and their geometrical implications, Riemann-Roch theorem,
the Atiyah-Singer theorem, Nash’s isometric embeddings, are but a few, now referred
to as classical among many remarkable examples. The work of S. T. Yau and his school, by using the theory of PDEs, has completely changed how one studies problems in geometry, besides solving some of the most difficult geometric problems. Donaldson and Seiberg-Witten's work on the topology of four-dimensional differential manifolds was based largely on the theory of PDEs. The theory of PDEs is also closely connected to many other fields of mathematics beside geometry and topology: for example, probability theory and statistical analysis (Brownian motion, many-particle hydrodynamics) and dynamical systems, especially Hamiltonian systems.

One should stress that as a subject, PDEs primary in analysis has always been in a central position in the development of analysis itself. Starting with Cauchy-Riemann equations and Fourier series, many most important topics developed in harmonic analysis (distributions, Sobolev spaces, singular integral operators, pseudo-differential operators, Fourier integral operators, para-differential calculus and microlocal analysis ...) are intimately connected to the theory of PDEs.

This said, I now wish to turn to somewhat more specific issues of current importance and interest. The choice of these issues and examples are rather biased partially due to my own views and research interests.

Analysis on Singularities

A small piece of every smooth curve resembles a straight line segment. A linearization is the local approximation of a curve by a straight line. In many instances such an approximation suffices at least locally in space (as well as in time for a dynamical problem), and much of the theory of PDEs in the 19th and early part of the 20th century was, in a rough sense, following such a philosophical point of view. For example, small displacements of an elastic string can be very well described by a solution of a linear wave equation. Thus smooth and slowly varying phenomena, though modeled by some nonlinear PDEs can often be understood by using linear ones which approximate them, and such phenomena are in principle linear in nature.

Nonlinear equations appear whenever such an approximation is not sufficient. For instance, light is often described as linear electro-magnetic oscillations, but with a growing intensity of light, nonlinear phenomena become crucial, and solution of PDEs model such phenomena which exhibited singular behavior—highly localized (or concentrated) in space. Thus laser optics is mainly described by nonlinear PDEs. It has a soliton (particle-like) solutions. Solitons attracted the attention of mathematical physicists after the inverse scattering method was devised by Gardner, Green, Kruskal, and Mimura, [GGKM], for solving the KdV equations and its extension to the nonlinear Schrödinger equation was established by Zakharov and Shabat [ZS]. These are basically in the framework of integrable systems. In high dimensions some PDE works on such type problems can be found in [M].

Another example is topological defects in nematic liquid crystals. Suppose a unit vector field $n(x)$ represents a static energy-minimizing configuration of a liquid crystal
molecule orientation. Here $x$ denotes the position of a molecule. (Rigorously speaking $n(x)$ should be a local average of molecule orientation at $x$.) Then the classical theory of Oseen-Frank tells us that $n(x)$ minimizes the elastic energy functional of the form $\int w(n, \nabla n)^2 \,dx$. In the simplest case, $w(n, \nabla n) = |\nabla n|^2$, that is the nonlinear $\sigma$-model. One deduces that the Euler-Lagrange equation for $n(x)$:

$$\Delta n + |\nabla n|^2 n = 0.$$ 

This is a rather difficult and truly nonlinear PDE. It has $n(x) = \frac{x}{|x|}$ as a solution. In fact, $\frac{x}{|x|}$ is an absolute energy-minimizing configuration [BCL] (hence it is stable). In this case, $x = 0$ is a singularity for the orientation. No matter how small a neighborhood of $x = 0$ one takes, one cannot approximate such an orientation by a smooth solution. Very much like a vertex in a conical surface, the singularity in this example is intrinsic. In fact, the name “nematic” comes from “nematoes” simply describe as dark thread-like singularities in the orientation of liquid crystals that can be observed under a polarized light.

Indeed, many interesting natural phenomena contain some type of singular behavior and they are often manifest through (energy, mass, momentum . . . ) concentrations and oscillations. Singularities of solutions of PDEs that describe these phenomena are, therefore, significant, and they often reflect the characteristics of these problems. Studies of singularities and their dynamics are indeed studies of those truly nonlinear phenomena.

One can divides these singularities into two basic categories: topological and non-topological. Energy (etc.) concentrations at these singularities may or may not be quantized. One example of non-topological singularities with local quantized energy is spike-layer in reaction-diffusion systems (see [Ni]). The existence of such spike-layer patterns may be formally argued via Turing’s idea of diffusion-driven instability. Consider the Gierer-Meinhardt (1972) system (suitably scaled and simplified) for two competing chemical substances

$$\begin{cases}
  u_t &= \varepsilon^2 \Delta u - u + \frac{u^2}{v} \\
  v_t &= M \Delta v - v + u^2
\end{cases} \quad \text{in } \Omega \times \mathbb{R}_+,$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega$$

(that is, spcies are confined in the region $\Omega$). Here $u$ and $v$ are positive functions that represent the densities of the activator and the inhibitor, respectively. If the activator diffuses rather slowly ($\varepsilon$ is very small) and the inhibitor diffuses rapidly ($M$ is very large), then one observes (see [Ni]) the formation of spike-layers in $u$.

Unlike defects in liquid crystals, there is no apparent topological reason for $u$ to have such singular behavior. Rather it is due to other mechanisms. Thus we shall
refer to such singularities as non-topological. It is, however, somewhat surprising that the energy of \( u \) at each such spikes is essentially a constant (see [Ni] for details). In other words, spike-layers in a reaction-diffusion system are non-topological singularities with each singularity carrying a quantized amount of energy. This energy quantization property helps our understanding of the problem. But due to the non-topological nature of such singularities, they are often rather unstable dynamically without any additional stabilizing mechanism. Similar types of singularities also arise in the KdV equations and in the nonlinear Schrödinger equations for particle-like solutions (see [M]).

Non-topological singularities also exist in the well-known Euler and Navier Stokes equations for incompressible fluid dynamics with their still unsolved problem of the regularity and the mathematical description of turbulence [F]. Vorticity concentrations may be viewed as non-topological singularities. In 2D flows, if the initial vorticity is given by \( \omega_0(x) = \sum_{i=1}^{N} \xi_i \delta(x - x_i) \), where \( \xi_i > 0 \) and \( \sum \xi_i < \infty \) (here \( \delta(*) \) denotes the Dirac function), then, at a later time \( t \), the vorticity is simply given by \( \omega(x, t) = \sum_{i=1}^{N} \xi_i \delta(x - x_i(t)) \). Where

\[
(*) \frac{d}{dt} x_i(t) = -\nabla_{x_i} W(x_1(t), \ldots, x_N(t)), \ i = 1, \ldots, N \text{ with } x_i(0) = x_i \text{ (see [BM]).}
\]

\( W \) is the standard logarithmic interaction potential. Singularities \( x_i(t), i = 1, \ldots, N \), of the flow at time \( t \) are non-topological and the vorticity concentration \( \xi_i \) at \( x_i(t) \) is by no means quantized. One notes (see [BM]) that a suitable hydrodynamic limit of the particle motion (*) is, in fact, the 2D incompressible Euler equation. This fact indicates that the Euler equation can have complex structure at arbitrary many scales.

The physical problems in 3D is even more difficult. On the other hand, for the 2D Navier Stokes equations, if the initial vorticity is the one described above, then in the time interval much smaller than \( 1/\mu \), here \( \mu \) is the viscosity of the fluids, the picture of vorticity concentrations remains as that for the Euler equation. But when the time surpasses \( \frac{1}{\mu} \), the dissipation effect takes over and completely smooths out flows in 2D. One should expect similar facts to remain true in 3D, though it is still an open problem (see [F]). Nonetheless, the viscosity (or dissipation) does seem to introduce a length scale so that flows are smooth in a region of diameter smaller than this length scale after a suitable magnification. Related to this is the partial regularity theorem of Caffarelli-Kohn-Nirenberg [CKN], which says that the normalized energy concentrations at possible singularities of the solutions of the Navier-Stokes equations have to be definite amounts.

Topological singularities have always been an important object of much study over the last half century. Among the known examples are magnetic bubbles in a ferromagnetic continuum, vortices in superfluids, and superconductivity, topological defects of liquid crystals, as well as skyrmions, monopoles and instantons which are particle-like solutions (solitons) in generic models of high energy physics. These singularities not only carry definite topological informations but also quantized amounts of energy. As a result, they are often more stable energetically and dynamically. The value of topological solitary solutions in particle physics consists in the possibility of going beyond the paradigm of perturbation theory. Indeed, soliton solutions correspond to full nonlinear equations.
and disappear in their linearized form.

An example of such topological singularities that has attracted a great deal of attention is vortices in superconductors described by the Ginzburg-Landau equations. A systematic study of 2D vortices for a simplified static model can be found in references [BBH], [PR]. Rigorous justification of the dynamical behavior of the Ginzburg-Landau vortices can be found in [L]. Another interesting related example is the Landau-Lifschitz equation that model ferromagnets. A remarkable phenomena observe in planar model of Landau-Lifshitz equations is the existence of particle-like solutions (vortices or topological solitons). Formal physics arguments indicate that the behavior of these vortices is very similar to that of ordinary vortices observed in classical fluids. When these magnetic vortices (bubbles) are far apart and they move slowly in time, in other words, it is in a regime of “particle + field”, they display characteristics of the 2D Hall motion of interacting electric charges in a uniform magnetic field. However, rigorous mathematical proofs (see [LX]) are not only subtle, they also give rise to some difficult questions at the heart of physics. In particular, numerical simulations and physical experiments are needed to study the effects of so-called “sound waves” or “radiations” associated with the dynamics of such magnetic vortices.

So far I have restricted myself mostly to two-dimensional situations. Filaments (vortices in 3D), for example, are much harder to study than point vortices in 2D. In the Ginzburg-Landau equations much has been done, see e.g. [B], [L], [LR].

As for the dual points of view we explained earlier, the study of singularities also has remarkable implications in pure mathematics. For example, energy concentration sets of the Seiberg-Witten equations on a symplectic four-manifold are pseudo-holomorphic curves introduced by Gromov (see [T]). The topology (Seiberg-Witten invariants) of the modulo space of solutions of the Seiberg-Witten equations (which reflects the topology of the underlying 4-manifold) reduces to that of Gromov invariants of pseudo-holomorphic curves. The recent study of Ricci flows and their singularities (Hamilton’s program) by G. Perelman (see [M]) is probably one of the most exciting developments in the history of mathematics. The famous Poincaré conjecture as well as Thurston’ s geometrization conjecture may be solved by the Hamilton-Perelman program for the study of Ricci flow equations and their singularities.

Finally, I should point out that we have discussed here only singularities in a physical space. For many PDE problems, particularly those that study phenomena associated with high oscillations in certain directions, the singular behavior may also be in the so-called phase variables. For example, it arises in the study of propagation of singularities for hyperbolic PDEs. The mathematical tools such as $H$-measure ([Ta]), microlocal analysis and Wigner transforms are introduced especially for such purposes.

Analysis on Multiscale Problems

It is well-known that both space and time scales have played fundamental roles in mathematical modelings of many physical phenomena and also have contributed greatly
to the complexity of problems. For instance, one may usually consider mechanical oscillations as almost linear phenomena, but nonlinear effects become important if one is interested in the behavior of almost nondamped oscillations after many oscillation periods because the accumulation of small nonlinear effects may completely alter the system's long time behavior.

Another simple example is viscoelastic fluids (which we shall explain in slightly more detail below). These fluids exhibit elastic behavior as well as the usual fluid properties. On a very small scale, the nonlinear effect of elasticity is negligible, but even with a linear elasticity at a small scale, the contribution on the macroscopic (large scale) flow is rather nonlinear. In fact, such fluids are very different from the usual ones, they are non-Newtonian.

Problems involving many scales have long been of interest and were studied in mathematical physics, and often the physics as well as the mathematics involved at each level of scale are very different, even though one probably deals with only one continuum. In the example of fluids, it is well known at a quantum-mechanical scale that one studies fine structures and properties of electrons, and very often Schrödinger equations are used to model them. At a molecular scale, dynamics and interactions are usually described by Newton’s law of motion. In order to understand the evolution of clouds of molecules (on a scale of the mean free path), it is necessary to use kinetic theory, say Boltzmann equations. Finally, at the macroscopic level, fluids can be described by the density, velocity, pressure, and temperature fields, which obey the continuum Euler and Navier Stokes equations [Li].

Even with very good understanding of problems at each scale, many difficult issues remain to be resolved. Some typical questions that one has to answer for each (class) of problems are: How does one pass information from a smaller scale to a larger one? How does one analyze global effects when contributions from several scales are present? Before further discussion of these questions I note the following:

The current excitement in multiscale problems is driven mainly by the use of mathematical models in applied sciences, in particular, biology, chemistry, material sciences, fluid dynamics, and ocean-atmosphere sciences .... The timing of the current trend and interest is particularly worthwhile pointing out. On the science side, we have reached a stage of relatively mature knowledge of physics at various levels of scales including molecular dynamics. From a historical perspective, we are in an age when computers (with impressive computational capability) have become a more dominating factor in our lives, as well as an important tool in the sciences. There are also urgent needs from the sciences and technologies for multiscale modelings, analysis, and computation (nano-science being a good example, astronomy exploring the frontier and secrets of our universe is another).

Reiterating our first questions, we wish to understand microscopic contributions in macroscopic equations. Various rigorous mathematical analyses have been carried out to solve many problems. There are works that use the so-called semiclassical limits and weak convergence methods (see for example [Gr], [BGL]) to pass from the nonlinear Schrödinger equations or Boltzmann equations to the classical fluids equations. There
are also remarkable contributions (some by my colleagues at the Courant Institute, see [V] and [Y]) that pass from many particle (with statistical effects) systems to kinetic models as well as classical fluids models through so-call hydrodynamics. Wigner transforms and microlocal measures have also been used to transform Schrödinger-type equations to kinetic models (say Boltzmann equations), etc. However, as I learned from H. T. Yau, the rigorous passage from quantum mechanics (quantum many body problems) to, say, kinetic models are particularly difficult and there is a lack of rigorous mathematical justification, though there are so-called renormalization group methods.

The answers to the first question will certainly also help to answer the second question, that is, to understand the coupled effects of several scales, though, in general, that may be much harder. There are some interesting recent mathematical developments, for example, numerical techniques for multiscale dynamical systems with stochastic effects, see [Ma], [V]. I shall now mention some PDE analytical issues.

Our first example is viscoelastic fluids flows. Consider a viscoelastic fluid with macroscopic density \( \rho \) and macroscopic velocity \( v \). For simplicity, we assume the density to be a constant everywhere, and so the local conservation of mass leads to the incompressibility condition \( \text{div} \ v = 0 \). The conservation law for momentum leads to the following (phenomenological) equations (also called Oldroyd models):

\[
v_t + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla (W_F(F) \cdot F^T),
\]

where \( \mu \) is the fluid viscosity and \( F \) is the deformation matrix given by the flow map,

\[
\begin{align*}
\frac{dx}{dt} &= v(t, x) \\
x(0) &= \xi
\end{align*}
\]

that is \( F(t, x) \equiv \frac{\partial x}{\partial \xi} \).

Here \( W(F) \) is the local stored elastic energy density associated with the deformation \( F \).

The elastic property of fluids particles contributes the term \( \nabla (W_F(F) \cdot F^T) \) in the macroscopic equations, and it makes the viscoelastic fluid non-Newtonian. We note that even the microlocal elasticity in linear, i.e., \( W(F) = |F|^2/2 \), the macroscopic effect is still nonlinear given by \( (F, F^T) \). It is such nonlinear terms that made the Oldroyd models of fluids difficult to analyze mathematically. We also note that \( F \) actually satisfies a transport equation (which is, therefore, hyperbolic) of the form

\[
F_t + v \cdot \nabla F = \nabla v \cdot F.
\]

From the transport equation for \( F \), one concludes that \( F \) will not gain any additional regularity under the flow (in fact, \( F \) may lose regularity sometime). This together with the quadratic contribution of \( F \) into the momentum equation may cause the formation of singularities.

Nonetheless, we are able to have some basic mathematical understanding of viscoelastic fluids. There are two philosophical points involved in our study of this scale-coupled problem, which I believe could also be useful for other related issues. The first
point is that there is a natural energy law that governs such fluids flows. If the micro-
scopica contribution in the macroscopic equation is well-behaved, the the PDE systems
of the Oldroyd model will possess a certain stability because the viscosity term in the
momentum equation together with the energy law will make systems dissipative. On
the other hand, if the microscopic contribution in the macroscopic equation is not well-
behaved, that is, part of the energy or momentum of the system gives rise to some
additional “defect” stress (such defect stress may actually arise in a hydrodynamic
limiting process or an averaging process in some other problems), then the nonlinearity
may actually also give rise to dissipation through a rather different mechanism.
This latter scenario occurs in some nonlinear evolution problems. A classical well-
known example is the Lax-Glimm theorem, [La], on certain BV-solution of hyperbolic
conservation laws.

The second point is that Oldroyd models that describe viscoelastic fluid flow al-
ways has two features: macroscopic motions of fluids and microscopic dynamics that
contribute to some additional internal stress. Such flows may often be viewed as a
composition of two dynamics: the dynamic of “center-manifolds” and the dynamics
approaching the “center-manifold”. The “center-manifolds” in those flows will obey
various conservation laws of the original systems (hence such “center-manifolds” are
quite nonlinear). The dynamics approaching the “center-manifolds” is more like the
linearization of PDEs along conservation laws. Obviously a much more detailed anal-
ysis is needed to justify this dynamical picture. Indeed, by following this view, we
actually show both v and F in the Oldroyd model of viscoelastic fluids satisfying the
same type of “damped wave equations” through in the original system v satisfies a
parabolic equation and F satisfies a hyperbolic equation.

Similar approaches have also emerged in the study of the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f).$$

Here one of the key techniques (used by DiPerna-Lions [Li]) is to take the velocity aver-
ages, which is well-known in statistical physics, as macroscopic quantities. Magically,
the transport equations induce some improved partial regularity on velocity averages
exactly by some kind of dissipative effect. This is consistent with our first point. On
the other hand, it is noticed in the work [G] that linearizations at the Maxwellian solu-
tions (which is precisely the center-manifold we referred to), the Boltzmann equation
has certain elliptic properties that lead to improved regularity as well as dissipation of
the motion toward such Maxwellian solutions.

Another analytic example is a kinetic-hydrodynamic coupled PDE system for mo-
tions of a polymer in a solvent. There are many such models, see e.g. [BIAL11], and we
take the following
\[ f_t + v \cdot \nabla_x f + \nabla u \cdot \nabla f = \nu \Delta f + \text{div}_Q (\nabla Q \psi(Q) f) \]
\[ \text{div} \ v = 0 \]
\[ v_t + v \cdot \nabla v + \nabla \rho = \mu \Delta v \]
\[ \text{div} \ \left( \int Q \otimes \nabla Q \psi(Q) f \, dQ \right). \]

Here \( f = f(t, x, Q) \) denotes the distribution of the polymer, \( v \) is the macroscopic velocity, \( p \) the pressure, \( \mu, \nu \) are viscosities, and \( \psi(Q) \) is the chemical potential which is convex in \( Q \).

If \( \psi(Q) = |Q|^2/2 \) and we let \( \tau = \int Q \otimes Q f \, dQ \), then the above coupled Fokker-Planck and Navier-Stokes system reduces to the Oldroyd model
\[
\begin{cases}
  v_t + v \cdot \nabla v + \nabla \rho = \mu \Delta v - \text{div} \ \tau, \\
  \text{div} \ v = 0 \\
  \tau_t + v \cdot \nabla \tau + \omega \tau - \tau \omega = D(v) \tau.
\end{cases}
\]

Here \( \omega = \frac{\nabla v - (\nabla v)^T}{2} \) and \( D(v) = \frac{\nabla v + (\nabla v)^T}{2} \).

Again there is a natural energy law governing the polymer-system. One can follow the two points we have described to employ this coupled system to establish, for example, global dynamics near Maxwellian distributions of \( f \) and small velocity \( v \).

Finally, we want to mention the Faddeev model, which was first employed in nuclear physics and more recently (by Faddeev and Nami [F]) to model knotted solitons such as DNAs. Consider the space time, a 4-dimensional Minkowisk space \( \mathbb{R}^{3,1} \) with coordinates \( x_\mu, \mu = 0, 1, 2, 3, x_0 \) being time and \( x_k, k = 1, 2, 3 \) space variables. The field \( \vec{n}(x) \) is defined on \( \mathbb{R}^{3,1} \) and has values in the 2-dimensional sphere \( \mathbb{S}^2 : \vec{n} : \mathbb{R}^{3,1} \rightarrow \mathbb{S}^2 \). The Lagragian is given by the relativistic actional function
\[
A = a \int |\partial_\mu \vec{n}|^2 \, d^4x + b \int |\partial_\mu \vec{n} \wedge \partial_\nu \vec{n}|^2 \, d^4x
\]
and the corresponding static Hamiltonian energy
\[
E = a \int |\partial_k \vec{n}|^2 \, d^3x + b \int |\partial_k \vec{n} \wedge \partial_\ell \vec{n}|^2 \, d^3x.
\]

Unlike many field models, the Faddeev model has infinitely many ground states in infinitely many topological classes (see [LY]). In particular, one can have some particle like solutions of size several hundreds and thousands of times as large as others. That is very characteristic of protein DNAs. It is obvious the usual mean-field theory (hydrodynamics) would not work well to model the dynamics and interactions of such particles. One of the difficulties is that there is no clear scale separations under these circumstances. It is interesting to note a simple reason for the Faddeev model to have
particles-like solutions in infinitely many topological classes. Suppose the topological charge of such a particle is \( Q \). Then the energy carried by such a particle is proportional to \( |Q|^{3/4} \). In particular, for some large \( Q \)'s, one prefers to have basic particles of topological charge \( Q \) instead of splitting them into smaller charged particles.

To end the discussion of multi-scale problem, I cannot resist mentioning that before the age of computers, the solutions of mathematical models were obtained by special techniques. This often restricted the study to very simplified (maybe over simplified) equations. The computational capabilities now available have made a paradigm change in the way of analysing problems in sciences. Sometimes the mathematical models used may be close to reality when analytic techniques are replaced by scientific computations. There are exciting developments in numerical and computational multi-scale problems (see for example [D], [EE]). Yet, the ultimate validity of the models used in sciences must be rigorously analyzed and justified and the theory of PDEs will always play a significant role in our investigations.

References


